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Mathematics

On Extremal Solutions of Two-Point Boundary Value Problems for Second Order Nonlinear Singular Differential Equations

Nino Partsvania

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University

(Presented by Academy Member Ivane Kiguradze)

ABSTRACT. Unimprovable in a certain sense conditions are established which guarantee, respectively, the unique and non-unique solvability and the existence of a minimal and a maximal solutions of two-point boundary value problems for second order nonlinear singular differential equations. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *second order nonlinear differential equation, strong singularity, two-point boundary value problem, minimal solution, maximal solution.*

In the present paper we consider the problems on the existence of a solution of the differential equation

$$u'' = f\left(t,u,u'\right), \tag{1}$$

defined on a finite open interval $]a,b[$ and satisfying either the conditions

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty, \tag{2}$$

or

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0, \quad \int_a^b u'^2(t)dt < +\infty. \tag{3}$$

Here the function $f:]a,b[\times R^2 \to R$ satisfies the local Carathéodory conditions, i.e. $f(\cdot,x,y):]a,b[\to R$ is measurable for any $(x,y)\in R^2$, $f(t,\cdot,\cdot): R^2 \to R$ is continuous for almost all $t\in]a,b[$ and the function

$$f_\rho^*(t) = \sup\left\{|f(t,x,y)| : |x|+|y| \le \rho\right\} \tag{4}$$

is Lebesgue integrable on $[a+\varepsilon, b-\varepsilon]$ for any $\rho \in]0,+\infty[$ and $\varepsilon\in]0,(b-a)/2[$.

A function $u:]a,b[\to R$ is said to be a solution of Eq. (1) if it is absolutely continuous together with its first derivative on each closed interval, contained in $]a,b[$, and satisfies Eq. (1) almost everywhere on $]a,b[$. The solution of Eq. (1), satisfying conditions (2) (conditions (3)), is said to be a solution of problem (1),(2) (of problem (1),(3)).

A solution u_0 of problem (1),(2) or problem (1),(3) is said to be maximal (minimal) if an arbitrary solution of that problem satisfies the inequality

$$u(t) \le u_0(t) \ \big(u(t) \ge u_0(t)\big) \quad \text{for} \quad a < t < b .$$

Eq. (1) is said to be **singular** if for some $\rho > 0$, the function f_ρ^*, defined by equality (4), is non-integrable on $[a,b]$, having the singularity at least at one of the boundary points of that interval. The singularity at the point a (at the point b) is said to be **strong** if

$$\int_a^t (s-a) f_\rho^*(s)ds = +\infty \quad \left(\int_t^b (b-s) f_\rho^*(s)ds = +\infty \right) \quad \text{for} \quad a < t < b .$$

The basics of the theory of boundary value problems for singular differential equations and systems, including the two-point problems

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u(t) = 0 \tag{2_0}$$

and

$$\lim_{t\to a} u(t) = 0, \quad \lim_{t\to b} u'(t) = 0 \tag{3_0}$$

for the singular differential equation (1), were laid by I. Kiguradze (see [1, 2] and the references therein). In [2-13], the optimal sufficient conditions are established for the solvability and unique solvability of problems (1),(2_0); (1),(3_0), and analogous problems for higher order singular differential equations. In [6] and [13], there are also obtained the conditions for the existence of extremal solutions of problems (1),(2_0) and (1),(3_0).

R. P. Agarwal and I. Kiguradze ([14], [15]) proposed new statements of two-point problems for higher order linear differential equations with strong singularities, and they found unimprovable sufficient conditions for the Fredholmicity and unique solvability of those problems. In particular, they proved that if $f(t,x,y) \equiv p(t)x + q(t)$ and the function p has a strong singularity at the point a or b, then problem (1),(2_0) (problem (1),(3_0)) may have an infinite set of solutions whereas problem (1), (2) (problem (1),(3)) is uniquely solvable. Therefore in the case of strong singularity it is expedient to replace conditions (2_0) (conditions (3_0)) by conditions (2) (conditions (3)). In that case problems (1),(2) and (1),(3) are studied insufficiently. In particular, the question on the existence of extremal solutions of those problems remains still open. The goal of the present paper is to fill this gap.

By $L_{loc}(]a,b[)$ (by $L_{loc}(]a,b])$) we denote the space of functions $q:]a,b[\to R$, Lebesgue integrable on $[a+\varepsilon, b-\varepsilon]$ (on $[a+\varepsilon, b]$) for any $\varepsilon \in]0,(b-a)/2]$.

The following theorem is valid.

Theorem 1. *Let on* $]a,b[\times R^2$ *the inequality*

$$f(t,x,y)\operatorname{sgn} x \ge -\left(\frac{l_0}{(t-a)^2(b-t)^2} + \frac{l_1}{(t-a)(b-t)} \right)|x| - \frac{l_2}{(t-a)(b-t)}|y| - q(t) \tag{5}$$

be fulfilled, where l_0, l_1, l_2 *are non-negative constants, and* $q \in L_{loc}(]a,b[)$ *is a non-negative function such that*

$$\frac{4l_0}{(b-a)^2} + \frac{l_1}{2} + \frac{2l_2}{b-a} < 1 \tag{6}$$

and

$$\int_a^b (t-a)^{1/2}(b-t)^{1/2}q(t)dt < +\infty\,. \tag{7}$$

Then problem (1),(2) *is solvable and it has a minimal and a maximal solutions.*

Remark 1. Condition (6) in Theorem 1 is unimprovable in the sense that it cannot be replaced by the condition

$$\frac{4l_0}{(b-a)^2}+\frac{l_1}{2}+\frac{2l_2}{b-a}<1+\varepsilon\,, \tag{6ε}$$

no matter how small ε>0 would be. Indeed, if

$$\alpha = 1-\frac{\varepsilon}{3}>\frac{1}{2}$$

and

$$f(t,x,y) \equiv -\left(\frac{\alpha(1-\alpha)(b+a-2t)^2}{(t-a)^2(b-t)^2}+\frac{2\alpha}{(t-a)(b-t)}\right)x\,, \tag{8}$$

then conditions (5), (6ε), and (7) are satisfied, where

$$l_0=\alpha(1-\alpha)(b-a)^2,\quad l_1=2\alpha,\quad l_2=0,\quad q(t)\equiv 0\,. \tag{9}$$

On the other hand, in that case the function

$$u(t)\equiv c(t-a)^{\alpha}(b-t)^{\alpha} \tag{10}$$

is a solution of problem (1), (2) for any $c\in R$. Therefore this problem does not have a minimal and a maximal solutions.

Remark 2. The conditions of Theorem 1 do not guarantee the existence of extremal solutions of problem (1),(2_0). Indeed, if

$$0<\alpha<\frac{1}{4}$$

and identity (8) holds, then on the one hand, conditions (5)-(7) are satisfied, where l_0, l_1, l_2 and q are the numbers and the function, given by equalities (9), and on the other hand, the function u, defined by equality (10), is a solution of problem (1),(2_0) for any $c\in R$. Therefore this problem has no extremal solution.

Theorem 2. *Let* $f_\rho^*\in L_{loc}(]a,b])$ *for any* $\rho>0$ *and let on* $]a,b[\times R^2$ *the inequality*

$$f(t,x,y)\operatorname{sgn}x \ge -\left(\frac{l_0}{(t-a)^2(2b-a-t)^2}+\frac{l_1}{(t-a)(2b-a-t)}\right)|x|-\frac{l_2}{(t-a)(2b-a-t)}|y|-q(t)$$

be fulfilled, where l_0, l_1, l_2 *are non-negative constants, and* $q\in L_{loc}(]a,b])$ *is a non-negative function such that*

$$\frac{l_0}{(b-a)^2}+\frac{l_1}{2}+\frac{l_2}{b-a}<1 \tag{11}$$

and

$$\int_a^b (t-a)^{1/2}q(t)dt<+\infty\,. \tag{12}$$

Then problem (1),(3) *is solvable and it has a minimal and a maximal solutions.*

Remark 3. If $\alpha\in(0,1)$ and

$$f(t,x,y) \equiv -\left(\frac{4\alpha(1-\alpha)(b-t)^2}{(t-a)^2(2b-a-t)^2} + \frac{2\alpha}{(t-a)(2b-a-t)} \right) x,$$

then for any $c \in R$, the function

$$u(t) \equiv c(t-a)^{\alpha}(2b-a-t)^{\alpha}$$

is a solution of problem (1),(3_0). From this fact it follows that:

(i) the conditions of Theorem 2 do not guarantee the existence of extremal solutions of problem (1),(3_0);

(ii) condition (11) in Theorem 2 is unimprovable and it cannot be replaced by the condition

$$\frac{l_0}{(b-a)^2} + \frac{l_1}{2} + \frac{l_2}{b-a} < 1 + \varepsilon,$$

no matter how small ε>0 would be.

Remark 4. Conditions (7) and (12) in Theorems 1 and 2 are also unimprovable and they cannot be replaced by the conditions

$$\int_a^b (t-a)^{1/2+\varepsilon}(b-t)^{1/2+\varepsilon} q(t)dt < +\infty \tag{7ε}$$

and

$$\int_a^b (t-a)^{1/2+\varepsilon} q(t)dt < +\infty, \tag{12ε}$$

no matter how small ε>0 would be. Indeed, if

$$f(t,x,y) \equiv \gamma_1(t-a)^{-\frac{3}{2}} + \gamma_2(t-a)^{-\frac{3}{2}}, \text{ where } |\gamma_1| + |\gamma_2| > 0 \quad \left(f(t,x,y) \equiv (t-a)^{-\frac{3}{2}} \right),$$

then problem (1),(2) (problem (1),(3)) has no solution, though in that case all the conditions of Theorem 1 (Theorem 2) are fulfilled except condition (7) (condition (12)) instead of which condition (7ε)(condition (12ε)) is satisfied.

Note that under the conditions of Theorem 1 (Theorem 2) problem (1),(2) (problem (1),(3)) may be both uniquely and non-uniquely solvable. This is evident from Theorems 3-6 below.

Theorems 3 and 4 concern the case where the function *f* with respect to the phase variables satisfies the Lipschitz one-sided condition. More precisely, we consider the case where one of the following two conditions is satisfied on $]a,b[\times R^2$:

$$\begin{aligned} &\big(f(t,x_1,y_1) - f(t,x_2,y_2)\big)\operatorname{sgn}(x_1-x_2) \geq \\ &\geq -\left(\frac{l_0}{(t-a)^2(b-t)^2} + \frac{l_1}{(t-a)(b-t)} \right)|x_1-x_2| - \frac{l_2}{(t-a)(b-t)}|y_1-y_2| \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\big(f(t,x_1,y_1) - f(t,x_2,y_2)\big)\operatorname{sgn}(x_1-x_2) \geq \\ &\geq -\left(\frac{l_0}{(t-a)^2(2b-a-t)^2} + \frac{l_1}{(t-a)(2b-a-t)} \right)|x_1-x_2| - \frac{l_2}{(t-a)(2b-a-t)}|y_1-y_2|, \end{aligned} \tag{14}$$

where l_0, l_1, l_2 are non-negative constants.

Theorem 3. *If along with* (13) *conditions* (6) *and* (7) *are fulfilled, where* $q(t) \equiv |f(t,0,0)|$, *then problem* (1),(2) *has a unique solution.*

Theorem 4. *Let along with* (14) *conditions* (11) *and* (12) *be fulfilled, where* $q(t) \equiv | f(t,0,0) |$. *If, moreover,* $f_\rho^* \in L_{loc}(]a,b])$ for any $\rho \geq 0$, *then problem* (1),(3) *has a unique solution.*

Finally, we consider the case where one of the following two conditions is satisfied on $]a,b[\times R^2$:

$$-r_0 \, |x|^\lambda \geq \big(f(t,x,y) - p(t)\,|y|\big)\operatorname{sgn} x \geq$$

$$\geq -\left(\frac{l_0}{(t-a)^2(b-t)^2} + \frac{l_1}{(t-a)(b-t)}\right)|x| - \frac{r}{(t-a)^\mu (b-t)^\mu}\,|x|^\lambda \tag{15}$$

and

$$-r_0 \, |x|^\lambda \geq \big(f(t,x,y) - p(t)\,|y|\big)\operatorname{sgn} x \geq$$

$$\geq -\left(\frac{l_0}{(t-a)^2(2b-a-t)^2} + \frac{l_1}{(t-a)(2b-a-t)}\right)|x| - \frac{r}{(t-a)^\mu}\,|x|^\lambda, \tag{16}$$

where

$$l_i \geq 0 \ \ (i=0,1), \quad r > r_0 > 0, \quad 0 < \lambda < 1, \quad \mu < \frac{3+\lambda}{2}, \tag{17}$$

and $p:]a,b[\to R$ is a measurable function such that

$$l_2 = \operatorname{vraimax}\{(t-a)(b-t)\,|p(t)| : a < t < b\} < +\infty, \tag{18}$$

or

$$l_2 = \operatorname{vraimax}\{(t-a)(2b-a-t)\,|p(t)| : a < t < b\} < +\infty. \tag{19}$$

Theorem 5. *If along with* (15), (17), *and* (18) *condition* (6) *is satisfied, then problem* (1),(2) *has: the trivial solution, the positive on* $]a,b[$ *maximal solution, and the negative on* $]a,b[$ *minimal solution.*

Theorem 6. *Let along with* (16), (17), *and* (19) *condition* (11) *be satisfied. If, moreover,* $f_\rho^* \in L_{loc}(]a,b])$ for any $\rho \geq 0$, *then problem* (1),(3) *has: the trivial solution, the positive on* $]a,b[$ *maximal solution, and the negative on* $]a,b[$ *minimal solution.*

As examples, we consider the differential equations

$$u'' = -\left(\frac{l_0}{(t-a)^2(b-t)^2} + \frac{l_1}{(t-a)(b-t)}\right)u - \frac{l_2}{(t-a)(b-t)}\,|u'| - \frac{r}{(t-a)^\mu (b-t)^\mu}\,|u|^\lambda \operatorname{sgn} u \tag{20}$$

and

$$u'' = -\left(\frac{l_0}{(t-a)^2(2b-a-t)^2} + \frac{l_1}{(t-a)(2b-a-t)}\right)u - \frac{l_2}{(t-a)(2b-a-t)}\,|u'| - \frac{r}{(t-a)^\mu}\,|u|^\lambda \operatorname{sgn} u, \tag{21}$$

where l_0, l_1, l_2 are non-negative constants, r is a positive constant, $0 < \lambda < 1$, and $\mu < (3+\lambda)/2$. From Theorems 5 and 6 it follows that if inequality (6) (inequality (11)) is satisfied, then problem (20),(2) (problem (21),(3)) has: the trivial solution, the positive on $]a,b[$ maximal solution, and the negative on $]a,b[$ minimal solution.

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ნ. ფარცვანია

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REFERENCES

1. *I.T. Kiguradze* (1975), Nekotorye singulyarnye kraevye zadachi dlya obyknovennykh differentsial'nykh uravneniy. Tbilisi University Press, Tbilisi (in Russian).
2. *I.T. Kiguradze, B. L. Shekhter* (1987), Itogi Nauki i Tekh., Ser. Sovrem. Probl. Mat., Noveyshie Dostizh., **30:** 105-201 (in Russian); English transl.: (1988), J. Sov. Math., **43,** 2: 2340-2417.
3. *R.P. Agarwal and D. O'Regan* (1998), J. Differential Equations, **143,** 1: 60-95.
4. *R.P. Agarwal and D. O'Regan* (2003), Singular differential and integral equations with applications, Kluwer Academic Publishers, The Netherlands.
5. *I. Kiguradze* (2003), Funct. Differ. Equ., **10**, 1-2: 259-281.
6. *I. Kiguradze, N. Partsvania* (2005), Mem. Differential Equations Math. Phys., **36:** 147-152.
7. *T. Kiguradze* (2009), Nonlinear Analysis, **71:** 789-798.
8. *T. Kiguradze* (2009), Mem. Differential Equations Math. Phys., **47:** 169-174.
9. *T.I. Kiguradze* (2010), Differ. Uravn., **46,** 2: 183-190 (in Russian); English transl.: (2010), Differ. Equ., **46,** 2: 187-194.
10. *A. Lomtatidze, L. Malaguti* (2003), Nonlinear Analysis, **52,** 6(A): 1553-1567.
11. *A. Lomtatidze, P. Torres* (2003), Czech. Math. J., **53,** 1: 19-43.
12. *I. Rachůnková, S. Staněk, M. Tvrdý* (2006), Handbook of differential equations, ordinary differential equations, **3:** 607-723. Elsevier.
13. *I. Rachůnková, S. Staněk, M. Tvrdý* (2009), Solvability of nonlinear singular problems for ordinary differential equations, Hindawi Publishing Corporation.
14. *R.P. Agarwal, I. Kiguradze* (2006), Boundary Value Problems: 1-32; Article ID 83910.
15. *I. Kiguradze* (2004), Mem. Differential Equations Math. Phys., **32:** 101-107.

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