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On transitional solutions of second order nonlinear differential equations ☆

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1. Introduction

This paper deals with the existence of solutions $u \in C^2(\mathbb{R})$ of the nonlinear differential equation

$$u'' = f(t, u, u') \tag{1.1}$$

satisfying the conditions

$$\lim_{t\to-\infty} u(t) = 0, \quad \lim_{t\to+\infty} u(t) = 1, \quad 0 \leqslant u(t) \leqslant 1 \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous and such that

$$f(t, 0, 0) = 0, \quad f(t, 1, 0) = 0 \quad \text{for } t \in \mathbb{R}. \tag{1.3}$$

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Due to (1.3), every solution of (1.1), (1.2) connects the stationary states $u_0(t) \equiv 0$ and $u_1(t) \equiv 1$ and lies between them. For this reason it is called a transitional solution.

Problems of the type (1.1), (1.2) originate from the investigation of traveling wave solutions of reaction–diffusion equations which model several biological phenomena (see [10]). Indeed recall that a traveling wave solution v has a constant profile, that is such that $v(t,x) = u(x - c\tau)$ and satisfies the equation

$$u'' - \big(c + h(u)\big)u' + g(u) = 0, \tag{1.4}$$

where $t := x - c\tau$ is the wave coordinate, $g(u)$ is the nonlinear reaction term which vanishes at 0 and 1 and h is a convective effect. The wave speed c is a further unknown of the problem.

A very wide literature is devoted to the study of the existence, uniqueness and stability of traveling wave solutions; we only quote the recent monograph [2] for quite general results and the large bibliography there contained. However, most of the results concerns dynamics without convective effects, i.e., for $h(t) \equiv 0$. In [6] the existence of a unique (up to space-shifts) monotone solution of (1.4), (1.2) is proven, when g and h are continuous with $0 < g(u) \leqslant L^2 u$ for all $u \in]0,1]$ and c is greater or equal than a certain threshold value.

Problems of the type (1.1), (1.2) also appear in the study of some physical processes when the variable transits from an unstable equilibrium state into a stable one. The contribution by Klokov [5] fits into this context and deals with the case when Eq. (1.1) has the form

$$u'' = g_1(u,u')u' - g_2(u). \tag{1.5}$$

In particular, in [5, Theorem 21] the unique solvability (up to space-shifts) of problem (1.5), (1.2) is proven when the functions g_1 and g_2 are continuous on [0, 1] and satisfy the conditions

$$\begin{gathered} g_2(0) = g_2(1) = 0, \quad g_1(u,u') \geqslant g_0(u) > 0, \\ 0 < g_2(u) \leqslant \frac{1}{4} g_0(u) \int_0^u g_0(s)\,ds \end{gathered} \tag{1.6}$$

for all $0 < u < 1$, $u' \geqslant 0$ and some nonnegative continuous function g_0.

The general case when (1.1) is autonomous, that is when $f = f(u,u')$, was recently investigated by the first two authors. In [7, Theorem 4.3] the existence of a monotone solution of problem (1.1), (1.2) is proven when assuming (1.3) and

$$f(u,0) < 0, \qquad f(u,u') \geqslant 2Lu' - L^2 u \tag{1.7}$$

or the symmetric conditions

$$f(u,0) > 0, \qquad f(u,u') \leqslant -2Lu'y + L^2(1-u) \tag{1.7$'$}$$

for all $0 < u < 1$, $u' \geqslant 0$ and some constant $L > 0$. This result also deals with the uniqueness and the nonexistence problems in the autonomous case. As it is easy to see, it includes the quoted one in [5] only in the case when g_1 is constant.

The nonautonomous problem was first extensively investigated by Volpert and Suhov [13] for equations having the structure

$$u'' - cu' + g(t, u) = 0 \tag{1.8}$$

with $g_-(u) \leqslant g(t, u) \leqslant g_+(u)$ for all $0 < u < 1$, $t \in \mathbb{R}$, $g_\mp(0) = g_\mp(1) = 0$ and $g_-(u) > 0$ for $0 < u < 1$. The problem arises when studying stationary nonconstant solutions of a semi-linear parabolic equation describing a chemical reaction. Under additional strong regularity conditions on all $g_\mp$ and g, the existence of infinitely many solutions satisfying (1.8), (1.2) is showed in [13, Theorem 3.3], for all sufficiently large c. We also mention the contribution by Sanchez [12] concerning again problem (1.8), (1.2) in the case when g has a product-type structure, that is $g(t, u) = a(t)g(u)$.

The same multiplicity result given in [13] was then obtained in [7, Theorem 5.1] for the general problem (1.1), (1.2) when assuming

$$2Lu' - L^2 u \leqslant \varphi_1(u, u') \leqslant f(t, u, u') \leqslant \varphi_2(u, u'), \qquad \varphi_2(u, 0) < 0 \tag{1.9}$$

whenever $0 < u < 1$, $u' \geqslant 0$, for some constant $L > 0$, and continuous functions φ_i: $\mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ satisfying $\varphi_i(0, 0) = \varphi_i(1, 0) = 0$ $(i = 1, 2)$.

The problem to find positive bounded solutions of a second order dynamics with assigned conditions at infinity also arises in other contexts and recent contributions appeared, dealing with different situations. We refer, in particular to [3,8,9] and the references there contained as well as to the books by Agarwal et al. [1] and by O'Regan [11].

The aim of this paper is to give new existence results for (1.1), (1.2) (see Theorems 2.1 and 2.1$'$) which generalize and unify all the previous quoted discussion concerning this problem. Precisely, they include the results in [5] and [7] (see Remark 2.1); moreover, in the nonautonomous case Theorems 2.1 and 2.1$'$ also allow to treat some cases when $f(\cdot, u, u')$ is unbounded in $\mathbb{R}$ or vanishes when $t \to \pm\infty$, which were never investigated in any previous quoted discussion (see Remark 2.2).

Theorems 2.1 and 2.1$'$ differ for symmetric sign conditions on the right-hand side f of (1.1). Instead, Theorem 2.4 provides an existence result for (1.1), (1.2) which is based on a different type of growth and sign conditions on the function f. Together with the other ones, it can be considered as a further achievement in the theory of boundary value problems on infinite domains which is, in our opinion, far from being completely investigated.

The main technique for proving all these results derives from the comparison-type theory introduced by Kiguradze and Shekhter [4] for studying the existence of solutions of (1.1) such that

$$\gamma_1(t) \leqslant u(t) \leqslant \gamma_2(t) \quad \text{for } t \in \mathbb{R}, \tag{1.10}$$

where $\gamma_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are prescribed continuous functions satisfying the inequality

$$\gamma_1(t) \leqslant \gamma_2(t) \quad \text{for } t \in \mathbb{R}. \tag{1.11}$$

In Remark 2.3 we discuss the conditions assumed in our results and show that, in all cases, they are optimal in a certain sense.

All our theorems allow us to obtain expressive sufficient conditions for the solvability of problem (1.1), (1.2) for differential equations having one of the following structures:

$$u'' = p_1(t)u' + p_2(t)u(1-u), \tag{1.12}$$

$$u'' = p_1(t)f_1(u,u')u' + p_2(t)f_2(u,u'), \tag{1.13}$$

$$u'' = f_1(t,u)u' + f_2(t,u). \tag{1.14}$$

This discussion is contained in Corollaries 2.2, 2.3, 2.5 and 2.6.

The paper is organized as follows. In Section 2 we state all the main results together with the connected remarks. Their proofs are given in Section 4. Section 3 is devoted to some auxiliary lemmas.

2. Statement of the main results

Our first existence result extends and unifies the quoted results in [5] and [7].

Theorem 2.1. *Let there exist a real number a, continuous functions $h:\mathbb{R}\to[0,+\infty[$, $\delta:]0,1/2[\to]0,+\infty[$ and a C^1-function $w:]-\infty,a]\times[0,1]\to[0,+\infty[$ such that, along with* (1.3) *the following conditions are satisfied, where $f^*(t,x) := \max\{f(t,s,y)\colon\ x\leqslant s\leqslant 1-x,\ 0\leqslant y\leqslant\delta(x)\}$:*

$$f(t,x,y) \geqslant -h(t)\left(1+y^2\right) \quad \textit{for } t\in\mathbb{R},\ 0\leqslant x\leqslant 1,\ y\geqslant 0, \tag{2.1}$$

$$f^*(t,x) < 0 \quad \textit{for } t\in\mathbb{R},\ 0<x<\frac{1}{2}, \tag{2.2}$$

$$\int_0^{+\infty} s f^*(s,x)\,ds = -\infty \quad \textit{for } 0<x<\frac{1}{2}, \tag{2.3}$$

$$w(t,0)=0, \qquad \int_{-\infty}^{a} w(s,x)\,ds = +\infty, \qquad \frac{\partial w(t,x)}{\partial x}\geqslant 0$$
$$\textit{for } t\leqslant a,\ 0<x<1, \tag{2.4}$$

$$f\big(t,x,w(t,x)\big) \geqslant w(t,x)\frac{\partial w(t,x)}{\partial x} + \frac{\partial w(t,x)}{\partial t} \quad \textit{for } t\leqslant a,\ 0\leqslant x\leqslant 1. \tag{2.5}$$

Then problem (1.1), (1.2) *has at least one solution such that*

$$u'(t) > 0 \quad \textit{whenever } 0<u(t)<1. \tag{2.6}$$

As in [7], the above result has a symmetric statement.

Theorem 2.1′. *Suppose there exist a real number a, continuous functions $h:\mathbb{R}\to[0,+\infty[$, $\delta:]0,1/2[\to]0,+\infty[$, and a C^1-function $w:[a,+\infty[\times[0,1]\to[0,+\infty[$ such that, along with* (1.3) *the following conditions are fulfilled, where $f_*(t,x) := \min\{f(t,s,y)\colon\ x\leqslant s\leqslant 1-x,\ 0\leqslant y\leqslant\delta(x)\}$:*

$$f(t,x,y) \leqslant h(t)\left(1+y^2\right) \quad \textit{for } t\in\mathbb{R},\ 0\leqslant x\leqslant 1,\ y\geqslant 0, \tag{2.1′}$$

$$f_*(t,x) > 0 \quad \textit{for } t\in\mathbb{R},\ 0<x<\frac{1}{2}, \tag{2.2′}$$

$$\int_{-\infty}^{0} s f_*(s,x)\,ds = -\infty \quad \text{for } 0 < x < \frac{1}{2}, \tag{2.3$'$}$$

$$w(t,1) = 0, \quad \int_{a}^{+\infty} w(s,x)\,ds = +\infty, \quad \frac{\partial w(t,x)}{\partial x} \leqslant 0$$
$$\text{for } t \geqslant a,\ 0 < x < 1, \tag{2.4$'$}$$

$$f\big(t,x,w(t,x)\big) \leqslant w(t,x)\frac{\partial w(t,x)}{\partial x} + \frac{\partial w(t,x)}{\partial t} \quad \text{for } t \geqslant a,\ 0 \leqslant x \leqslant 1. \tag{2.5$'$}$$

Then problem (1.1), (1.2) *has at least one solution satisfying condition* (2.6).

As applications of the previous results, we now provide some simple sufficient conditions for the solvability of problem (1.1), (1.2) when the right-hand side f has one of the structures (1.12), (1.13).

Corollary 2.2. *Let us consider Eq.* (1.13) *with* $p_1, p_2 \in C(\mathbb{R})$ *and* $f_1, f_2 \in C([0,1] \times \mathbb{R})$ *given functions such that*

$$f_2(0,0) = f_2(1,0) = 0, \quad f_2(x,0) > 0 \quad \text{for } 0 < x < 1 \tag{2.7}$$

and

$$f_1(x,y) \geqslant f_0(x) > 0, \quad f_2(x,y) \leqslant f_0(x)\int_0^x f_0(s)\,ds \quad \text{for } x \in\,]0,1[,\ y \geqslant 0, \tag{2.8}$$

for some nonnegative function $f_0 \in C([0,1])$. *Moreover, suppose that constants* $a \in \mathbb{R}$ *and* $\alpha > 0$ *exist in such a way that conditions*

$$\int_0^{+\infty} s p_1(s)\,ds = +\infty, \qquad p_1(t) > 0, \quad p_2(t) \leqslant -\alpha p_1(t) \quad \text{for } t \in \mathbb{R}, \tag{2.9}$$

$$p_1 \in C^1\big(]-\infty, a]\big), \quad p_1'(t) \leqslant 0 \quad \text{for } t \leqslant a \tag{2.10}$$

are satisfied together with

$$p_2(t)\,\operatorname{sgn} t \leqslant \frac{1}{4}p_1^2(t) - \frac{1}{2\beta}p_1'(t) \quad \text{for } t \leqslant a, \tag{2.11}$$

where $\beta := \sup\{f_0(x):\ 0 \leqslant x \leqslant 1\}$. *Then problem* (1.13), (1.2) *admits solutions satisfying condition* (2.6). *The same assertion holds when replacing* (2.8), (2.9), *and* (2.10), *respectively with the following conditions*:

$$f_1(x,y) \geqslant f_0(x) > 0, \quad f_2(x,y) \leqslant f_0(x)\int_x^1 f_0(s)\,ds$$
$$\text{for } 0 < x < 1,\ y \geqslant 0, \tag{2.8$'$}$$

$$\int_{-\infty}^{0} sp_1(s)\,ds = +\infty, \qquad p_1(t) < 0, \quad p_2(t) \geqslant \alpha\bigl|p_1(t)\bigr| \quad \textit{for } t \in \mathbb{R}, \tag{2.9$'$}$$

$$p_1 \in C^1\bigl([a, +\infty[\bigr), \quad p_1'(t) \leqslant 0 \quad \textit{for } t \geqslant a \tag{2.10$'$}$$

and (2.11) *is satisfied for* $t \geqslant a$.

Corollary 2.3. *Let us consider Eq.* (1.12) *with* $p_1, p_2 \in C(\mathbb{R})$ *given functions. Suppose that constants* $a \in \mathbb{R}$ *and* $\alpha > 0$ *exist in such a way that* (2.9), (2.10) *or* (2.9′), (2.10′) *are satisfied. Moreover, assume that*

$$p_2(t)\operatorname{sgn} t \leqslant \frac{1}{4}p_1^2(t) - \frac{1}{2}p_1'(t) \quad \textit{for } |t| \geqslant a. \tag{2.12}$$

Then problem (1.12), (1.2) *is solvable and each solution* u *is such that*

$$0 < u(t) < 1, \quad u'(t) > 0 \quad \textit{for } t \in \mathbb{R}.$$

If instead of (2.12) *we have in* $]-\infty, a]$ *or in* $[a, +\infty[$ (*according to what of the pair of conditions* (2.9), (2.10) *or* (2.9′), (2.10′) *is satisfied*)

$$p_2(t)\operatorname{sgn} t \geqslant \frac{1+\varepsilon}{4}\,p_1^2(t) - \frac{1+\varepsilon}{2}\,p_1'(t) \tag{2.13}$$

for some $\varepsilon > 0$, *then problem* (1.12), (1.2) *has no solution.*

Remark 2.1. If $p_1(t) \equiv 2$, $p_2(t) \equiv -1$, $f_1(x, y) = \frac{1}{2}g_1(x, y)$, $f_2(x, y) \equiv g_2(x)$, and conditions (1.6) are fulfilled, then the functions p_i and f_i $(i = 1, 2)$ satisfy conditions (2.7)–(2.11). Moreover, if $p_1(t) \equiv 2L$, $p_2(t) \equiv -L^2$ $(p_1(t) \equiv -2L$, $p_2(t) \equiv L^2)$, $f_1(x, y) \equiv 1$ and $f_2(x, y) = 2L^{-1}y - L^{-2}g(x, y)$ $(f_2(x, y) = L^{-2}g(x, y) - 2L^{-1}y)$, and conditions (1.7) (conditions (1.7′)) are fulfilled, then the functions p_i and f_i $(i = 1, 2)$ satisfy conditions (2.7)–(2.11) (conditions (2.7), (2.8′), (2.9′), (2.10′) and (2.11)). Therefore, Corollary 2.2 generalizes the results of [5] and [7] concerning the existence of solutions respectively of (1.5) and $u'' = f(u, u')$ which satisfies (1.2). As for [7, Theorem 5.1], it follows from Theorem 2.1 since inequalities (1.9) guarantee the fulfillment of (1.3), (2.1)–(2.5) with $h(t) \equiv L^2 + 2L$, $w(t, x) \equiv Lx$.

Remark 2.2. Suppose conditions (2.7)–(2.10) are satisfied and inequality (2.11) holds in some interval $]-\infty, a]$. Moreover, either

$$\sup\bigl\{\bigl|p_2(t)\bigr|\colon t \in \mathbb{R}\bigr\} = +\infty, \tag{2.14}$$

or

$$f_2(x, y) > 0 \quad \text{for } 0 < x < 1,\ y \geqslant 0, \text{ and } \inf\bigl\{p_1(t)\colon t \in \mathbb{R}\bigr\} = 0. \tag{2.15}$$

Then, by virtue of Corollary 2.2, problem (1.13), (1.2) is solvable. On the other hand, this problem cannot be studied by [7, Theorem 5.1] since the function $f(t, x, y) = p_1(t)f_1(x, y)y + p_2(t)f_2(x, y)$ does not satisfy condition (1.9). Indeed, otherwise for some $\ell > 0$ we would have $p_1(t)f_1(x, y)y + p_2(t)f_2(x, y) \geqslant 2\ell y - \ell^2 x$ for $0 < x < 1$,

$y \geqslant 0$. Thus $|p_2(t)| \leqslant \frac{\ell^2}{2f_2(1/2,0)}$ for $t \in \mathbb{R}$, and $p_1(t) f_1(x, \ell x)\ell x \geqslant \ell^2 x + |p_2(t)| f_2(x, \ell x)$ for $t \in \mathbb{R}$, $x \in\,]0, 1[$. However, the first of these last two inequalities contradicts condition (2.14), and the second one contradicts condition (2.15).

As an example, consider Eq. (1.12), where either $p_1(t) = 1$, $p_2(t) = -\frac{1}{8}(1 + \exp(t))$ for $t \in \mathbb{R}$, or $p_1(t) = 1$ for $t \leqslant 0$, $p_1(t) = \frac{1}{1+t}$ for $t \geqslant 0$, and $p_2(t) = -\frac{1}{4}$ for $t \in \mathbb{R}$. According to what just observed, in these cases problem (1.12), (1.2) is solvable, although [7, Theorem 5.1] does not give an answer on the solvability of that problem.

The next existence result is based on different growth and sign conditions on the right-hand side f and allows us to treat also differential equations having structures not included in the previous ones, such as (1.14).

Theorem 2.4. *Let there exist a positive number a, a continuous function $h : \mathbb{R} \to [0, +\infty[$ and C^1-functions $w_1 :]-\infty, -a] \times [0, 1] \to [0, +\infty[$, $w_2 : [a, +\infty[\times [0, 1] \to [0, +\infty[$ such that along with* (1.3) *the following conditions are fulfilled*:

$$f(t, x, y) \geqslant -h(t)(1 + y^2) \quad \text{for } t \in \mathbb{R},\ 0 \leqslant x \leqslant 1,\ y \in \mathbb{R}, \tag{2.16}$$

$$w_1(t, 0) = 0, \quad \int_{-\infty}^{-a} w_1(s, x)\, ds = +\infty, \quad \frac{\partial w_1(t, x)}{\partial x} \geqslant 0 \quad \text{for } t \leqslant -a,\ x \in\,]0, 1[, \tag{2.17_1}$$

$$f\big(t, x, w_1(t, x)\big) \geqslant w_1(t, x)\frac{\partial w_1(t, x)}{\partial x} + \frac{\partial w_1(t, x)}{\partial t} \quad \text{for } t \leqslant -a,\ x \in [0, 1], \tag{2.18_1}$$

$$w_2(t, 1) = 0, \quad \int_{a}^{+\infty} w_2(s, x)\, ds = +\infty, \quad \frac{\partial w_2(t, x)}{\partial x} \leqslant 0 \quad \text{for } t \geqslant a,\ x \in\,]0, 1[, \tag{2.17_2}$$

$$f\big(t, x, w_2(t, x)\big) \leqslant w_2(t, x)\frac{\partial w_2(t, x)}{\partial x} + \frac{\partial w_2(t, x)}{\partial t} \quad \text{for } t \geqslant a,\ x \in [0, 1]. \tag{2.18_2}$$

Then problem (1.1), (1.2) *is solvable.*

We now provide applications of the above result to differential equations of the types (1.12), (1.14).

Corollary 2.5. *Let us consider Eq.* (1.14) *with f_1, f_2 continuous functions, $f_1(\cdot, u) \in C^1(\mathbb{R})$. Assume that*

$$f_2(t, 0) = f_2(t, 1) = 0 \quad \text{for } t \in \mathbb{R} \tag{2.19}$$

and there exists a positive number a such that

$$tf_1(t,x) < 0, \quad \frac{\partial f_1(t,x)}{\partial t} \leqslant 0 \quad \textit{for } |t| \geqslant a,\ 0 < x < 1, \tag{2.20}$$

and for any $x \in [0,1]$,

$$f_2(t,x) \geqslant -\frac{1}{4} f_1(t,x) \int_0^x f_1(t,s)\,ds + \frac{1}{2} \int_0^x \frac{\partial f_1(t,s)}{\partial t}\,ds \quad \textit{for } t \leqslant -a, \tag{2.21$_1$}$$

$$f_2(t,x) \leqslant \frac{1}{4} f_1(t,x) \int_x^1 f_1(t,s)\,ds - \frac{1}{2} \int_x^1 \frac{\partial f_1(t,s)}{\partial t}\,ds \quad \textit{for } t \geqslant a. \tag{2.21$_2$}$$

Then problem (1.14), (1.2) *has at least one solution.*

Corollary 2.6. *Let us consider Eq.* (1.12) *with $p_1 \in C^1(\mathbb{R})$ and $p_2 \in C(\mathbb{R})$. Assume that there exists a positive number a such that* (2.12) *holds together with*

$$tp_1(t) < 0, \quad p_1'(t) \leqslant 0 \quad \textit{for } |t| \geqslant a. \tag{2.22}$$

Then problem (1.12), (1.2) *is solvable, and its arbitrary solution u satisfies $0 < u(t) < 1$ for $t \in \mathbb{R}$. Moreover, whenever in the interval $]-\infty, -a]$ or in the interval $[a, +\infty[$ condition* (2.13) *holds for some $\varepsilon > 0$, then problem* (1.12), (1.2) *has no solution.*

Remark 2.3. According to Corollary 2.3, condition (2.5) in Theorem 2.1 and condition (2.5′) in Theorem 2.1′ cannot be improved in the sense that they cannot be replaced respectively by the conditions

$$f\big(t,x,w(t,x)\big) \geqslant (1-\varepsilon)\left[w(t,x)\frac{\partial w(t,x)}{\partial x} + \frac{\partial w(t,x)}{\partial t}\right] \quad \text{for } t \leqslant a,\ 0 \leqslant x \leqslant 1,$$
$$f\big(t,x,w(t,x)\big) \leqslant (1+\varepsilon)\left[w(t,x)\frac{\partial w(t,x)}{\partial x} + \frac{\partial w(t,x)}{\partial t}\right] \quad \text{for } t \geqslant a,\ 0 \leqslant x \leqslant 1,$$

no matter how small $\varepsilon > 0$ would be.

Similarly, according to Corollary 2.6, conditions (2.18$_1$) and (2.18$_2$) in Theorem 2.4 cannot be improved in the sense that they cannot be replaced by the inequalities

$$f\big(t,x,w_1(t,x)\big) \geqslant (1-\varepsilon)\left[w_1(t,x)\frac{\partial w_1(t,x)}{\partial x} + \frac{\partial w_1(t,x)}{\partial t}\right]$$
$$\text{for } t \leqslant -a,\ 0 \leqslant x \leqslant 1,$$
$$f\big(t,x,w_2(t,x)\big) \leqslant (1+\varepsilon)\left[w_2(t,x)\frac{\partial w_2(t,x)}{\partial x} + \frac{\partial w_2(t,x)}{\partial t}\right]$$
$$\text{for } t \geqslant a,\ 0 \leqslant x \leqslant 1,$$

no matter how small $\varepsilon > 0$ would be.

Analogously, conditions (2.21$_1$) and (2.21$_2$) in Corollary 2.5 cannot be replaced by the inequalities

$$f_2(t,x) \geqslant -\frac{1+\varepsilon}{4} f_1(t,x) \int_0^x f_1(t,s)\,ds + \frac{1+\varepsilon}{2} \int_0^x \frac{\partial f_1(t,s)}{\partial t}\,ds$$
$$\text{for } t \leqslant -a,\ 0 \leqslant x \leqslant 1,$$
$$f_2(t,x) \leqslant \frac{1+\varepsilon}{4} f_1(t,x) \int_x^1 f_1(t,s)\,ds - \frac{1+\varepsilon}{2} \int_x^1 \frac{\partial f_1(t,s)}{\partial t}\,ds$$
$$\text{for } t \geqslant a,\ 0 \leqslant x \leqslant 1.$$

Remark 2.4. Regarding the possible existence of transitional solutions that reach the stable equilibrium or leave the unstable one in a finite time, observe that if there exists a continuous function $\ell : \mathbb{R} \to]0, +\infty[$ such that

$$\bigl|f(t,x,y)\bigr| \leqslant \ell(t)\bigl(x(1-x)+|y|\bigr) \quad \text{for } t \in \mathbb{R},\ 0 \leqslant x \leqslant 1,\ |y| \leqslant 1, \tag{2.23}$$

then every solution of problem (1.1), (1.2) satisfies the condition

$$0 < u(t) < 1 \quad \text{for } t \in \mathbb{R}. \tag{2.24}$$

Indeed, if this would be the case, an interval $[t_1, t_2]$ could be found such that

$$0 < u(t) < 1, \quad \bigl|u'(t)\bigr| < 1 \quad \text{for } t_1 < t < t_2 \tag{2.25}$$

and

$$\text{either} \quad u(t_1) = 0, \quad u'(t_1) = 0, \quad \text{or} \quad u(t_2) = 1, \quad u'(t_2) = 0. \tag{2.26}$$

On the other hand, according to (2.23) we have $|u''(t)| \leqslant \ell(t)(u(t)(1-u(t)) + |u'(t)|)$ for $t_1 \leqslant t \leqslant t_2$. Hence, taking into account (2.26) and applying the Gronwall–Bellman lemma, we find that either $u(t) = 0$ for $t_1 \leqslant t \leqslant t_2$, or $u(t) = 1$ for $t_1 \leqslant t \leqslant t_2$. But this contradicts condition (2.25).

Instead, if (2.23) is not valid and all the assumptions of Theorem 2.4 are fulfilled, problem (1.1), (1.2) may have a solution not satisfying condition (2.24). Indeed, if

$$h_1(t) := \begin{cases} 3t^2 & \text{for } t < 0, \\ 0 & \text{for } t \geqslant 0, \end{cases}$$
$$h_2(t) := \begin{cases} 6|t|(1-\exp(t^3))^{-1/3} & \text{for } t < 0, \\ 6 & \text{for } t \geqslant 0, \end{cases}$$
$$\omega(t,x) := \begin{cases} (1-\exp(t^3))^{1/3} & \text{for } 0 \leqslant x \leqslant \exp(t^3),\ t < 0, \\ (1-x)^{1/3} & \text{for } \exp(t^3) < x \leqslant 1,\ t < 0, \\ (1-x)^{1/3} & \text{for } 0 \leqslant x \leqslant 1,\ t \geqslant 0, \end{cases}$$

then the differential equation

$$u'' = h_1(t)u' - h_2(t)u\omega(t,u)$$

has the solution

$$u(t) = \begin{cases} \exp(t^3) & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0, \end{cases}$$

which satisfies conditions (1.2), but not condition (2.24). On the other hand, the function $f(t,x,y) = h_1(t)y - h_2(t)x\omega(t,x)$ satisfies conditions (1.3), (2.16), (2.17_i) and (2.18_i) ($i = 1, 2$) with $h(t) = h_1(t) + h_2(t)$, $a = 0$, $w_1(t,x) = 3t^2x$, $w_2(t,x) = \int_x^1 s(1-s)^{1/3}\,ds$, but (2.23) is not valid.

3. Auxiliary statements

Following [4] let us give the definition of a lower (an upper) function of Eq. (1.1).

Definition 3.1. A function $\gamma : \mathbb{R} \to \mathbb{R}$ is said to be a lower (an upper) function of Eq. (1.1) if it is continuous and there exists a set $I \subset \mathbb{R}$, containing at most a finite number of points, such that $\gamma \in C^2(\mathbb{R} \setminus I)$,

$$f\bigl(t,\gamma(t),\gamma'(t)\bigr) \leqslant \gamma''(t) \quad \bigl(f\bigl(t,\gamma(t),\gamma'(t)\bigr) \geqslant \gamma''(t)\bigr) \quad \text{for } t \in \mathbb{R} \setminus I,$$

and at every $t_0 \in I$ the left and the right limits $\gamma'(t_0-)$, $\gamma'(t_0+)$ satisfying $\gamma'(t_0-) \leqslant \gamma'(t_0+)$ ($\gamma'(t_0-) \geqslant \gamma'(t_0+)$) exist.

For Eq. (1.1) let us consider also the following problems:

$$u(a_0) = c, \quad \gamma_1(t) \leqslant u(t) \leqslant \gamma_2(t) \quad \text{for } t \geqslant a_0, \tag{3.1}$$

$$u(a_0) = c, \quad \gamma_1(t) \leqslant u(t) \leqslant \gamma_2(t) \quad \text{for } t \leqslant a_0. \tag{3.2}$$

Theorems 5.1 and 5.3_1 from [4] immediately implies the following two lemmas.

Lemma 3.1. *Let γ_1 and γ_2 be a lower and an upper function of Eq.* (1.1) *satisfying inequality* (1.11). *Let, moreover, there exist a continuous function $h : \mathbb{R} \to [0, +\infty[$ such that either*

$$f(t,x,y)\operatorname{sgn} x \geqslant -h(t)(1+y^2) \quad \textit{for } t \in \mathbb{R},\ \gamma_1(t) \leqslant x \leqslant \gamma_2(t),\ y \in \mathbb{R}, \tag{3.3}$$

or

$$\sigma f(t,x,y)\operatorname{sgn} y \geqslant -h(t)(1+y^2) \quad \textit{for } t \in \mathbb{R},\ \gamma_1(t) \leqslant x \leqslant \gamma_2(t),\ y \in \mathbb{R}, \tag{3.4}$$

with $\sigma \in \{-1, 1\}$. Then problem (1.1), (1.10) *is solvable.*

Lemma 3.2. *Let γ_1 and γ_2 be a lower and an upper function of Eq.* (1.1) *satisfying inequality* (1.11). *Let, moreover, condition* (3.4) *hold, where $h : \mathbb{R} \to [0, +\infty[$ is a continuous function and $\sigma = 1$ ($\sigma = -1$). Then for any $a_0 \in \mathbb{R}$ and $c \in [\gamma_1(a_0), \gamma_2(a_0)]$ problem* (1.1), (3.1) (*problem* (1.1), (3.2)) *is solvable.*

Now for Eq. (1.1) we consider the following two auxiliary problems:

$$u(a_0) = c, \quad 0 \leqslant u(t) \leqslant 1 \quad \text{for } t \geqslant a_0, \tag{3.5}$$

$$u(a_0) = c, \quad 0 \leqslant u(t) \leqslant 1 \quad \text{for } t \leqslant a_0, \tag{3.5$'$}$$

where $a_0 \in \mathbb{R}$ and $c \in \,]0, 1[$ are arbitrarily fixed numbers.

Lemma 3.3. *Let conditions* (1.3), (2.1)–(2.3) *hold and*

$$f(t,x,y)=f(t,x,0) \quad \textit{for } t\in\mathbb{R},\ 0\leqslant x\leqslant 1,\ y\leqslant 0. \tag{3.6}$$

Then problem (1.1), (3.5) *is solvable and each arbitrary solution satisfies the conditions*

$$u'(t)>0 \quad \textit{for } t\in\bigl\{s\geqslant a_0\colon\ u(s)<1\bigr\}, \tag{3.7}$$

$$\lim_{t\to+\infty}u(t)=1. \tag{3.8}$$

In order to prove the above lemma, we need Lemmas 3.4 and 3.5 below.

Lemma 3.4. *Let conditions* (2.2) *and* (3.6) *hold, where* $\delta :]0,1/2[\to]0,+\infty[$ *is a continuous function. Let, moreover,* $u:[t_0,t_1]\to\mathbb{R}$ *be a solution of* (1.1) *such that*

$$u'(t_0)<\delta(x), \quad x\leqslant u(t)\leqslant 1-x \quad \textit{for } t_0\leqslant t\leqslant t_1, \tag{3.9}$$

for some $x\in]0,1/2[$. *Then*

$$u'(t)<u'(t_0), \quad u''(t)\leqslant f^*(t,x) \quad \textit{for } t_0<t\leqslant t_1. \tag{3.10}$$

Proof. In view of (3.9) we can set $t^*=\sup\{t\in]t_0,t_1]\colon\ u'(s)<\delta(x) \text{ for } s\in[t_0,t]\}$. Then due to conditions (2.2) and (3.6) we have $u''(t)\leqslant f^*(t,x)<0$ for $t_0\leqslant t\leqslant t^*$, and consequently, $u'(t^*)<u'(t_0)<\delta(x)$. Hence by the definition of t^* it follows that $t^*=t_1$ and inequalities (3.10) hold. □

Lemma 3.5. *If conditions* (2.2) *and* (3.6) *hold, then every solution of problem* (1.1), (3.5) *satisfies inequality* (3.7).

Proof. Assume by contradiction that (3.7) does not hold. Then there exists $t_0\geqslant a_0$ such that $0<u(t_0)<1$ and $u'(t_0)\leqslant 0$. Therefore, we can find $\bar t>t_0$ satisfying $0<u(t)<1$ for $t_0\leqslant t\leqslant \bar t$. On account of conditions (2.2), (3.6), by Lemma 3.4 we obtain $0<u(t)<1$, $u'(t)<0$, $u''(t)<0$ for $t_0<t\leqslant\bar t$. But on the other hand, by (3.5) the above inequalities hold in the interval $]t_0,+\infty[$. Therefore, we find $0<u(t)<u(t_1)+u'(t_1)(t-t_1)<1-|u'(t_1)|(t-t_1)$ for $t>t_1$, where $t_1>t_0$, a contradiction. □

Proof of Lemma 3.3. Take $\gamma_1(t)\equiv 0$ and $\gamma_2(t)\equiv 1$. Then, according to conditions (1.3), (2.1), (2.2) and (3.6), γ_1 and γ_2 are a lower and an upper function of Eq. (1.1), and inequality (3.4) holds with $\sigma=1$. Hence by Lemma 3.2 we have the solvability of problem (1.1), (3.5).

Let u be an arbitrary solution of that problem. Then by Lemma 3.5 inequality (3.7) is satisfied. Consequently, there exists the limit $u(+\infty):=\lim_{t\to+\infty}u(t)$. Let us show that $u(+\infty)=1$. Indeed, otherwise there exist $x\in]0,1/2[$ and $t_0\in]a_0,+\infty[$ such that $u'(t_0)<\delta(x)$ and $x\leqslant u(t)\leqslant 1-x$ for $t\geqslant t_0$. Hence due to Lemma 3.4 and condition (3.7) we get $0<u'(t)<\delta(x)$ and $u''(t)\leqslant f^*(t,x)$ for $t\geqslant t_0$. If we multiply the last inequality by t and then integrate, we obtain

$$tu'(t)-u(t)-t_0u'(t_0)+u(t_0)\leqslant\int_{t_0}^{t}sf^*(s,x)\,ds \quad \text{for } t\geqslant t_0,$$

and consequently $\int_{t_0}^{+\infty} sf^*(s,x)\,ds > -1 - t_0u'(t_0) > -\infty$. But this contradicts equality (2.3). □

The lemma below can be proved in an analogous way as Lemma 3.3.

Lemma 3.3′. *Suppose conditions* (1.3), (2.1′)–(2.3′), *and* (3.6) *are fulfilled. Then problem* (1.1), (2.5′) *is solvable and each arbitrary solution satisfies the conditions* $u'(t) > 0$ *for* $t \in \{s \leqslant a_0\colon u(s) > 0\}$ *and* $\lim_{t\to-\infty} u(t) = 0$.

Lemma 3.6. *Suppose conditions* (1.3), (2.1)–(2.3), (3.6) *are fulfilled, and* u *is a solution of problem* (1.1), (3.5) *defined on its maximal existence interval. Then either* u *is a solution of problem* (1.1), (1.2), *or there exists* $t_0 < a_0$ *such that*

$$u(t_0) = 0, \quad u'(t_0) > 0, \quad 0 \leqslant u(t) \leqslant 1 \quad \textit{for } t_0 \leqslant t < +\infty, \ \lim_{t\to+\infty} u(t) = 1. \tag{3.11}$$

Proof. Let $]t_*, +\infty[$ be the interval where u is defined. Then, by virtue of Lemma 3.3, conditions (3.7), (3.8) are satisfied and, moreover, either

$$u'(t) > 0, \quad 0 < u(t) < 1 \quad \text{for } t_* < t \leqslant a_0, \tag{3.12}$$

or there exists $t_0 \in\,]t_*, a_0[$ such that the restriction of u to $[t_0, +\infty[$ is a solution of (1.1), (3.11).

Assume inequalities (3.12). Then on account of (2.1) we deduce

$$\begin{aligned}\ln \frac{1+u'(t)}{1+u'(a_0)} &= -\int_t^{a_0} \frac{du'(s)}{1+u'(s)} \leqslant h^*(t)\int_t^{a_0} \bigl(1+u'(s)\bigr)\,ds \\ &= h^*(t)\bigl(a_0 - t + u(a_0) - u(t)\bigr) < (1 + a_0 - t)h^*(t) \quad \text{for } t_* < t \leqslant a_0,\end{aligned}$$

where $h^*(t) = \max\{h(s)\colon t \leqslant s \leqslant a_0\}$. Consequently, $0 < u'(t) < (1+u'(a_0))\exp((1 + a_0 - t)h^*(t))$ for $t_* < t \leqslant a_0$. Hence due to the definition of the interval $]t_*, +\infty[$ it is clear that $t_* = -\infty$. Let us show that in this case u is a solution of problem (1.1), (1.2), i.e., $\lim_{t\to-\infty} u(t) = 0$. Assume the contrary. Then there exists $x \in\,]0, 1/2[$ such that $x < u(t) < 1 - x$ for $t \leqslant a_0$. On the other hand, since $\liminf_{t\to-\infty} u'(t) = 0$, there exists a decreasing sequence of points $\{t_n\}$ satisfying $t_n \to -\infty$ and $u'(t_n) \to 0$ as $n \to +\infty$. Thus for sufficiently large n we have $u'(t_n) < \varepsilon$, where ε is an arbitrarily small positive number satisfying $\varepsilon \leqslant \delta(x)$, and δ is the function appearing in (2.2). Then by Lemma 3.4 we get $u'(t) < u'(t_n) < \varepsilon$ for $t_n \leqslant t \leqslant a_0$. Hence, in view of the arbitrariness of ε, we deduce $u'(t) \leqslant 0$ for $t \leqslant a_0$, in contradiction with the first inequality in (3.12). □

The following lemma can be proved analogously.

Lemma 3.6′. *Let conditions* (1.3), (2.1′)–(2.3′), (3.6) *hold, and let* u *be a solution of problem* (1.1), (2.5′) *defined on its maximal existence interval. Then either* u *is a solution of problem* (1.1), (1.2), *or there exists* $t_0 \in\,]a_0, +\infty[$ *such that* $u(t_0) = 1$, $0 \leqslant u(t) \leqslant 1$ *for* $t \leqslant t_0$, $\lim_{t\to-\infty} u(t) = 0$.

We conclude this section with two lemmas concerning initial problems for the first order differential equation

$$\frac{du}{dt} = w(t, u). \tag{3.13}$$

Lemma 3.7. *Let* $w \in C(]-\infty, a] \times [0, 1])$ *be a nonnegative function,* $w(t, \cdot) \in C^1([0, 1])$, *and conditions* (2.4) *hold. Then the differential equation* (3.13) *has a unique solution, defined on* $]-\infty, a]$, *such that*

$$u(a) = 1, \quad 0 < u(t) \leqslant 1 \quad \textit{for } t \leqslant a, \ \lim_{t\to-\infty} u(t) = 0. \tag{3.14}$$

Proof. Let us extend the function w to $]-\infty, a] \times \mathbb{R}$ by defining $w(t, x) = w(t, 0)$ for $x \leqslant 0$ and $w(t, x) = w(t, 1)$ for $x \geqslant 1$. Then Eq. (3.13) has a unique solution, defined on $]-\infty, a]$, such that $u(a) = 1$. On the other hand, in view of (2.4) we have $u'(t) \geqslant 0$, $0 < u(t) \leqslant 1$ for $t \leqslant a$. Moreover, $\int_{-\infty}^{a} w(s, x)\, ds \leqslant \int_{-\infty}^{a} w(s, u(s))\, ds = 1 - u(-\infty) \leqslant 1$, where $x = \lim_{t\to-\infty} u(t)$. From the last inequality it follows that $x = 0$ since $\int_{-\infty}^{a} w(s, x)\, ds = +\infty$ for $x > 0$. Therefore conditions (3.14) are satisfied. □

Lemma 3.7′. *Let* $w \in C([a, +\infty[\times [0, 1])$ *be a nonnegative function,* $w(t, \cdot) \in C^1([0, 1])$, *and conditions* (2.4′) *hold. Then the differential equation* (3.13) *has a unique solution, defined on* $[a, +\infty[$, *such that* $u(a) = 0$, $0 \leqslant u(t) < 1$ *for* $t \geqslant a$, $\lim_{t\to+\infty} u(t) = 1$.

The proof of this lemma is similar to that of Lemma 3.7.

4. Proof of the main results

Proof of Theorem 2.1. Without loss of generality since we are searching for monotone solutions, we will assume below that the function f satisfies condition (3.6).

First let us note that if problem (1.1), (1.2) is solvable, according to Lemma 3.5 it is easy to see that every solution of that problem satisfies condition (2.6).

Taking into account (1.3), (2.1)–(2.3) and (3.6), by Lemma 3.3 we deduce the solvability of problem (1.1), (3.5). Denote by u_1 the solution of that problem with $a_0 = 0$, $c = 1/2$. We will assume that u_1 is maximally extended to the left as a solution of Eq. (1.1). By Lemma 3.6 either u_1 is a solution of problem (1.1), (1.2), or there exists $t_0 \in]-\infty, 0[$ such that $u_1(t_0) = 0$, $0 \leqslant u_1(t) \leqslant 1$ for $t \geqslant t_0$, and $\lim_{t\to+\infty} u_1(t) = 1$. Obviously, it remains to consider the second case. Moreover, without loss of generality it can be assumed that $a \leqslant t_0$.

Due to Lemma 3.7, conditions (2.4) guarantee the existence of a solution u_2 of Eq. (3.13), defined in the interval $]-\infty, a]$ and satisfying the conditions $u_2(a) = 1$, $0 < u_2(t) \leqslant 1$ for $t \leqslant a$, and $\lim_{t\to-\infty} u_2(t) = 0$. Set

$$\gamma_1(t) := \begin{cases} 0 & \text{for } t \leqslant t_0, \\ u_1(t) & \text{for } t > t_0, \end{cases} \qquad \gamma_2(t) := \begin{cases} u_2(t) & \text{for } t < a, \\ 1 & \text{for } t \geqslant a. \end{cases}$$

Of course, γ_1 and $\gamma_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$0 \leqslant \gamma_1(t) \leqslant \gamma_2(t) \leqslant 1 \quad \text{for } t \in \mathbb{R},$$
$$\lim_{t\to-\infty} \gamma_i(t) = 0, \quad \lim_{t\to+\infty} \gamma_i(t) = 1 \quad (i = 1, 2). \tag{4.1}$$

On the other hand, by virtue of conditions (1.3), (2.5) and (2.1), γ_1 and γ_2 are respectively a lower and an upper function of Eq. (1.1), and inequality (3.3) holds. Then by Lemma 3.1 problem (1.1), (1.10) has a solution u, which in view of (4.1) satisfies conditions (1.2). □

The proof of Theorem 2.1′ is similar to that of Theorem 2.1. The only difference is that instead of Lemmas 3.3, 3.6 and 3.7, Lemmas 3.3′, 3.6′ and 3.7′ have to be used.

Proof of Theorem 2.4. By virtue of Lemma 3.7 and conditions (2.17_1) (Lemma 3.7′ and conditions (2.17_2)), the differential equation

$$\frac{du}{dt} = w_1(t, u) \quad \left(\frac{du}{dt} = w_2(t, u)\right)$$

has a solution u_1 (a solution u_2), defined in the interval $]-\infty, -a]$ (in the interval $[a, +\infty[$) and satisfies the conditions

$$u_1(-a) = 1, \quad 0 < u_1(t) \leqslant 1 \quad \text{for } t \leqslant -a, \quad \lim_{t\to-\infty} u_1(t) = 0$$
$$\left(u_2(a) = 0, \quad 0 \leqslant u_2(t) < 1 \quad \text{for } t \geqslant a, \quad \lim_{t\to+\infty} u_2(t) = 1\right).$$

Set

$$\gamma_1(t) := \begin{cases} 0 & \text{for } t < a, \\ u_2(t) & \text{for } t \geqslant a, \end{cases} \qquad \gamma_2(t) := \begin{cases} u_1(t) & \text{for } t \leqslant -a, \\ 1 & \text{for } t > -a. \end{cases} \tag{4.2}$$

Of course, $\gamma_i : \mathbb{R} \to [0, 1]$ $(i = 1, 2)$ are continuous functions satisfying (1.11). Moreover, $\gamma_1 \in C^2(\mathbb{R} \setminus \{a\})$, $\gamma_2 \in C^2(\mathbb{R} \setminus \{-a\})$ and $\gamma_1'(a-) \leqslant \gamma_1'(a+)$, $\gamma_2'(-a-) \geqslant \gamma_2'(-a+)$. If now we take into account conditions (1.3), (2.18_1), (2.18_2) and Definition 3.1, then it becomes clear that γ_1 and γ_2 are a lower and an upper function of Eq. (1.1). On the other hand, inequality (2.16) yields inequality (3.3). By Lemma 3.1 the above-mentioned conditions guarantee the solvability of problem (1.1), (1.10). However, by virtue of equalities (4.2), inequalities (1.10) imply conditions (1.2). □

Proof of Corollary 2.5. Put

$$f(t, x, y) := f_1(t, x)y + f_2(t, x),$$
$$h(t) := \max\bigl\{\bigl|f_1(t, x)\bigr| + \bigl|f_2(t, x)\bigr| : 0 \leqslant x \leqslant 1\bigr\},$$
$$w_1(t, x) := \frac{1}{2}\int_0^x f_1(t, s)\,ds, \qquad w_2(t, x) := -\frac{1}{2}\int_x^1 f_1(t, s)\,ds.$$

Obviously, f satisfies inequality (2.16). Moreover, equalities (2.19) yield equalities (1.3), and conditions (2.20) imply conditions (2.17_i) $(i = 1, 2)$. Further, due to (2.21_1) we find

$$\begin{aligned} f\big(t,x,w_1(t,x)\big) &= \frac{1}{2} f_1(t,x) \int\limits_0^x f_1(t,s)\,ds + f_2(t,x) \\ &\geqslant \frac{1}{4} f_1(t,x) \int\limits_0^x f_1(t,s)\,ds + \frac{1}{2} \int\limits_0^x \frac{\partial f_1(t,s)}{\partial t}\,ds \\ &= w_1(t,x) \frac{\partial w_1(t,x)}{\partial x} + \frac{\partial w_1(t,x)}{\partial t} \quad \text{for } t \leqslant -a,\ 0 \leqslant x \leqslant 1, \end{aligned}$$

i.e., condition (2.18_1). Analogously, in view of (2.21_2), we can show that (2.18_2) holds. The assertion follows by applying Theorem 2.4. □

Proof of Corollary 2.6. Equation (1.12) is derived from Eq. (1.14) in the case where $f_1(t,x) = p_1(t)$ and $f_2(t,x) = p_2(t)x(1-x)$. In that case inequalities (2.22) and (2.12) imply inequalities (2.20) and (2.21_i) $(i = 1, 2)$. Therefore, all the conditions of Corollary 2.5 are fulfilled, which guarantee the solvability of problem (1.12), (1.2). On the other hand, according to Remark 2.4, an arbitrary solution u of problem (1.12), (1.2) satisfies (2.24).

Now let us show that if along with (2.22) condition (2.13) holds in the interval $[a, +\infty[$, then problem (1.12), (1.2) has no solution. Assume by contradiction the existence of a solution u of this problem. Clearly, u satisfies (2.24). On the other hand, by virtue of inequality (2.13) without loss of generality we can assume that

$$p_2(t)u(t) > \left(\frac{1}{4} + \varepsilon_0\right) p_1^2(t) - \frac{1}{2} p_1'(t) \quad \text{for } t \geqslant a, \tag{4.3}$$

where ε_0 is a sufficiently small positive number. Put

$$v(t) = \big(1 - u(t)\big) \exp\left(-\frac{1}{2} \int\limits_a^t p_1(s)\,ds\right). \tag{4.4}$$

Then v is a solution of the equation

$$v'' + p(t)v = 0, \tag{4.5}$$

where $p(t) = p_2(t)u(t) - \frac{1}{4} p_1^2(t) + \frac{1}{2} p_1'(t)$. Moreover, in view of conditions (2.22) and (4.3) we find $p(t) > \varepsilon_0 p_1^2(t) \geqslant \varepsilon_0 p_1^2(a)$ for $t \geqslant a$. Therefore, $\int^{+\infty} p(s)\,ds = +\infty$, and so all solutions of Eq. (4.5) have sequences of zeros tending to $+\infty$. On the other hand, (2.24) and (4.4) imply that $v(t) > 0$ for $t \geqslant a$, a contradiction.

Analogously it can be proved that this problem has no solution also in the case where inequality (2.13) holds in the interval $]-\infty, -a]$. □

Proof of Corollary 2.2. We limit ourselves to consider only the case where conditions (2.8)–(2.11) are fulfilled; the other one being analogous.

Put $f(t,x,y) = p_1(t) f_1(x,y) y + p_2(t) f_2(x,y)$. Evidently, f satisfies condition (2.1), where $h(t) = \max\{|p_2(t)| f_0(x) \int_0^x f_0(s)\,ds\colon\ 0 \leqslant x \leqslant 1\}$. Further, by virtue of conditions (2.7) and (2.8), equalities (1.3) are satisfied and there exists a continuous function $\delta_0 :]0, 1/2[\to]0, +\infty[$ such that

$$f_2(s,y) > 0 \quad \text{for } x \leqslant s \leqslant 1-x,\ 0 < x < \frac{1}{2},\ 0 \leqslant y \leqslant \delta_0(x),$$
$$\delta_1(x) := \min\left\{\frac{f_2(s,y)}{f_1(s,y)}\colon x \leqslant s \leqslant 1-x,\ 0 \leqslant y \leqslant \delta_0(x)\right\} > 0 \quad \text{for } 0 < x < \frac{1}{2}.$$

Suppose $\delta(x) := \min\{\frac{\alpha\delta_1(x)}{2}, \delta_0(x)\}$, $\rho(x) := \min\{\frac{\alpha\delta_1(x) f_0(s)}{2}\colon x \leqslant s \leqslant 1-x\}$. Then in view of (2.8) and (2.9) we find

$$f(t,s,y) \leqslant p_1(t) f_1(s,y) y - \alpha p_1(t) f_2(s,y) \leqslant p_1(t) f_1(s,y)\big(y - \alpha\delta_1(x)\big)$$
$$\leqslant -\rho(x) p_1(t) \quad \text{for } x \leqslant s \leqslant 1-x,\ 0 \leqslant y \leqslant \delta(x),$$

and consequently, conditions (2.2) and (2.3) are satisfied.

Put $w(t,x) := \frac{1}{2} p_1(t) \int_0^x f_0(s)\,ds$. Then assumptions (2.8), (2.9) and (2.11) yield conditions (2.4) and (2.5). Now if we apply Theorem 2.1, we conclude that problem (1.13), (1.2) has at least one solution satisfying condition (2.6).

The case where conditions $(2.8')$–$(2.10')$ hold can be proved analogously but applying Theorem $2.1'$ instead of Theorem 2.1. □

The proof of Corollary 2.3 is analogous to that of Corollary 2.6. The only difference is that Corollary 2.2 is used instead of Corollary 2.5.

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