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NONLOCAL BOUNDARY VALUE PROBLEMS FOR TWO–DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH STRONG SINGULARITIES

For the linear differential system

$$u_1' = p_1(t)u_2 + q_1(t), \quad u_2' = p_2(t)u_1 + q_2(t), \tag{1}$$

the boundary value problems

$$u_1(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u_1(b_k) = 0, \quad \int_a^b p_1(t)u_2^2(t)dt < +\infty \tag{2}$$

and

$$u_1(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u_2(b_k) = 0, \quad \int_a^b p_1(t)u_2^2(t)dt < +\infty \tag{3}$$

are considered. Here m is a natural number,

$$a < b_1 \le b_k \le b, \quad \beta_k > 0 \ \ (k = 1, \dots, m),$$

$p_1 : (a, b) \to [0, +\infty)$ and $q_1 : (a, b) \to R$ are Lebesgue integrable functions, while p_2, $q_2 : (a, b) \to R$ are functions, integrable on $(a+\varepsilon, b)$ for arbitrarily small $\varepsilon > 0$, but not integrable on (a, b), having singularities at the point a. We are mainly interested in the case where the function p_2 has a strong singularity at a, i.e.,

$$\int_a^b (t-a)\big(|p_2(t)| - p_2(t)\big)dt = +\infty.$$

In this case, conditions (2) and (3) are not equivalent to the conditions

$$u_1(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u_1(b_k) = 0 \tag{$2'$}$$

and

$$u_1(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u_2(b_k) = 0, \tag{$3'$}$$

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i.e. from the unique solvability of problem $(1),(2)$ (problem $(1),(3)$) it does not follow the unique solvability of problem $(1),(2')$ (problem $(1),(3')$).

Problems $(1),(2)$ and $(1),(3)$ are investigated under the assumptions

$$0 \le p_1(t) \le \ell_0 \;\text{ for almost all }\; t \in (a,b), \quad \int_a^{b_1} p_1(t)dt > 0, \tag{4}$$

$$\int_a^b (t-a)\Big(p_2(t)\int_a^t |q_1(s)|ds\Big)^2 dt < +\infty, \tag{5}$$

$$\int_a^b p_1(t)\Big(\int_t^b q_2(s)ds\Big)^2 dt < +\infty, \tag{6}$$

where ℓ_0 is a positive constant.

The Agarwal–Kiguradze type conditions [1] are found, guaranteeing, respectively, the Fredholmicity and unique solvability of the above-mentioned problems.

In particular, the following theorems are proved.

Theorem 1. *Let along with* (4) *the inequality*

$$\liminf_{t\to a}\big((t-a)^2 p_2(t)\big) > -\frac{1}{4\ell_0}$$

be fulfilled, and let the homogeneous system

$$u_1' = p_1(t)u_2, \quad u_2' = p_2(t)u_1 \tag{1_0}$$

under the boundary conditions (2) *have only the trivial solution. If, moreover, the functions* q_1 *and* q_2 *satisfy conditions* (5) *and* (6)*, then problem* $(1),(2)$ *has one and only one solution.*

Theorem 2. *Let along with* (4) *the inequality*

$$p_2(t) \ge -\ell\Big(\frac{1}{(t-a)^2} + \frac{1}{(b-t)^2}\Big) \;\textit{ for almost all }\; t \in (a,b)$$

be fulfilled, where ℓ *is a positive constant such that*

$$\ell < \frac{1}{4\ell_0}. \tag{7}$$

If, moreover, the functions q_1 *and* q_2 *satisfy conditions* (5) *and* (6)*, then problem* $(1),(2)$ *has one and only one solution.*

Theorem 3. *Let along with* (4) *the inequality*

$$\liminf_{t\to a}\big((t-a)^2 p_2(t)\big) > -\frac{1}{4\ell_0}$$

be fulfilled, and let the homogeneous problem $(1_0),(3)$ *have only the trivial solution. If, moreover, the functions* q_1 *and* q_2 *satisfy conditions* (5) *and* (6)*, then problem* $(1),(3)$ *has one and only one solution.*

Theorem 4. *Let along with* (4) *the condition*

$$p_2(t) \geq -\frac{\ell}{(t-a)^2} \quad \textit{for almost all } t \in (a,b)$$

be fulfilled, where ℓ is a positive constant, satisfying inequality (7)*. If, moreover, the functions q_1 and q_2 satisfy conditions* (5) *and* (6)*, then problem* (1), (3) *has one and only one solution.*

Condition (7) in Theorems 2 and 4 is unimprovable and it cannot be replaced by the condition

$$\ell = \frac{1}{4\ell_0}.$$

For $p_1(t) \equiv \ell_0$ and $q_1(t) \equiv 0$, system (1) is equivalent to the equation

$$u'' = p(t)u + q(t), \tag{8}$$

where $p(t) \equiv \ell_0 p_2(t)$, $q(t) \equiv \ell_0 q_2(t)$.

As for the boundary conditions (2) and (3), they are equivalent to the conditions

$$u(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u(b_k) = 0, \quad \int_a^b u'^2(t)\,dt < +\infty \tag{9}$$

and

$$u(a+) = 0, \quad \sum_{k=1}^{m} \beta_k u'(b_k) = 0, \quad \int_a^b u'^2(t)\,dt < +\infty, \tag{10}$$

respectively.

From Theorems 1–4 it follow new results on the unique solvability of problems (8), (9) and (8), (10), which, in contrast to the previous well-known results on the unique solvability of nonlocal problems for Eq. (8) (see [2–10] and the references therein), cover the case, where

$$\int_a^b (t-a)\big(|p(t)| - p(t)\big)\,dt = +\infty.$$

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