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Mathematics

On Homology Monoids of Simplicial Abelian Monoids

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ABSTRACT. The homology semimodules $H_n(S)$ of a presimplicial semimodule *S* are studied in the case where *S* is a simplicial abelian monoid. In particular, it is shown that if a simplicial abelian monoid *A* satisfies the Kan condition and the monoid of path components of *A* is a group, then the homology monoids $H_n(A)$ are isomorphic to the homotopy groups $®_n(A)$. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: simplicial abelian monoid, Kan condition, homology monoid, homotopy group

In [1] a version of homological algebra for semimodules and its applications are given (for further developments and applications see [2-4]). In the present paper, the homology semimodules $H_n(S)$ of a presimplicial semimodule *S*, introduced in [1], are studied in the case where *S* is a simplicial abelian monoid.

Let us begin with the following two definitions and theorem, the semimodule versions of which are given in [1].

Definition 1. We say that a sequence of abelian monoids and monoid homomorphisms

$$X : \cdots \rightrightarrows X_{n+1} \overset{\partial^+_{n+1}}{\underset{\partial^-_{n+1}}{\rightrightarrows}} X_n \overset{\partial^+_{n}}{\underset{\partial^-_{n}}{\rightrightarrows}} X_{n-1} \rightrightarrows \cdots, \qquad n \in \mathbb{Z},$$

written $X = \{X_n, \partial^+_n, \partial^-_n\}$ for short, is a *chain complex* if

$$\partial^+_n \partial^+_{n+1} + \partial^-_n \partial^-_{n+1} = \partial^+_n \partial^-_{n+1} + \partial^-_n \partial^+_{n+1}$$

for each integer *n*. For every chain complex *X*, we define the monoid

$$Z_n(X) = \left\{ x \in X_n \mid \partial^+_n(x) = \partial^-_n(x) \right\},$$

the *n-cycles*, and the *n-th homology* monoid

$$H_n(X) = Z_n(X) / \circ_n(X),$$

where $\circ_n(X)$ is a congruence on $Z_n(X)$ defined as follows:

$$x \circ_n(X)\, y \;\Leftrightarrow\; x + \partial^+_{n+1}(u) + \partial^-_{n+1}(v) = y + \partial^+_{n+1}(v) + \partial^-_{n+1}(u) \text{ for some } u, v \text{ in } X_{n+1}.$$

The homomorphisms $\partial^+_n, \partial^-_n$ are called *differentials* of the chain complex *X*.

Definition 2. Let $X=\{X_n,\partial_n^+,\partial_n^-\}$ and $X'=\{X'_n,\partial_n'^+,\partial_n'^-\}$ be chain complexes of abelian monoids. We say that a sequence $f=\{f_n\}$ of monoid homomorphisms $f_n:X_n\to X'_n$ is a $\pm$ - *morphism* from X to X' if

$$f_{n-1}\partial_n^+=\partial_n'^+f_n \quad \text{and} \quad f_{n-1}\partial_n^-=\partial_n'^-f_n \quad \text{for all} \quad n.$$

If $f=\{f_n\}:X\to X'$ is a $\pm$-morphism of chain complexes, then $f_n(Z_n(X))\subset Z_n(X')$, and the map

$$H_n(f):H_n(X)\to H_n(X'),\ \ H_n(f)(cl(x))=cl(f_n(x)),$$

is a homomorphism of monoids. Thus, H_n is a covariant additive functor from the category of chain complexes and their $\pm$-morphisms to the category of abelian monoids.

Recall that a presimplicial abelian monoid A is a sequence of abelian monoids $A_0,A_1,A_2,\dots$ together with monoid homomorphisms, called face operators,

$$\partial_n^i:A_n\to A_{n-1},\quad n\geq 1,\quad 0\leq i\leq n,$$

such that

$$\partial_n^i\partial_{n+1}^j=\partial_n^{j-1}\partial_{n+1}^i \quad \text{if} \quad 0\leq i<j\leq n+1.$$

Suppose $A=\{A_n,\partial_n^i\}$ and $B=\{B_n,¢_n^i\}$ are presimplicial abelian monoids. A morphism (or a presimplicial map) $f:A\to B$ is a collection of monoid homomorphisms $f_n:A_n\to B_n$ satisfying $f_{n-1}\partial_n^i=¢_n^i f_n$ for all i and for all n.

If A is a presimplicial abelian monoid, then

$$\underline{A}:\cdots\rightrightarrows A_n\overset{\partial_n^+}{\underset{\partial_n^-}{\rightrightarrows}}A_{n-1}\rightrightarrows\cdots\rightrightarrows A_2\overset{\partial_2^+}{\underset{\partial_2^-}{\rightrightarrows}}A_1\overset{\partial_1^+}{\underset{\partial_1^-}{\rightrightarrows}}A_0\rightrightarrows 0,$$

where

$$\partial_n^+=\partial_n^0+\partial_n^2+\cdots,\qquad \partial_n^-=\partial_n^1+\partial_n^3+\cdots,$$

is a nonnegative chain complex of abelian monoids. Using the greatest integer function, one can write

$$\partial_n^+=\sum_{k=0}^{[\frac{n}{2}]}\partial_n^{2k},\qquad \partial_n^-=\sum_{k=0}^{[\frac{n-1}{2}]}\partial_n^{2k+1}.$$

We define the *n-th homology monoid of the presimplicial abelian monoid A* by

$$H_n(A)=H_n(\underline{A}).$$

Clearly, if $f=\{f_n\}$ is a morphism from a presimplicial abelian monoid $A=\{A_n,\partial_n^i\}$ to a presimplicial abelian monoid $B=\{B_n,¢_n^i\}$, then $¢_n^+f_n=f_{n-1}\partial_n^+$ and $¢_n^-f_n=f_{n-1}\partial_n^-$ for all $n\geq 1$, that is, f can be regarded as a $\pm$-morphism from $\underline{A}$ to $\underline{B}$. Consequently, $H_n(A)$ is a covariant additive functor from the category of presimplicial abelian monoids and their morphisms to the category of abelian monoids.

Let $f=\{f_n\}$ and $g=\{g_n\}$ be morphisms from a presimplicial abelian monoid $A=\{A_n,\partial_n^i\}$ to a presimplicial abelian monoid $B=\{B_n,¢_n^i\}$. One says that f is presimplicially homotopic to g if there is a family h of monoid homomorphisms $h_n^i:A_n\to B_{n+1}$, $0\leq i\leq n$, $n=0,1,\dots$, such that

$$¢_{n+1}^0h_n^0=f_n,\quad ¢_{n+1}^{n+1}h_n^n=g_n,$$

$$\partial^i_{n+1} h^j_n = h^{j-1}_{n-1} \partial^i_n \quad \text{if} \quad i < j,$$

$$\partial^{j+1}_{n+1} h^{j+1}_n = \partial^{j+1}_{n+1} h^j_n \quad \text{if} \quad 0 \le j < n,$$

$$\partial^i_{n+1} h^j_n = h^j_{n-1} \partial^{i-1}_n \quad \text{if} \quad i > j+1.$$

Theorem 3. *Suppose that* $f : A \to B$ *and* $g : A \to B$ *are morphisms of presimplicial abelian monoids. If* f *is presimplicially homotopic to g, then*

$$H_n(f) = H_n(g) : H_n(A) \to H_n(B)$$

for all n.

For any presimplicial abelian monoid A, we define the *normalized chain complex* associated to A as follows:

$$N(A): \cdots \rightrightarrows N_n(A) \overset{d_n^+}{\underset{d_n^-}{\rightrightarrows}} N_{n-1}(A) \rightrightarrows \cdots \rightrightarrows N_2(A) \overset{d_2^+}{\underset{d_2^-}{\rightrightarrows}} N_1(A) \overset{d_1^+}{\underset{d_1^-}{\rightrightarrows}} N_0(A) \rightrightarrows 0,$$

where

$$N_0(A) = A_0, \quad N_1(A) = A_1 \cap Ker(\partial^0_1), \quad N_2(A) = A_2 \cap Ker(\partial^0_2) \cap Ker(\partial^1_2), \cdots,$$

$$N_n(A) = A_n \cap Ker(\partial^0_n) \cap \cdots \cap Ker(\partial^{n-1}_n), \cdots,$$

$$d_n^+ = \begin{cases} \partial^n_n \mid N_n(A) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \qquad d_n^- = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \partial^n_n \mid N_n(A) & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that the inclusions of abelian monoids $i_n : N_n(A) \to A_n$, $n \ge 0$, assemble into a $\pm$-morphism of nonnegative chain complexes

$$\begin{array}{cccccccccccccc}
N(A): & \cdots \rightrightarrows & N_n(A) & \overset{d_n^+}{\underset{d_n^-}{\rightrightarrows}} & N_{n-1}(A) & \rightrightarrows \cdots \rightrightarrows & N_2(A) & \overset{d_2^+}{\underset{d_2^-}{\rightrightarrows}} & N_1(A) & \overset{d_1^+}{\underset{d_1^-}{\rightrightarrows}} & N_0(A) & \rightrightarrows 0 \\
{\scriptstyle i_A}\downarrow & & {\scriptstyle i_n}\downarrow & & {\scriptstyle i_{n-1}}\downarrow & & {\scriptstyle i_2}\downarrow & & {\scriptstyle i_1}\downarrow & & {\scriptstyle i_0}\downarrow & \\
\underline{A}: & \cdots \rightrightarrows & A_n & \overset{\partial_n^+}{\underset{\partial_n^-}{\rightrightarrows}} & A_{n-1} & \rightrightarrows \cdots \rightrightarrows & A_2 & \overset{\partial_2^+}{\underset{\partial_2^-}{\rightrightarrows}} & A_1 & \overset{\partial_1^+}{\underset{\partial_1^-}{\rightrightarrows}} & A_0 & \rightrightarrows 0.
\end{array}$$

Consequently, for each n, we have a natural homomorphism of abelian monoids

$$H_n(i_A) : H_n(N(A)) \to H_n(\underline{A}) = H_n(A), \quad H_n(i_A)(cl(a)) = cl(a).$$

Now recall that a simplicial abelian monoid is a presimplicial abelian monoid A together with degeneracy homomorphisms

$$s^i_n : A_n \to A_{n+1}, \quad 0 \le i \le n,$$

satisfying

$$\partial^i_{n+1} s^j_n = \begin{cases} s^{j-1}_{n-1} \partial^i_n, & i < j, \\ \text{id}, & i = j,\ j+1, \\ s^j_{n-1} \partial^{i-1}_n, & i > j+1, \end{cases}$$

and

$$s^i_{n+1} s^j_n = s^{j+1}_{n+1} s^i_n, \qquad i \le j.$$

Elements of A_n are called n-simplices.

Let A and A' be simplicial abelian monoids. A simplicial map $f : A \to A'$ is a family of homomorphisms $(f_n : A_n \to A'_n)_{n\geq 0}$ which commute with the face and degeneracy operators.

One says that a simplicial abelian monoid A satisfies the Kan condition if for every collection of $n+1$ n-simplices $a_0, a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{n+1}$ satisfying the compatibility condition $\partial_n^i(a_j) = \partial_n^{j-1}(a_i)$, $i < j$, $i \neq k$, $j \neq k$, there exists an $(n+1)$-simplex a such that $\partial_{n+1}^i(a) = a_i$ for $i \neq k$.

Let A be a Kan simplicial abelian monoid. The abelian monoid $®_0(A)$, the monoid of path components of A, and the abelian group $®_n(A)$, $n \geq 1$, the n-th homotopy group of A, are defined as follows:

$$®_0(A) = A_0 / °_0, \quad ®_n(A) = \{a \in A_n \mid \partial_n^i(a) = 0,\ 0 \leq i \leq n\} / °_n, \quad n \geq 1,$$

where $°_n$, $n \geq 0$, is a congruence given by

$$a \, °_n \, b \iff \text{there is } c \in A_{n+1} \text{ with } \partial_{n+1}^n(c) = a,$$

$$\partial_{n+1}^{n+1}(c) = b, \text{ and } \partial_{n+1}^i(c) = 0 \quad \text{for } 0 \leq i < n.$$

(See e.g. [5] for details.)

Theorem 4. *Let A be a Kan simplicial abelian monoid. Then the monoid $H_n(N(A))$, $n \geq 1$, is a group, and coincides with the group $®_n(A)$. If, in addition, $®_0(A)$ is a group, then $H_0(N(A))$ is also a group, and coincides with $®_0(A)$.*

Before we continue, we recall that the group completion of an abelian monoid S can be constructed in the following way. Define an equivalence relation ~ on $S \times S$ as follows:

$$(u, v) \sim (x, y) \Leftrightarrow u + y + z = v + x + z \quad \text{for some} \quad z \in S.$$

Let $[u, v]$ denote the equivalence class of (u, v). The quotient set $(S \times S)/\sim$ with the addition $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ is an abelian group $(0 = [x, x], -[x, y] = [y, x])$. This group, denoted by $K(S)$, is the group completion of S. The canonical homomorphism $k_S : S \to K(S)$ sends x to $[x,0]$.

Proposition 5. *Suppose that A is a simplicial abelian monoid, $K(A)$ its group completion and $k_A : A \to K(A)$ the canonical simplicial map. Then the induced homomorphism*

$$H_n(k_A) : H_n(A) \to H_n(K(A)), \qquad H_n(k_A)(cl(a)) = cl([a, 0]),$$

is an injection for each n.

This proposition, Theorem 4 and Theorem 1.7 of [6] lead to

Theorem 6. *Let A be a Kan simplicial abelian monoid such that $®_0(A)$ is a group. Then the map*

$$H_n(i_A) : H_n(N(A)) \to H_n(A), \quad H_n(i_A)(cl(a)) = cl(a),$$

is an isomorphism for all n.

Combining Theorems 4 and 6, we have

Corollary 7. *For any Kan simplicial abelian monoid A with $®_0(A)$ a group, $H_n(A)$ is isomorphic to $®_n(A)$ for all n.*

Note that all the results stated here for simplicial abelian monoids admit straightforward generalizations to the case of simplicial semimodules.

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(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

პრესიმპლიციალური S ნახევრადმოდულის ჰომოლოგიის $H_n(S)$ ნახევრადმოდულები შესწავლილია იმ შემთხვევაში, როცა S არის სიმპლიციალური აბელის მონოიდი. კერძოდ, დადგენილია, რომ თუ სიმპლიციალური აბელის მონოიდი A აკმაყოფილებს კანის პირობას და $\text{®}_0(A)$ ჯგუფია, მაშინ $H_n(A)$ მონოიდი $\text{®}_n(A)$ ჯგუფის იზომორფულია ($n \geq 0$).

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