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Mathematics

Boundary Value Problems on an Infinite Interval for Singular in Phase Variables Two-Dimensional Differential Systems

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ABSTRACT. For singular in phase variables two-dimensional differential systems, optimal in a certain sense conditions guaranteeing the existence of positive solutions of boundary value problems on an infinite interval are found. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *two-dimensional differential system, singular in phase variables, boundary value problem on an infinite interval, the nonlinear Kneser problem, positive solution.*

Let $a>0$, $R_-=]-\infty,0]$, $R_+=[0,+\infty[$, and $R_{0+}=]0,+\infty[$. On a positive semi-axis R_{0+}, we consider the differential system

$$\frac{du_i}{dt}=f_i\big(t,u_1,u_2\big)\quad(i=1,2) \tag{1}$$

with one of the following two types boundary conditions:

$$\lim_{t\to+\infty}u_i(t)=c_i\quad(i=1,2) \tag{2}$$

and

$$\int_0^a\varphi\big(u_1(s)\big)d\sigma(s)=c_0,\quad \lim_{t\to+\infty}u_2(t)=c\,. \tag{3}$$

Below everywhere, unless the contrary is stated, it is assumed that c_0 is a positive constant, c and c_i $(i=1,2)$ are nonnegative constants, $f_i:R_{0+}\times R_{0+}{}^2\to R_-$ $(i=1,2)$ are continuous functions, and $\varphi:R_+\to R_+$ is a continuous nondecreasing function such that

$$\lim_{x\to+\infty}\varphi(x)=+\infty\,.$$

As for $\sigma:[0,a]\to R_+$, it is a nondecreasing function satisfying the equality

$$\sigma(a)-\sigma(0)=1\,.$$

A continuously differentiable vector function $(u_1,u_2): R_{0+} \to R_{0+}{}^2$, satisfying system (1) in R_{0+}, is said to be **a positive solution** of that system.

If the component u_i of a positive solution (u_1,u_2) at the point 0 has the right-hand limit

$$u_i(0+) = \lim_{t>0,\, t\to 0} u_i(t),$$

then we put $u_i(0) = u_i(0+)$.

A positive solution (u_1,u_2) of system (1) is said to be **a positive solution of problem (1),(2)** (**of problem (1),(3)**) if it satisfies conditions (2) (there exists $u_1(0+)$ and equalities (3) are satisfied).

If $c_1 = c_2 = 0$, then a positive solution of problem (1),(2) is said to be **a vanishing at infinity positive solution of system (1)**.

If

$$f_1(t,x,y) \equiv -y, \quad f_2(t,x,y) \equiv -f(t,x,-y),$$

and $c = c_2 = 0$, then the differential system (1) is equivalent to the differential equation

$$u'' = f(t,u,u'), \tag{4}$$

and conditions (2) and (3) are equivalent to the conditions

$$\lim_{t\to+\infty} u(t) = c_1 \tag{5}$$

and

$$\int_0^a \varphi\big(u(s)\big)\, d\sigma(s) = c_0, \tag{6}$$

respectively. Consequently, problem (1),(2) (problem (1),(3)) has a positive solution if and only if problem (4),(5) (problem (4),(6)) has a so-called Kneser solution, i.e. a solution satisfying the inequalities

$$u(t) > 0, \quad u'(t) < 0 \quad \text{for} \quad t \in R_{0+}.$$

Problem (1),(3), as problem (4),(6), is said to be the nonlinear Kneser problem. These problems are investigated in detail in the case where the functions f_i $(i = 1,2)$ and f have no singularities in phase variables (see, e.g., [1]-[9], and the references therein).

In [10], for the singular in a phase variable equation (4), sufficient conditions for the existence of a Kneser solution satisfying one of conditions (5) and (6) are established. In the present paper, these results are generalized for system (1), having the singularities in phase variables. More precisely, sufficient conditions for the existence of at least one positive solution for the singular problem (1),(2), as well as for the singular problem (1),(3), are established.

Throughout the paper, it is assumed that the functions f_i $(i = 1,2)$ on the set $R_{0+} \times R_{0+}{}^2$ admit the estimates

$$g_{10}(t) \le -x^{\lambda_1} y^{-\mu_1} f_1(t,x,y) \le g_1(t),$$
$$g_{20}(t) \le -x^{\lambda_2} y^{\mu_2} f_2(t,x,y) \le g_2(t),$$

where λ_i and μ_i $(i = 1,2)$ are nonnegative constants, and $g_{i0}: R_{0+} \to R_{0+}$, $g_i: R_{0+} \to R_{0+}$ $(i = 1,2)$ are continuous functions. Moreover, we use the following notations.

$$\nu_0 = \frac{\mu_1}{1+\mu_2}, \quad \nu = 1 + \lambda_1 + \lambda_2 \nu_0.$$

If $\lambda_i > 0$ for some $i \in \{1,2\}$, then

$$\lim_{x\to 0} f_i(t,x,y) = +\infty \quad \text{for} \quad t>0,\ \ y>0 .$$

And if $\mu_2 > 0$, then

$$\lim_{y\to 0} f_2(t,x,y) = +\infty \quad \text{for} \quad t>0,\ \ x>0 .$$

Consequently, in both cases system (1) has the singularity in at least one phase variable.

First we consider problem (1),(2). The following theorem is valid.

Theorem 1. *If*

$$\int_t^{+\infty} g_i(s)ds < +\infty \quad \text{for} \quad t>0 \ \ (i=1,2),$$

then for any $c_1 > 0$ *and* $c_2 \ge 0$, *problem* (1),(2) *has at least one positive solution.*

Theorem 2. *If*

$$c_1 > 0, \quad \int_{t_0}^{+\infty} g_{10}(s)ds = +\infty, \tag{7}$$

where $t_0 > 0$, then the condition

$$c_2 = 0, \quad \int_{t_0}^{+\infty} g_{20}(s)ds < +\infty, \quad \int_{t_0}^{+\infty} g_{10}(t)\left(\int_t^{+\infty} g_{20}(s)ds\right)^{\nu_0} dt < +\infty \tag{8}$$

is necessary, and the condition

$$c_2 = 0, \quad \int_{t_0}^{+\infty} g_2(s)ds < +\infty, \quad \int_{t_0}^{+\infty} g_1(s)\left(\int_s^{+\infty} g_2(\tau)d\tau\right)^{\nu_0} ds < +\infty$$

is sufficient for the existence of at least one positive solution of problem (1),(2).

Corollary 1. *Let along with* (7) *the following condition*

$$\limsup_{t\to+\infty} \frac{g(t)}{g_0(t)} < +\infty \tag{9}$$

hold. Then for the existence of at least one positive solution of problem (1),(2) *it is necessary and sufficient condition* (8) *to be fulfilled* .

Theorem 3. *If*

$$\int_t^{+\infty} g_{20}(s)ds < +\infty, \quad w_0(t) \equiv \int_t^{+\infty} g_{10}(s)\left(\int_s^{+\infty} g_{20}(\tau)d\tau\right)^{\nu_0} ds < +\infty \quad \text{for} \quad t>0,$$

and

$$w(t) \equiv \int_t^{+\infty} w_0^{-\frac{\lambda_2}{\nu}}(s) g_2(s)ds < +\infty, \quad \int_t^{+\infty} g_1(s) w^{\nu_0}(s)ds < +\infty \quad \text{for} \quad t>0,$$

then system (1) *has at least one vanishing at infinity positive solution.*

Theorems 2 and 3 imply the following statement.

Corollary 2. *Let*

$$\liminf_{t\to+\infty}\left(t^{1-\alpha}g_{10}(t)\right)>0,\quad \limsup_{t\to+\infty}\left(t^{1-\alpha}g_1(t)\right)<+\infty, \tag{10}$$

$$\liminf_{t\to+\infty}\left(t^{\beta}g_{20}(t)\right)>0,\quad \limsup_{t\to+\infty}\left(t^{\beta}g_2(t)\right)<+\infty, \tag{11}$$

where α *and* β *are nonnegative constants. Then for the existence of at least one vanishing at infinity positive solution of system* (1) *it is necessary and sufficient that*

$$\beta>\frac{1+\mu_2}{\mu_1}\alpha+1.$$

Consider now problem (1),(3).

If

$$\int_0^{+\infty} g_i(s)ds<+\infty \quad (i=1,2), \tag{12}$$

then on the set $R_+\times R_{0+}\times R_+$ we put

$$w_0(t,x,y)=\left[x^{\nu}+\nu\int_t^{+\infty}g_{10}(s)\left(x^{\lambda_2}y^{1+\mu_2}+(1+\mu_2)\int_s^{+\infty}g_{20}(\tau)d\tau\right)^{\nu_0}ds\right]^{\frac{1}{\nu}},$$

$$w(t,x,y)=\left[y^{1+\mu_2}+(1+\mu_2)\int_t^{+\infty}w_0^{-\lambda_2}(s,x,y)g_2(s)ds\right]^{\frac{1}{1+\mu_2}},$$

$$w_1(t,x,y)=\left[x^{1+\lambda_1}+(1+\lambda_1)\int_t^{+\infty}w^{\mu_1}(s,x,y)g_1(s)ds\right]^{\frac{1}{1+\lambda_1}}.$$

And if

$$\int_t^{+\infty}g_2(s)ds<+\infty \quad \text{for } t>0,\quad \int_0^{+\infty}g_1(s)\left(\int_s^{+\infty}g_2(\tau)d\tau\right)^{\nu_0}ds<+\infty, \tag{13}$$

then on the set $R_+\times R_{0+}$ we put

$$v_0(t,x)=\left[x^{\nu}+\nu(1+\mu_2)^{\nu_0}\int_t^{+\infty}g_{10}(s)\left(\int_s^{+\infty}g_{20}(\tau)d\tau\right)^{\nu_0}ds\right]^{\frac{1}{\nu}},$$

$$v_1(t,x)=\left[x^{1+\lambda_1}+(1+\lambda_1)\int_t^{+\infty}v^{\mu_1}(s,x)g_1(s)ds\right]^{\frac{1}{1+\lambda_1}},$$

where

$$v(t,x)=\left[(1+\mu_2)\int_t^{+\infty}v_0^{-\lambda_2}(s,x)g_2(s)ds\right]^{\frac{1}{1+\mu_2}} \quad \text{for } t>0,\ x>0.$$

Theorem 4. *If along with* (12) *the condition*

$$\inf\left\{\int_0^a \varphi\big(w_1(s,x,c)\big)d\sigma(s):\ x>0\right\}<c_0$$

is satisfied, then problem (1),(3) *has at least one positive solution.*

Theorem 5. *Let either*

$$c>0,\quad \int_t^{+\infty} g_{10}(s)ds=+\infty \quad \text{for}\ \ t>0,$$

or

$$c\ge 0,\quad \int_t^{+\infty} g_{20}(s)ds<+\infty \ \ \text{for}\ \ t>0,\quad \int_0^{+\infty} g_{10}(s)\left(\int_s^{+\infty} g_{20}(\tau)d\tau\right)^{\nu_0} ds<+\infty$$

and

$$\int_0^a \varphi\big(v_0(s,0)\big)d\sigma(s)>c_0\,.$$

Then problem (1),(3) *has no solution.*

Theorem 6. *Let* $c=0$, *and let along with* (13) *the condition*

$$\inf\left\{\int_0^a \varphi\big(v_1(s,x)\big)d\sigma(s):\ x>0\right\}<c_0$$

be satisfied. Then problem (1),(3) *has at least one positive solution.*

Theorems 5 and 6 yield the following propositions.

Corollary 3. *Let*

$$\int_{t_0}^{+\infty} g_{10}(s)ds=+\infty\,,$$

where $t_0>0$. *Then for the existence of at least one positive solution of problem* (1),(3) *for every sufficiently large* c_0, *it is necessary and sufficient that*

$$c=0,\quad \int_t^{+\infty} g_{20}(s)ds<+\infty \ \ \text{for}\ \ t>0,\quad \int_0^{+\infty} g_{10}(s)\left(\int_s^{+\infty} g_{20}(\tau)d\tau\right)^{\nu_0} ds<+\infty\,.$$

Corollary 4. *Let conditions* (10) *and* (11) *hold, where* α *and* β *are nonnegative constants. Then for the existence of at least one positive solution of problem* (1),(3) *for every sufficiently large* c_0, *it is necessary and sufficient that*

$$c=0,\quad \beta>\frac{1+\mu_2}{\mu_1}\alpha+1\,.$$

Finally we note that the proofs of the above formulated theorems are based on the results of [11].

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