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SOME OPTIMAL CONDITIONS FOR THE UNIQUE SOLVABILITY OF THE DIRICHLET PROBLEM FOR SECOND ORDER SINGULAR LINEAR DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Abstract. Unimprovable in a certain sense conditions are established guaranteeing the unique solvability of the Dirichlet problem and the weighted Dirichlet problem for the differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t),$$

where p and $q :]a, b[\to \mathbb{R}$ are locally integrable functions with nonintegrable singularities at the points a and b, and $\tau :]a, b[\to]a, b[$ is a measurable function.

On a finite open interval $]a, b[$, we consider the differential equation

$$u''(t) = p(t)u(\tau(t)) + q(t) \tag{1}$$

with the Dirichlet boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{2}$$

and the weighted Dirichlet boundary conditions

$$u(a) = u(b) = 0, \quad \sup\big\{(t-a)^{-\alpha}(b-t)^{-\beta}|u(t)| : \ a < t < b\big\} < +\infty. \tag{3}$$

Here p and $q :]a, b[\to \mathbb{R}$ are Lebesgue integrable on every closed interval contained in $]a, b[$ functions, $\tau :]a, b[\to]a, b[$ is a measurable function, α and $\beta \in [0, 1]$, and $c_i \in \mathbb{R}$ $(i = 1, 2)$. The functions p and q may have nonintegrable singularities at the boundary points of the interval $]a, b[$. More precisely, the results below on the unique solvability of problems $(1), (2)$ and $(1), (3)$ cover the cases, where

$$\int\limits_a^b \big(|p(t)| + |q(t)|\big)dt = +\infty,$$

i.e. the case where Eq. (1) is singular.

We use the following notation.

$C([a, b])$ and $L([a, b])$ are the spaces of continuous on $[a, b]$ and Lebesgue integrable on $[a, b]$ real functions, respectively;

$L_{loc}(]a, b[)$ is the space of real functions which are Lebesgue integrable on $[a+\varepsilon, b-\varepsilon]$ for every $\varepsilon \in]0, (b-a)/2[$;

$$h_{\alpha,\beta}(t, s) = \begin{cases} \dfrac{s-a}{(t-a)^{\alpha}}(b-t)^{1-\beta} & \text{for } \ a \le s \le t, \ \ a < t \le b, \\[2ex] \dfrac{b-s}{(b-t)^{\beta}}(t-a)^{1-\alpha} & \text{for } \ a \le t < b, \ \ t \le s \le b. \end{cases}$$

A continuous function $u : [a, b] \to \mathbb{R}$ is said to be a solution of Eq. (1) if it is absolutely continuous together with u' on every closed interval contained in $]a, b[$ and satisfies Eq. (1) almost everywhere on $]a, b[$.

A solution of Eq. (1) satisfying the boundary conditions (2) (the boundary conditions (3)) is said to be a solution of problem $(1), (2)$ (of problem $(1), (3)$).

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In the case, where $\tau(t) \equiv t$, i.e when the Eq. (1) has the form

$$u''(t) = p(t)u(t) + q(t), \tag{4}$$

singular problems $(1),(2)$ and $(1),(3)$ have been studied in sufficient detail (see, e.g., [1, 2, 6–11] and the references therein). However, in the general case the above mentioned problems are still not well studied. The present paper is devoted to filling the existing gap.

We have proved theorems on the Fredholmicity of problems $(1),(2)$ and $(1),(3)$, based on which we established unimprovable in a certain sense conditions for the unique solvability of these problems.

First we consider problem $(1),(2)$.

Theorem 1. *If the functions p and q satisfy the conditions*

$$\int_a^b (t-a)(b-t)|p(t)|dt < +\infty, \tag{5}$$

$$\int_a^b (t-a)(b-t)|q(t)|dt < +\infty, \tag{6}$$

then for the unique solvability of problem $(1),(2)$ it is necessary and sufficient that the corresponding homogeneous problem

$$u''(t) = p(t)u(\tau(t)), \tag{1_0}$$

$$u(a) = 0, \quad u(b) = 0 \tag{2_0}$$

has only a trivial solution.

This theorem is an analogue of I. Kiguradze's theorem on the Fredholmicity of problem $(4),(2)$ (see [9, Theorem 1.1]).

Remark 1. Let

$$\lim_{t\to a} \tau(t) = a, \quad \lim_{t\to b} \tau(t) = b.$$

If, moreover, the function p is of constant sign, and the function q satisfies condition (6) (the function p satisfies condition (5), and the function q is of constant sign), then for the solvability of problem $(1),(2)$ it is necessary that condition (5) (condition (6)) is satisfied.

Theorem 2. *Let there exist numbers α and $\beta \in [0,1]$ such that*

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}|p(t)|dt < +\infty, \tag{7}$$

$$\int_a^b h_{\alpha,\beta}(t,s)(\tau(s)-a)^{\alpha}(b-\tau(s))^{\beta}|p(s)|ds < b-a \quad \text{for } a \le t \le b. \tag{8}$$

If, moreover, the function q satisfies condition (6), then problem $(1),(2)$ has a unique solution.

Corollary 1. *Let the functions τ and p satisfy the conditions*

$$\int_a^b \frac{dt}{(\tau(t)-a)(b-\tau(t))} < +\infty, \tag{9}$$

$$(\tau(t)-a)(b-\tau(t))|p(t)| \le 2 \text{ for almost all } t \in \,]a,b[\,, \quad \operatorname{mes}\big\{t \in \,]a,b[\,: (\tau(t)-a)(b-\tau(t)) < 2\big\} > 0, \tag{10}$$

and let the function q satisfy condition (6). Then problem $(1),(2)$ has a unique solution.

Example 1. Let $\tau :]a,b[\to]a,b[$ be an arbitrary measurable function satisfying condition (9), and

$$p(t) = -\frac{2}{(\tau(t)-a)(b-\tau(t))} \quad \text{for almost all } t \in]a,b[\,. \tag{11}$$

Then the homogeneous problem $(1_0), (2_0)$ has the nontrivial solution $u(t) \equiv (t-a)(b-t)$. Thus in this case the nonhomogeneous problem $(1), (2)$ either has no solution or has an infinite set of solutions no matter how $q \in L_{loc}(]a,b[)$ and $c_i \in \mathbb{R}$ $(i=1,2)$ are. On the other hand, by conditions (9) and (11) for $\alpha = \beta = 1$ condition (7) is satisfied, and instead of (8) we have

$$\int_a^b h_{\alpha,\beta}(t,s)(\tau(s)-a)^\alpha (b-\tau(s))^\beta |p(s)|ds = b-a \ \text{ for } \ a \le t \le b.$$

Consequently, conditions (8) and (10) in Theorem 2 and Corollary 1 are unimprovable and they cannot be replaced by the conditions

$$\int_a^b h_{\alpha,\beta}(t,s)(\tau(s)-a)^\alpha (b-\tau(s))^\beta |p(s)|ds \le b-a \ \text{ for } \ a \le t \le b,$$

and

$$(\tau(t)-a)(b-\tau(t))|p(t)| \le 2 \quad \text{for almost all } t \in]a,b[\,,$$

respectively.

Corollary 2. *Let there exist* α *and* $\beta \in [0,1]$ *such that along with* (7) *the condition*

$$\int_a^b (t-a)^{1-\alpha}(\tau(t)-a)^\alpha (b-t)^{1-\beta}(b-\tau(t))^\beta |p(t)|dt \le b-a \tag{12}$$

holds. If, moreover, the function q *satisfies condition* (6)*, then problem* $(1), (2)$ *has a unique solution.*

The analogous result for problem $(4), (2)$ was first obtained by A. M. Lyapunov [12]. In [12], it is proved that if the function $p \in C([a,b])$ satisfies the integral inequality

$$\int_a^b |p(t)|dt \le \frac{4}{b-a}, \tag{13}$$

then the homogeneous differential equation

$$u''(t) = p(t)u(t) \tag{4_0}$$

under the boundary conditions (2_0) has only the trivial solution.

P. Hartman and A. Wintner [4] (see, also [3, Ch. XI, Theorem 5.1]) have proved that the homogeneous boundary value problem $(4_0), (2_0)$ has only the trivial solution also in the case where the function $p \in C([a,b])$ satisfies more general than (13) integral condition

$$\int_a^b (t-a)(b-t)[p(t)]_- dt \le b-a, \tag{14}$$

where $[p(t)]_- = (|p(t)| - p(t))/2$.

E. R. van Kampen and A. Wintner [5] have shown that the Lyapunov condition (13) is unimprovable and it cannot be replaced by the condition

$$\int_a^b [p(t)]_- dt \le \frac{4+\varepsilon}{b-a},$$

no matter how small $\varepsilon > 0$ is (see, also [7, Example 1.1]). In view of the inequality

$$(t-a)(b-t) < 4(b-a)^2 \quad \text{for } \ t \ne (a+b)/2,$$

it is evident that the examples constructed by E. R. van Kampen, A. Wintner, and I. Kiguradze confirm the unimprovability of condition (14) as well which is often referred to as the Lyapunov–Hartman–Wintner condition.

If p and $q \in L([a,b])$, and the function p satisfies the Lyapunov–Hartman–Wintner condition, then problem $(1),(2)$, due to its Fredholmicity, is uniquely solvable for arbitrarily fixed $c_i \in \mathbb{R}$ $(i = 1,2)$.

I. Kiguradze ([7, Corollary 1.1]) has proved that condition (14) guarantees the unique solvability of problem $(1),(2)$ in the case where the functions p and $q \in L_{loc}(]a,b[)$ are not integrable on $[a,b]$ but satisfy conditions (5) nad (6).

If p is a non-positive function, then Corollary 2 is a generalization of the above mentioned result by I. Kiguradze for the singular equation (1).

Remark 2. Condition (12) in Corollary 2 cannot be replaced by the condition

$$\int_a^b (t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}(b-t)^{1-\beta}(b-\tau(t))^{\beta}[p(t)]_-\,dt \le b-a. \tag{15}$$

Indeed, if

$$\tau(t) = b+a-t, \quad p(t) = \frac{\pi^2}{(b-a)^2} \quad \text{for } a \le t \le b,$$

then the homogeneous problem $(1_0),(2_0)$ has the nontrivial solution

$$u(t) = \sin\frac{\pi(t-a)}{b-a}.$$

Thus for arbitrarily fixed $p \in L([a,b])$ and $c_i \in \mathbb{R}$ $(i = 1,2)$ problem $(1),(2)$ either has no solution or has an infinite set of solutions. On the other hand, in this case all the conditions of Corollary 2 are satisfied except condition (12) instead of which the Lyapunov–Hartman–Wintner type condition (15) holds.

Now we consider the weighted boundary value problem $(1),(3)$.

Theorem 3. *If the functions p and q satisfy the conditions*

$$\int_a^b (t-a)^{1-\alpha}(\tau(t)-a)^{\alpha}(b-t)^{1-\beta}(b-\tau(t))^{\beta}|p(t)|\,dt < +\infty, \tag{16}$$

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}|q(t)|\,dt < +\infty, \tag{17}$$

then for the unique solvability of problem $(1),(3)$ it is necessary and sufficient that the corresponding homogeneous problem $(1_0),(3)$ has only a trivial solution.

Example 2. Let

$$\tau(t) = a + (b-a)\exp\left(-\frac{1}{(t-a)(b-t)}\right),$$

$$p(t) = (t-a)^{\alpha-1}(b-t)^{\beta-1}\exp\left(\frac{\alpha}{(t-a)(b-t)}\right)p_0(t) \ \text{ for almost all } \ t \in \,]a,b[\,, \tag{18}$$

where $p_0 \in L([a,b])$ is an arbitrary function, satisfying the condition

$$\operatorname{ess\,min}\big\{|p_0(t)| \,:\, a < t < b\big\} > 0. \tag{19}$$

Then the function p along with (16) satisfies also the condition

$$|p(t)| \ge \delta\exp\left(\frac{\alpha}{(t-a)(b-t)}\right) \ \text{ for almost all } \ t \in \,]a,b[\,,$$

where $\delta = const > 0$. Consequently, unlike Theorem 1, Theorem 3 covers the case where the function p has the singularity of an infinite order at the points a and b.

Theorem 4. *If the function p along with* (16) *satisfies condition* (8)*, and the function q satisfies condition* (17)*, then problem* (1), (3) *has a unique solution.*

Corollary 3. *Let $\alpha = \beta = 1$, $q \in L([a,b])$, and let the function p satisfy condition* (10)*. Then problem* (1), (3) *has a unique solution.*

Example 3. Let $\alpha = \beta = 1$,

$$\tau(t) = a + (b-a)\exp\left(-\frac{1}{(t-a)(b-t)}\right), \quad p(t) = \exp\left(\frac{1}{(t-a)(b-t)}\right)p_0(t) \ \text{ for almost all } \ t \in \,]a,b[\,,$$

where $p_0 :]a,b[\to \mathbb{R}$ is an arbitrary measurable function which along with (19) satisfies the condition

$$|p_0(t)| \le 2(b-a)^{-2} \ \text{ for almost all } \ t \in \,]a,b[\,.$$

Then the function p has the singularity of an infinite order at the points a and b. Nevertheless, by virtue of Corollary 3, problem (1), (3) is uniquely solvable.

Corollary 4. *If the function p satisfies condition* (12)*, and the function q satisfies condition* (17)*, then problem* (1), (3) *has a unique solution.*

Example 4. Let τ and p be functions given by equalities (18), where p_0 is an arbitrary measurable function which along with (9) satisfies the condition

$$\int_a^b |p_0(t)|dt \le (b-a)^{2-\alpha-\beta}.$$

Then the function p has the singularity of an infinite order at the points a and b. Nevertheless, according to Corollary 4, problem (1), (3) is uniquely solvable.

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