STRESS SINGULARITY ANALYSIS IN INTERFACE CRACK PROBLEMS FOR COMPOSITE STRUCTURES

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STRESS SINGULARITY ANALYSIS IN INTERFACE CRACK PROBLEMS FOR COMPOSITE STRUCTURES

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Abstract. *We investigate three–dimensional interface crack problems (ICP) for metallic-piezoelectric composite bodies with regard to thermal effects. We give a mathematical formulation of the physical problem when the metallic and piezoelectric bodies are bonded along some proper parts of their boundaries where interface cracks occur. By potential methods the ICP is reduced to an equivalent strongly elliptic system of pseudodifferential equations on manifolds with boundary. We study the solvability of this system in different function spaces and prove uniqueness and existence theorems for the original ICP. We analyse the regularity properties of the corresponding thermo-mechanical and electric fields near the crack edges and near the curves where the different boundary conditions collide. In particular, we characterize the stress singularity exponents and show that they can be explicitly calculated with the help of the principal homogeneous symbol matrices of the corresponding pseudodifferential operators. We expose some numerical calculations which demonstrate that the stress singularity exponents depend on the material parameters essentially.*

1 INTRODUCTION

In this paper we investigate a mathematical model describing the interaction of the elastic, thermal and electric fields in a three-dimensional composite consisting of a piezoelectric (ceramic) matrix and metallic inclusions (electrodes) bonded along some proper parts of their boundaries where interface cracks occur. Modern industrial and technological processes apply widely such composite materials.

In spite of the fact that the piezoelectric phenomena were discovered long ago [22], the practical use of piezoelectric effects became possible only when piezoceramics and other materials (*metamaterials*) with pronounced piezoelectric properties were constructed. Nowadays, sensors and actuators made of such materials are widely used in medicine, in aerospace, in various industrial and domestic appliances, in measuring and controlling devices. Therefore investigation of the mathematical models for such composite materials and analysis of the corresponding thermo-mechanical and electric fields became very actual and important for fundamental research and practical applications (for details see [8], [10]-[12], [17], [18] and the references therein).

Due to great theoretical and practical importance, problems of thermopiezoelectricity became very popular among mathematicians and engineers. Due to the references [9], during recent years more then 1000 scientific papers have been published annually! Most of them are engineering-technical papers dealing with the two-dimensional case.

Here we consider a general three-dimensional interface crack problem (ICP) for an anisotropic piezoelectric-metallic composite with regard to thermal effects and perform a rigorous mathematical analysis by the potential method. Similar problems for different type of metallic-piezoelastic composites without cracks have been considered in [1]-[2].

In our analysis, we apply the Voigt's linear model in the piezoelectric part and the usual classical model of thermoelasticity in the metallic part to write the corresponding coupled systems of governing partial differential equations (see, e.g., [15]-[16], [22]). As a result, in the piezoceramic part the unknown field is represented by a 5-component vector (three components of the displacement vector, the electric potential function and the temperature), while in the metallic part the unknown field is described by a 4-component vector (three components of the displacement vector and the temperature). Therefore, the mathematical modeling becomes complicated since we have to find reasonable efficient boundary, transmission and crack conditions for the physical fields possessing different dimensions in adjacent domains.

Since the crystal structures with central symmetry, in particular isotropic structures, do not reveal the piezoelectric properties in Voight's model, we have to consider anisotropic piezoelectric media. This also complicates the investigation. Thus, we have to take into account the composed anisotropic structure and the diversity of the fields in the ceramic and metallic parts.

The essential motivation for the choice of the interface crack problem treated in the paper is that in a piezoceramic material, due to its brittleness, often arise cracks, especially when a piezoelectric device works at high temperature regime or under an intensive mechanical loading. The influence of the electric field on the crack growth has a very complex character. Experiments revealed that the electric field can either promote or retard the crack growth, depending on the direction of polarization and can even close an open crack [17].

As it is well known from the classical mathematical physics and the classical elasticity theory, in general, solutions to crack type and mixed boundary value problems have singularities near the crack edges and near the lines where different boundary conditions collide, regardless of the smoothness of given boundary data. The same effect can be observed in the case of our interface crack problem, namely, singularities of electric, thermal and stress fields appear near the crack edges and near lines, where the boundary conditions collide and where the interfaces intersect the exterior boundary. Throughout the paper we shall refer to such lines as *exceptional curves*.

In this paper, we apply the potential method and reduce the ICPs to the equivalent system of pseudodifferential equations (ΨDEs) on a proper part of the boundary of the composed body. We analyse the solvability of the resulting boundary-integral equations in Sobolev-Slobodetski (W^s_p), Bessel potential (H^s_p), and Besov ($B^s_{p,t}$) spaces and prove the corresponding uniqueness and existence theorems for the original ICPs. Moreover, our main goal is a detailed theoretical investigation of regularity properties of thermo-mechanical and electric fields near the exceptional curves and qualitative description of their singularities.

The paper is organized as follows. In Section 2, we collect the field equations of the linear theory of thermoelasticity and thermopiezoelasticity, introduce the corresponding matrix partial differential operators and the generalized matrix boundary stress operators generated by the field equations, and formulate the boundary-transmission problems for a composed body consisting of metallic and piezoelectric parts with interface cracks. Depending on the physical properties of the metallic and piezoelectric materials and on surrounding media, one can consider different boundary, transmission and crack conditions for the thermal and electric fields. In particular, depending on the thermal insulation and dielectric properties of the crack gap, we present and discuss here a model interface crack problems (ICP-A). In Section 3, we derive special representation formulas of solutions in terms of generalized layer potentials. Sections 4 is the main part of the present paper. Here the original interface crack problem (ICP-A) is reduced to the system of ΨDEs on manifolds with boundary and full analysis of solvability of these equations is given. Properties of the principal homogeneous symbol matrices are studied in detail and the existence, regularity and asymptotic properties of the solution fields are established. In particular, in Theorem 4.4, the global C^α-regularity results are shown with some $\alpha \in \left(0, \frac{1}{2}\right)$ depending on the eigenvalues of these symbol matrices. Note that the eigenvalues depend on the material parameters, in general, and actually they define the singularity exponents for the first order derivatives of solutions. In particular, they define stress singularity exponents. We calculate these exponents for particular cases explicitly, demonstrate their dependence on the material parameters and discuss problems related to the oscillating stress singularities. We present also some numerical results and compare stress singularities at different type exceptional curves. As computations have shown, the stress singularities at the exceptional curves are different from -0.5 and essentially depend on the material parameters. We recall that for interior cracks the stress singularities do not depend on the material parameters and equal to -0.5 (see, e.g., [5], [7], [18].

2 Formulation of the interface crack problem

2.1 Geometrical description of the composite configuration.

Let $\Omega^{(m)}$ and Ω be bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(m)}$ and $\partial\Omega$, respectively. Moreover, let $\partial\Omega$ and $\partial\Omega^{(m)}$ have a nonempty, simply connected intersection $\overline{\Gamma^{(m)}}$ with a positive measure, i.e., $\partial\Omega\cap\partial\Omega^{(m)} = \overline{\Gamma^{(m)}}$, mes$\Gamma^{(m)} > 0$. From now on $\Gamma^{(m)}$ will be referred to as an interface surface. Throughout the paper n and $\nu = n^{(m)}$ stand for the outward unit normal vectors to $\partial\Omega$ and to $\partial\Omega^{(m)}$, respectively. Evidently, $n(x) = -\nu(x)$ for $x \in \Gamma^{(m)}$.

Further, let $\overline{\Gamma^{(m)}} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}}$, where $\Gamma_C^{(m)}$ is an open, simply connected proper part of $\Gamma^{(m)}$. Moreover, $\Gamma_T^{(m)} \cap \Gamma_C^{(m)} = \varnothing$ and $\partial\Gamma^{(m)} \cap \overline{\Gamma_C^{(m)}} = \varnothing$.

We set $S_N^{(m)} := \partial\Omega^{(m)} \setminus \overline{\Gamma^{(m)}}$ and $S^* := \partial\Omega \setminus \overline{\Gamma^{(m)}}$. Further, we denote by S_D some open, nonempty, proper sub-manifold of S^* and let $S_N := S^* \setminus \overline{S_D}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1)

$$\partial\Omega = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N} \cup \overline{S_D}, \quad \partial\Omega^{(m)} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N^{(m)}}.$$

Throughout the paper, for simplicity, we assume that $\partial\Omega^{(m)}$, $\partial\Omega$, $\partial S_N^{(m)}$, $\partial\Gamma_T^{(m)}$, $\partial\Gamma_C^{(m)}$, ∂S_D, ∂S_N are C^∞-smooth and $\partial\Omega^{(m)} \cap \overline{S_D} = \varnothing$, if not otherwise stated. Some results, obtained in the paper, also hold true when these manifolds and their boundaries are Lipschitz and we formulate them separately.

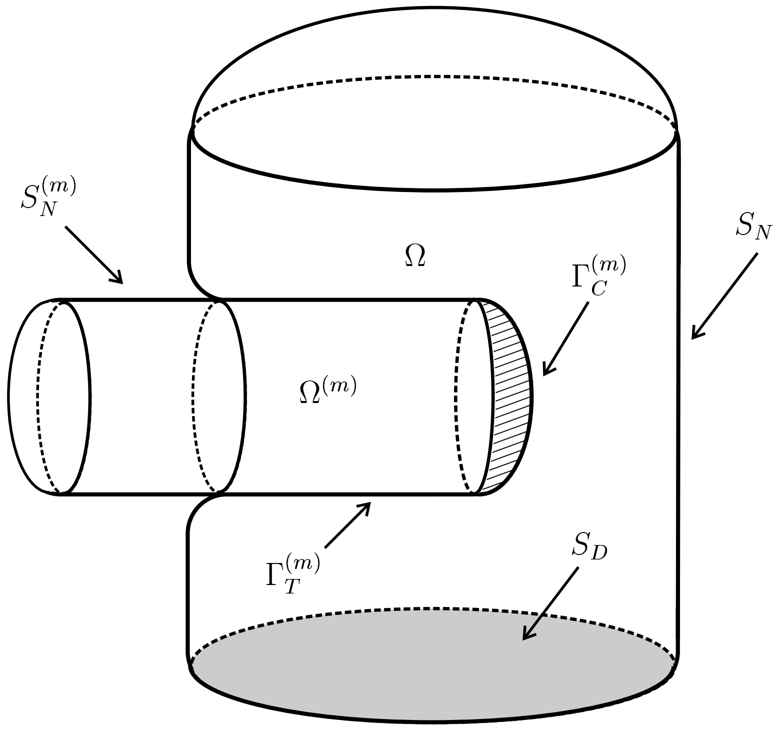

Figure 1: Metallic-piezoelectric composite

Let Ω be filled by an anisotropic homogeneous piezoelectric medium (ceramic matrix) and $\Omega^{(m)}$ be occupied by an isotropic or anisotropic homogeneous elastic medium (metallic inclusion). These two bodies interact along the interface $\Gamma^{(m)}$, where the interface crack $\Gamma_C^{(m)}$ occurs. Moreover, it is assumed that the composed body is fixed along the sub-surface S_D (the Dirichlet part of the boundary), while the sub-manifolds $S_N^{(m)}$ and S_N are the Neumann parts of the boundary. In the metallic domain $\Omega^{(m)}$ we have a four-dimensional thermoelastic field represented by the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and temperature distribution function $u_4^{(m)} = \vartheta^{(m)}$, while in the piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$, temperature distribution function $u_4 = \vartheta$ and the electric potential $u_5 := \varphi$.

2.2 Thermoelastic field equations.

Here we collect the field equations of the linear theory of thermoelasticity and introduce the corresponding matrix partial differential operators (see [16]). We will treat the general anisotropic case.

The basic governing equations of the classical thermoelasticity read as follows:
Constitutive relations:

$$\sigma_{ij}^{(m)} = \sigma_{ji}^{(m)} = c_{ijlk}^{(m)} \, s_{lk}^{(m)} - \gamma_{ij}^{(m)} \, \vartheta^{(m)} = c_{ijlk}^{(m)} \, \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \, \vartheta^{(m)}, \tag{2.1}$$

$$\mathcal{S}^{(m)} = \gamma_{ij}^{(m)} \, s_{ij}^{(m)} + \alpha^{(m)} \, [\, T_0^{(m)} \,]^{-1} \, \vartheta^{(m)}, \quad s_{lk}^{(m)} = 2^{-1} \, (\partial_l \, u_k^{(m)} + \partial_k \, u_l^{(m)}) \, . \tag{2.2}$$

Fourier Law: $q_j^{(m)} = -\varkappa_{jl}^{(m)} \, \partial_l T^{(m)}$.
Equations of motion: $\partial_i \sigma_{ij}^{(m)} + X_j^{(m)} = \varrho^{(m)} \, \partial_t^2 u_j^{(m)}$.
Equation of the entropy balance: $T^{(m)} \, \partial_t \mathcal{S}^{(m)} = -\partial_j q_j^{(m)} + X_4^{(m)}$.
Here $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ is the displacement vector, $\vartheta^{(m)} = T^{(m)} - T_0^{(m)}$ is the relative temperature (temperature increment); $\sigma_{kj}^{(m)}$ is the stress tensor in the theory of thermoelasticity, $s_{kj}^{(m)}$ is the strain tensor, $q^{(m)} = (q_1^{(m)}, q_2^{(m)}, q_3^{(m)})^\top$ is the heat flux vector; $\mathcal{S}^{(m)}$ is the entropy density, $\varrho^{(m)}$ is the mass density, $c_{ijkl}^{(m)}$ are the elastic constants, $\varkappa_{kj}^{(m)}$ are the thermal conductivity constants; $T_0^{(m)}$ is the initial temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields; $\gamma_{kj}^{(m)}$ are the thermal strain constants; $\alpha^{(m)} := \varrho^{(m)} \, \widetilde{c}^{(m)}$ are the thermal material constants; $\widetilde{c}^{(m)}$ is the specific heat per unit mass; $X^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)})^\top$ is a mass force density; $X_4^{(m)}$ is the heat source density; we employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation.

Throughout the paper the Einstein convention about the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

Constants involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)}, \quad \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)}, \; \varkappa_{ij}^{(m)} = \varkappa_{ji}^{(m)}, \quad i, j, k, l = 1, 2, 3. \tag{2.3}$$

Note that for an isotropic medium the thermomechanical coefficients are

$$c_{ijlk}^{(m)} = \lambda^{(m)} \, \delta_{ij} \, \delta_{lk} + \mu^{(m)} \, (\, \delta_{il} \, \delta_{jk} + \delta_{ik} \, \delta_{jl} \,), \quad \gamma_{ij}^{(m)} := \gamma^{(m)} \, \delta_{ij}, \quad \varkappa_{ij}^{(m)} = \varkappa^{(m)} \, \delta_{ij},$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are the Lamé constants (see the list of notation).

We assume that there are positive constants c_0 and c_1 such that

$$c^{(m)}_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq c_0\,\xi_{ij}\,\xi_{ij}, \quad \varkappa^{(m)}_{ij}\xi_i\xi_j \geq c_1\,\xi_i\,\xi_i \text{ for all } \xi_{ij}=\xi_{ji}\in\mathbb{R},\ \xi_j\in\mathbb{R},\ i,j=1,2,3. \tag{2.4}$$

If all the functions involved in the above equations are harmonic time dependent, that is they represent a product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau\, t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the theory of thermoelasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau = i\omega$ is a pure imaginary number, with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$ we get the *equations of statics*.

In this paper we will mainly consider the system of pseudo-oscillations

$$\begin{aligned}
&c^{(m)}_{ijlk}\,\partial_i\partial_l\,u^{(m)}_k - \varrho^{(m)}\,\tau^2\,u^{(m)}_j - \gamma^{(m)}_{ij}\,\partial_i\vartheta^{(m)} + X^{(m)}_j = 0, \quad j=1,2,3,\\
&-\tau\,T^{(m)}_0\,\gamma^{(m)}_{il}\,\partial_l u^{(m)}_i + \varkappa^{(m)}_{il}\,\partial_i\partial_l\vartheta^{(m)} - \tau\,\alpha^{(m)}\,\vartheta^{(m)} + X^{(m)}_4 = 0.
\end{aligned} \tag{2.5}$$

In matrix form these equations can be rewritten as

$$A^{(m)}(\partial,\tau)\,U^{(m)}(x) + \tilde{X}^{(m)}(x) = 0,$$

where $U^{(m)} := (u^{(m)}, \vartheta^{(m)})^\top$ is the unknown vector, $\tilde{X}^{(m)} = (X^{(m)}_1, X^{(m)}_2, X^{(m)}_3, X^{(m)}_4)^\top$ is a given vector, $A^{(m)}(\partial,\tau)$ is the nonselfadjoint matrix differential operator generated by equations (2.5),

$$\begin{aligned}
&A^{(m)}(\partial,\tau) = [\,A^{(m)}_{jk}(\partial,\tau)\,]_{4\times4}, \quad A^{(m)}_{jk}(\partial,\tau) = c^{(m)}_{ijlk}\,\partial_i\,\partial_l - \varrho^{(m)}\,\tau^2\,\delta_{jk},\\
&A^{(m)}_{4k}(\partial,\tau) = -\tau\,T^{(m)}_0\,\gamma^{(m)}_{kl}\,\partial_l, \quad A^{(m)}_{j4}(\partial,\tau) = -\gamma^{(m)}_{ij}\,\partial_i,\\
&A^{(m)}_{44}(\partial,\tau) = \varkappa^{(m)}_{il}\,\partial_i\,\partial_l - \alpha^{(m)}\,\tau,
\end{aligned} \tag{2.6}$$

where $j, k = 1, 2, 3$, and δ_{jk} is Kronecker's delta.

Denote by $A^{(m,0)}(\partial)$ the principal homogeneous part of the operator (2.6),

$$A^{(m,0)}(\partial) = \begin{bmatrix} [c^{(m)}_{ijlk}\,\partial_i\,\partial_l]_{3\times3} & [0]_{3\times1} \\ [0]_{1\times3} & \varkappa^{(m)}_{il}\,\partial_i\,\partial_l \end{bmatrix}_{4\times4}. \tag{2.7}$$

By $A^{(m)*}(\partial,\tau)$ we denote the 4×4 matrix differential operator formally adjoint to $A^{(m)}(\partial,\tau)$, that is $A^{(m)*}(\partial,\tau) := [\overline{A^{(m)}(-\partial,\tau)}]^\top$, where the over-bar denotes the complex conjugation.

With the help of the symmetry conditions (2.3) and inequalities (2.4) it can easily be shown that $A^{(m,0)}(\partial)$ is a selfadjoint elliptic operator with a positive definite principal homogeneous symbol matrix, that is,

$$A^{(m,0)}(\xi)\,\eta\cdot\eta \geq c^{(m)}\,|\xi|^2\,|\eta|^2 \text{ for all } \xi\in\mathbb{R}^3 \text{ and all } \eta\in\mathbb{C}^4$$

with some positive constant $c^{(m)} > 0$ depending on the material parameters.

Components of the mechanical thermostress vector acting on a surface element with a normal $\nu = (\nu_1, \nu_2, \nu_3)$ read as follows $\sigma^{(m)}_{ij}\,\nu_i = c^{(m)}_{ijlk}\,\nu_i\,\partial_l u^{(m)}_k - \gamma^{(m)}_{ij}\,\nu_i\,\vartheta^{(m)}$, $j = 1,2,3$, while the normal component of the heat flux vector (with opposite sign) has the form $-q^{(m)}_i\,\nu_i = \varkappa^{(m)}_{il}\,\nu_i\,\partial_l\vartheta^{(m)}$. We introduce the following generalized thermo-stress operator

$$\mathcal{T}^{(m)}(\partial,\nu) = [\ \mathcal{T}^{(m)}_{jk}(\partial,\nu)\ ]_{4\times4}, \tag{2.8}$$

where (for $j,k = 1,2,3$)

$$\mathcal{T}^{(m)}_{jk}(\partial,\nu) = c^{(m)}_{ijlk}\,\nu_i\,\partial_l,\quad \mathcal{T}^{(m)}_{j4}(\partial,\nu) = -\gamma^{(m)}_{ij}\,\nu_i,\quad \mathcal{T}^{(m)}_{4k}(\partial,\nu) = 0,\quad \mathcal{T}^{(m)}_{44}(\partial,\nu) = \varkappa^{(m)}_{il}\,\nu_i\,\partial_l.$$

For a four–vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$ we have

$$\mathcal{T}^{(m)}\,U^{(m)} = (\ \sigma^{(m)}_{i1}\,\nu_i,\ \sigma^{(m)}_{i2}\,\nu_i,\ \sigma^{(m)}_{i3}\,\nu_i,\ -q^{(m)}_i\,\nu_i\ )^\top. \tag{2.9}$$

Clearly, the components of the vector $\mathcal{T}^{(m)}\,U^{(m)}$ given by (2.9) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelasticity, while the forth one is the normal component of the heat flux vector (with opposite sign).

We also introduce the boundary operator associated with the adjoint operator $A^{(m)*}(\partial,\tau)$ which appears in Green's formulae,

$$\widetilde{\mathcal{T}}^{(m)}(\partial,\nu,\tau) = [\ \widetilde{\mathcal{T}}^{(m)}_{jk}(\partial,\nu,\tau)\ ]_{4\times4},$$

where (for $j,k = 1,2,3$)

$$\begin{aligned} &\widetilde{\mathcal{T}}^{(m)}_{jk}(\partial,\nu,\tau) = c^{(m)}_{ijlk}\,\nu_i\,\partial_l, && \widetilde{\mathcal{T}}^{(m)}_{j4}(\partial,\nu,\tau) = \mp T^{(m)}_0\,\gamma^{(m)}_{ij}\,\nu_i,\\ &\widetilde{\mathcal{T}}^{(m)}_{4k}(\partial,\nu,\tau) = 0, && \widetilde{\mathcal{T}}^{(m)}_{44}(\partial,\nu,\tau) = \varkappa^{(m)}_{il}\,\nu_i\,\partial_l. \end{aligned}$$

2.3 Thermopiezoelastic field equations.

In this subsection we collect the field equations of the linear theory of thermopiezoelasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators (cf. [18]).

In the thermopiezoelasticity we have the following governing equations (see the list of notation):
Constitutive relations:

$$\sigma_{ij} = \sigma_{ji} = c_{ijkl}\,s_{kl} - e_{lij}E_l - \gamma_{ij}\,\vartheta = c_{ijkl}\,\partial_l u_k + e_{lij}\,\partial_l\varphi - \gamma_{ij}\,\vartheta,\ i,j = 1,2,3, \tag{2.10}$$

$$\mathcal{S} = \gamma_{ij}\,s_{ij} + g_l E_l + \alpha\,[\,T_0\,]^{-1}\,\vartheta,\quad s_{kj} = 2^{-1}(\partial_k\,u_j + \partial_j\,u_k), \tag{2.11}$$

$$D_j = e_{jkl}\,s_{kl} + \varepsilon_{jl}\,E_l + g_j\,\vartheta = e_{jkl}\,\partial_l u_k - \varepsilon_{jl}\,\partial_l\varphi + g_j\,\vartheta,\ j = 1,2,3. \tag{2.12}$$

Fourier Law: $q_i = -\varkappa_{il}\,\partial_l T, \quad i = 1,2,3.$
Equations of motion: $\partial_i\sigma_{ij} + X_j = \varrho\,\partial_t^2 u_j, \quad j = 1,2,3.$
Equation of the entropy balance: $T\,\partial_t\mathcal{S} = -\partial_j q_j + X_4.$
Equation of static electric field: $\partial_i D_i - X_5 = 0.$
Here $u = (u_1,u_2,u_3)^\top$ is the displacement vector, φ is the electric potential, ϑ is the temperature

increment, σ_{kj} is the stress tensor in the theory of thermoelectroelasticity, s_{kj} is the strain tensor, D is the electric displacement vector, $E=(E_1,E_2,E_3):=-\mathrm{grad}\,\varphi$ is the electric field vector, $q=(q_1,q_2,q_3)$ is the heat flux vector, $\mathcal{S}$ is the entropy density, ϱ is the mass density, c_{ijkl} are the elastic constants, e_{kij} are the piezoelectric constants, ε_{kj} are the dielectric (permittivity) constants, γ_{kj} are thermal strain constants, $\varkappa_{kj}$ are thermal conductivity constants, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $\alpha:=\varrho\,\widetilde{c}$ with $\widetilde{c}$ as the specific heat per unit mass, g_i are pyroelectric constants characterizing the relation between thermodynamic processes and piezoelectric effects, $X=(X_1,X_2,X_3)^\top$ is a mass force density, X_4 is a heat source density, X_5 is a charge density.

From the relations above we derive the linear system of the corresponding pseudo-oscillation equations

$$\begin{aligned}
&c_{ijlk}\,\partial_i\,\partial_l\,u_k-\varrho\,\tau^2\,u_j-\gamma_{ij}\,\partial_i\vartheta+e_{lij}\,\partial_l\partial_i\varphi+X_j=0,\quad j=1,2,3,\\
&-\tau\,T_0\,\gamma_{il}\,\partial_l u_i+\varkappa_{il}\,\partial_i\,\partial_l\vartheta-\tau\,\alpha\,\vartheta+\tau\,T_0\,g_i\,\partial_i\varphi+X_4=0,\\
&-e_{ikl}\,\partial_i\partial_l u_k-g_i\,\partial_i\vartheta+\varepsilon_{il}\,\partial_i\partial_l\varphi+X_5=0,
\end{aligned}\tag{2.13}$$

or in matrix form

$$A(\partial,\tau)\,U(x)+\widetilde{X}(x)=0\ \text{ in }\ \Omega,\tag{2.14}$$

where $U:=(u,\vartheta,\varphi)^\top$, $\widetilde{X}=(X_1,X_2,X_3,X_4,X_5)^\top$, $A(\partial,\tau)$ is the matrix differential operator generated by equations (2.13)

$$\begin{aligned}
&A(\partial,\tau)=[A_{jk}(\partial,\tau)]_{5\times5}\,,\quad A_{jk}(\partial,\tau)=c_{ijlk}\,\partial_i\,\partial_l-\varrho\,\tau^2\,\delta_{jk},\\
&A_{j4}(\partial,\tau)=-\gamma_{ij}\,\partial_i,\quad A_{j5}(\partial,\tau)=e_{lij}\partial_l\partial_i,\quad A_{4k}(\partial,\tau)=-\tau\,T_0\,\gamma_{kl}\,\partial_l,\\
&A_{44}(\partial,\tau)=\varkappa_{il}\,\partial_i\,\partial_l-\alpha\,\tau,\quad A_{45}(\partial,\tau)=\tau\,T_0\,g_i\,\partial_i,\quad A_{5k}(\partial,\tau)=-e_{ikl}\,\partial_i\partial_l,\\
&A_{54}(\partial,\tau)=-g_i\partial_i,\quad A_{55}(\partial,\tau)=\varepsilon_{il}\,\partial_i\partial_l,\quad j,k=1,2,3.
\end{aligned}\tag{2.15}$$

Constants involved in these equations satisfy the symmetry conditions: $c_{ijkl}=c_{jikl}=c_{klij},\ \ e_{ijk}=e_{ikj},\ \ \varepsilon_{ij}=\varepsilon_{ji},\ \ \gamma_{ij}=\gamma_{ji},\ \ \varkappa_{ij}=\varkappa_{ji},\ \ i,j,k,l=1,2,3$. Moreover, from the physical considerations it follows that (see, e.g., [15]):

$$c_{ijkl}\,\xi_{ij}\,\xi_{kl}\geq c_0\,\xi_{ij}\,\xi_{ij}\ \text{ for all }\ \xi_{ij}=\xi_{ji}\in\mathbb{R},\tag{2.16}$$

$$\varepsilon_{ij}\,\eta_i\,\eta_j\geq c_1\,\eta_i\,\eta_i\,,\ \ \varkappa_{ij}\,\eta_i\,\eta_j\geq c_2\,\eta_i\,\eta_i\ \text{ for all }\ \eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3,\tag{2.17}$$

where c_0, c_1, and c_2 are positive constants.

By $A^*(\partial,\tau)$ we denote the operator formally adjoint to $A(\partial,\tau)$, that is $A^*(\partial,\tau):=[\overline{A(-\partial,\tau)}]^\top$.

In the theory of thermopiezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form $\sigma_{ij}\,n_i=c_{ijlk}\,n_i\,\partial_l u_k+e_{lij}\,n_i\,\partial_l\varphi-\gamma_{ij}\,n_i\,\vartheta$ for $j=1,2,3$, while the normal components of the electric displacement vector and the heat flux vector (with opposite sign) read as $-D_i\,n_i=-e_{ikl}\,n_i\,\partial_l u_k+\varepsilon_{il}\,n_i\,\partial_l\varphi-g_i\,n_i\,\vartheta,\quad -q_i\,n_i=\varkappa_{il}\,n_i\,\partial_l\vartheta$.

Let us introduce the following matrix differential operator

$$\mathcal{T}(\partial,n)=\big[\ \mathcal{T}_{jk}(\partial,n)\ \big]_{5\times5},\tag{2.18}$$

where (for $j,k=1,2,3$)

$$\begin{aligned}
&\mathcal{T}_{jk}(\partial,n)=c_{ijlk}\,n_i\,\partial_l,\quad \mathcal{T}_{j4}(\partial,n)=-\gamma_{ij}\,n_i,\quad \mathcal{T}_{j5}(\partial,n)=e_{lij}\,n_i\,\partial_l,\\
&\mathcal{T}_{4k}(\partial,n)=0,\quad \mathcal{T}_{44}(\partial,n)=\varkappa_{il}\,n_i\,\partial_l,\quad \mathcal{T}_{45}(\partial,n)=0,\\
&\mathcal{T}_{5k}(\partial,n)=-e_{ikl}\,n_i\,\partial_l,\quad \mathcal{T}_{54}(\partial,n)=-g_i\,n_i,\quad \mathcal{T}_{55}(\partial,n)=\varepsilon_{il}\,n_i\,\partial_l.
\end{aligned}\tag{2.19}$$

For a vector $U=(u,\varphi,\vartheta)^\top$ we have $\mathcal{T}(\partial,n)\,U=(\,\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -q_i\,n_i,\ -D_i\,n_i\,)^\top$. Clearly, the components of the vector $\mathcal{T}\,U$ have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermoelectroelasticity, the forth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

In Green's formulas there appear also the following boundary operator associated with the differential operator $A^*(\partial,\tau)$, $\widetilde{\mathcal{T}}(\partial,n,\tau)=\big[\,\widetilde{\mathcal{T}}_{jk}(\partial,n,\tau)\,\big]_{5\times5}$, where (for $j,k=1,2,3$)

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{jk}(\partial,n,\tau)=c_{ijlk}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{j4}(\partial,n,\tau)=\overline{\tau}\,T_0\,\gamma_{ij}\,n_i,\quad \widetilde{\mathcal{T}}_{j5}(\partial,n,\tau)=-e_{lij}\,n_i\,\partial_l,\\
&\widetilde{\mathcal{T}}_{4k}(\partial,n,\tau)=0,\quad \widetilde{\mathcal{T}}_{44}(\partial,n,\tau)=\varkappa_{il}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{45}(\partial,n,\tau)=0,\\
&\widetilde{\mathcal{T}}_{5k}(\partial,n,\tau)=e_{ikl}\,n_i\,\partial_l,\quad \widetilde{\mathcal{T}}_{54}(\partial,n,\tau)=-\,\overline{\tau}\,T_0\,g_i\,n_i,\quad \widetilde{\mathcal{T}}_{55}(\partial,n,\tau)=\varepsilon_{il}\,n_i\,\partial_l.
\end{aligned}$$

2.4 Formulation of the interface crack problems.

Let us consider the metallic-piezoelectric composite structure described in Subsection 2.1 (see Figure 1). We assume that

(1) the composed body is fixed along the sub-surface S_D, i.e, there are given homogeneous Dirichlet data for the vector $U=(u,\vartheta,\varphi)^\top$;

(2) the sub-surface $S_N^{(m)}$ is either traction free or there is applied some surface force, i.e., the components of the mechanical stress vector $\sigma_{ij}^{(m)}\,\nu_i$, $j=1,2,3$, are given on $S_N^{(m)}$;

(3) the sub-surface S_N is either traction free or there is applied some surface force, i.e., the components of the mechanical stress vector $\sigma_{ij}\,n_i$, $j=1,2,3$, are given on S_N;

(4) along the transmission interface submanifold $\Gamma_T^{(m)}$ the piezoelectric and metallic solids are bonded, i.e., the rigid contact conditions are fulfilled which means that the displacement and mechanical stress vectors are continuous across $\Gamma_T^{(m)}$;

(5) the faces of the interface crack $\Gamma_C^{(m)}$ are traction free, i.e., the components of the mechanical stress vectors $\sigma_{ij}^{(m)}\,\nu_i$ and $\sigma_{ij}\,n_i$, $j=1,2,3$, vanish on $\Gamma_C^{(m)}$.

Depending on the physical properties of the metallic and piezoelectric materials and also surrounding media, one can consider different boundary, transmission and crack conditions for the thermal and electric fields.

Here we formulate mathematically the following mixed interface crack problems. Without loss of generality, throughout the paper we assume that the initial reference temperatures T_0 and $T_0^{(m)}$ in the adjacent domains Ω and $\Omega^{(m)}$ are the same: $T_0=T_0^{(m)}$.

Problem (ICP-A) - the crack gap is thermally insulated dielectric:
Find vector-functions

$$U^{(m)}=(u_1^{(m)},u_2^{(m)},u_3^{(m)},u_4^{(m)})^\top\ :\ \Omega^{(m)}\to\mathbb{C}^4\quad\text{and}\quad U=(u_1,u_2,u_3,u_4,u_5)^\top\ :\ \Omega\to\mathbb{C}^5$$

belonging respectively to the spaces $[W^1_p(\Omega^{(m)})]^4$ and $[W^1_p(\Omega)]^5$ with $1<p<\infty$ and satisfying

(i) *the systems of partial differential equations*:

$$\big[\,A^{(m)}(\partial_x,\tau)\,U^{(m)}\,\big]_j=0 \quad \text{in} \quad \Omega^{(m)},\ \ j=1,2,3,4, \tag{2.20}$$

$$\big[\,A(\partial_x,\tau)\,U\,\big]_k=0 \quad \text{in} \quad \Omega,\ \ k=1,2,3,4,5, \tag{2.21}$$

(ii) *the boundary conditions*:

$$r_{S_N^{(m)}}\{[\,\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}\,]_j\}^+=Q_j^{(m)} \quad \text{on} \quad S_N^{(m)},\ \ j=1,2,3,4, \tag{2.22}$$

$$r_{S_N}\{[\,\mathcal{T}(\partial,n)\,U\,]_k\}^+=Q_k \quad \text{on} \quad S_N,\ \ k=1,2,3,4,5, \tag{2.23}$$

$$r_{S_D}\{u_k\}^+=f_k \quad \text{on} \quad S_D,\ \ k=1,2,3,4,5, \tag{2.24}$$

$$r_{\Gamma_T^{(m)}}\{u_5\}^+=f_5^{(m)} \quad \text{on} \quad \Gamma_T^{(m)}, \tag{2.25}$$

(iii) *the transmission conditions*:

$$r_{\Gamma_T^{(m)}}\{u_j\}^+-r_{\Gamma_T^{(m)}}\{u_j^{(m)}\}^+=f_j^{(m)} \ \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{2.26}$$

$$r_{\Gamma_T^{(m)}}\big\{[\,\mathcal{T}(\partial,n)U\,]_j\big\}^+ + r_{\Gamma_T^{(m)}}\big\{[\,\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}\,]_j\big\}^+=F_j^{(m)} \ \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{2.27}$$

(iv) *the interface crack conditions*:

$$r_{\Gamma_C^{(m)}}\big\{[\,\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}\,]_j\big\}^+=\widetilde{Q}_j^{(m)} \quad \text{on} \quad \Gamma_C^{(m)},\ \ j=1,2,3,4, \tag{2.28}$$

$$r_{\Gamma_C^{(m)}}\{[\,\mathcal{T}(\partial,n)\,U\,]_k\}^+=\widetilde{Q}_k \quad \text{on} \quad \Gamma_C^{(m)},\ \ k=1,2,3,4,5, \tag{2.29}$$

where $n=-\nu$ on $\Gamma^{(m)}$,

$$\begin{aligned}
&Q_k\in B^{-1/p}_{p,p}(S_N),\quad Q_j^{(m)}\in B^{-1/p}_{p,p}(S_N^{(m)}),\quad f_k\in B^{1/p'}_{p,p}(S_D),\quad f_k^{(m)}\in B^{1/p'}_{p,p}(\Gamma_T^{(m)}),\\
&F_j^{(m)}\in B^{-1/p}_{p,p}(\Gamma_T^{(m)}),\quad \widetilde{Q}_j^{(m)}\in B^{-1/p}_{p,p}(\Gamma_C^{(m)}),\quad \widetilde{Q}_k\in B^{-1/p}_{p,p}(\Gamma_C^{(m)}),\\
&\frac{1}{p'}+\frac{1}{p}=1,\quad k=1,2,3,4,5,\quad j=1,2,3,4.
\end{aligned} \tag{2.30}$$

Note that the functions $F_j^{(m)}$, Q_j, $\widetilde{Q}_j$, $\widetilde{Q}_j^{(m)}$ and $Q_j^{(m)}$ have to satisfy some evident compatibility conditions. We set

$$\begin{aligned}
&Q=(Q_1,\cdots,Q_5)^\top,\ \ \widetilde{Q}=(\widetilde{Q}_1,\cdots,\widetilde{Q}_5)^\top,\ Q^{(m)}=(Q_1^{(m)},\cdots,Q_4^{(m)})^\top,\\
&\widetilde{Q}^{(m)}=(\widetilde{Q}_1^{(m)},\cdots,\widetilde{Q}_4^{(m)})^\top,\ \ f=(f_1,\cdots,f_5)^\top,\ \ F^{(m)}=(F_1^{(m)},\cdots,F_4^{(m)})^\top,\\
&f^{(m)}=(f_1^{(m)},\cdots,f_5^{(m)})^\top.
\end{aligned} \tag{2.31}$$

A pair $(U^{(m)},U)\in[W^1_p(\Omega^{(m)})]^4\times[W^1_p(\Omega)]^5$ will be called a solution to the boundary-transmission problem (2.20)-(2.29).

Theorem 2.1 *Let $\Omega^{(m)}$ and Ω be Lipschitz and either $\tau=\sigma+i\omega$ with $\sigma>0$ or $\tau=0$. The interface crack problem (ICP-A) has at most one solution in the space $[W^1_2(\Omega^{(m)})]^4\times[W^1_2(\Omega)]^5$, provided* $\mathrm{mes}S_D>0$.

Below we apply the potential method to study the existence of solutions in different function spaces and to establish their regularity properties.

3 Representation formulas of solutions

Here we derive integral representation formulas of solutions to the homogeneous differential equations (2.20)-(2.21) by means of the layer potentials and certain boundary integral (pseudodifferential) operators generated by them.

3.1 Layer potentials.

Let $\Psi^{(m)}(\cdot\,,\tau)=\big[\,\Psi^{(m)}_{kj}(\cdot\,,\tau)\,\big]_{4\times 4}$ and $\Psi(\cdot\,,\tau)=\big[\,\Psi_{kj}(\cdot\,,\tau)\,\big]_{5\times 5}$ be the fundamental matrix-functions of the differential operators $A^{(m)}(\partial_x,\tau)$ and $A(\partial_x,\tau)$ and introduce the single and double layer potentials:

$$V^{(m)}_\tau(h^{(m)})(x)=\int\limits_{\partial\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)\,h^{(m)}(y)\,d_yS, \tag{3.32}$$

$$W^{(m)}_\tau(h^{(m)})(x)=\int\limits_{\partial\Omega^{(m)}}\big[\,\widetilde{\mathcal{T}}^{(m)}(\partial_y,\nu(y),\overline{\tau})\,[\Psi^{(m)}(x-y,\tau)]^\top\,\big]^\top\,h^{(m)}(y)\,d_yS, \tag{3.33}$$

$$V_\tau(h)(x)=\int\limits_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS, \tag{3.34}$$

$$W_\tau(h)(x)=\int\limits_{\partial\Omega}\big[\,\widetilde{\mathcal{T}}(\partial_y,n(y),\overline{\tau})\,[\Psi(x-y,\tau)]^\top\,\big]^\top\,h(y)\,d_yS, \tag{3.35}$$

where $h^{(m)}=(h^{(m)}_1,h^{(m)}_2,h^{(m)}_3,h^{(m)}_4)^\top$ and $h=(h_1,h_2,h_3,h_4,h_5)^\top$ are densities of the potentials. For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}^{(m)}_\tau(h^{(m)})(x):=\int\limits_{\partial\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)\,h^{(m)}(y)\,d_yS,\qquad x\in\partial\Omega^{(m)},$$

$$\mathcal{K}^{(m)}_\tau(h^{(m)})(x):=\int\limits_{\partial\Omega^{(m)}}\big[\,\mathcal{T}^{(m)}(\partial_x,\nu(x))\Psi^{(m)}(x-y,\tau)\,\big]\,h^{(m)}(y)\,d_yS,\qquad x\in\partial\Omega^{(m)},$$

$$\mathcal{H}_\tau(h)(x):=\int\limits_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS,\qquad x\in\partial\Omega,$$

$$\mathcal{K}_\tau(h)(x):=\int\limits_{\partial\Omega}\big[\,\mathcal{T}(\partial_x,n(x))\Psi(x-y,\tau)\,\big]\,h(y)\,d_yS,\qquad x\in\partial\Omega.$$

The layer boundary operators $\mathcal{H}^{(m)}_\tau$, $\mathcal{H}_\tau$ and $\mathcal{L}^{(m)}_\tau$, $\mathcal{L}_\tau$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}^{(m)}_\tau$, $\widetilde{\mathcal{K}}^{(m)\,*}_\tau$, $\mathcal{K}_\tau$ and $\widetilde{\mathcal{K}}^{*}_\tau$ are singular integral operators (pseudodifferential operators of order 0) (for details see [1], [2], [13]).

3.2 Auxiliary problems and representation formulas of solutions.

Here we assume that $\mathrm{Re}\,\tau = \sigma > 0$ and consider two auxiliary boundary value problems needed for our further purposes.

3.2.1. *Auxiliary problem* I: Find a vector function $U^{(m)} = (u_1^{(m)},\ u_2^{(m)},\ u_3^{(m)},\ u_4^{(m)})^\top : \Omega^{(m)} \to \mathbb{C}^4$ which belongs to the space $[\,W_2^1(\Omega^{(m)})\,]^4$ and satisfies the following conditions:

$$A^{(m)}(\partial,\tau)\,U^{(m)} = 0 \quad \text{in} \quad \Omega^{(m)}, \tag{3.36}$$

$$\big\{\,\mathcal{T}^{(m)}\,U^{(m)}\,\big\}^+ = \chi^{(m)} \quad \text{on} \quad \partial\Omega^{(m)}, \tag{3.37}$$

where $\chi^{(m)} = (\,\chi_1^{(m)},\,\chi_2^{(m)},\,\chi_3^{(m)},\,\chi_4^{(m)})^\top \in \big[\,H_2^{-\frac{1}{2}}(\partial\Omega^{(m)})\,\big]^4$. With the help of Green's formula it can easily be shown that the homogeneous version of this auxiliary BVP possesses only the trivial solution. Moreover, we have the following existence result.

Lemma 3.1 *Let* $\mathrm{Re}\,\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(m)} \in [\,W_p^1(\Omega^{(m)})\,]^4$ *to the homogeneous equation* (3.36) *can be uniquely represented by the single layer potential*

$$U^{(m)}(x) = V_\tau^{(m)}\big(\,\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\chi^{(m)}\,\big)(x), \quad x \in \Omega^{(m)}, \tag{3.38}$$

where

$$\mathcal{P}_\tau^{(m)} := -2^{-1}\,I_4 + \mathcal{K}_\tau^{(m)} \quad \textit{and} \quad \chi^{(m)} = \big\{\,\mathcal{T}^{(m)}U^{(m)}\,\big\}^+ \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})\,\big]^4. \tag{3.39}$$

3.2.2. *Auxiliary problem* II: Find a vector function $U = (u_1, u_2, u_3, u_4, u_5)^\top : \Omega \to \mathbb{C}^5$ which belongs to the space $[\,W_2^1(\Omega)\,]^5$ and satisfies the following conditions:

$$A(\partial,\tau)U = 0 \quad \text{in} \quad \Omega, \qquad \big\{\,\mathcal{T}U\,\big\}^+ + \beta\,\{\,U\,\}^+ = \chi \quad \text{on} \quad \partial\Omega, \tag{3.40}$$

where $\chi := (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5)^\top \in \big[\,H_2^{-\frac{1}{2}}(\partial\Omega)\,\big]^5$, β is a smooth real valued scalar function which does not vanish identically and

$$\beta \geq 0, \quad \mathrm{supp}\,\beta \subset S_D. \tag{3.41}$$

By the same arguments as in the proof of Theorem 2.1 we can easily show that the homogeneous version of this boundary value problem possesses only the trivial solution in the space $[\,W_2^1(\Omega)\,]^5$.

We look for a solution to the auxiliary BVP as a single layer potential, $U(x) = V_\tau(f)(x)$, where $f = (f_1,\,f_2,\,f_3,\,f_4,\,f_5)^\top \in \big[\,H_2^{-\frac{1}{2}}(\partial\Omega)\,\big]^5$ is an unknown density. The boundary condition (3.40) leads then to the system of equations:

$$\big(\,-\,2^{-1}\,I_5 + \mathcal{K}_\tau\,\big)\,f + \beta\,\mathcal{H}_\tau\,f = \chi \ \text{ on } \ \partial\Omega.$$

Denote the matrix operator generated by the left hand side expression of this equation by $\mathcal{P}_\tau$ and rewrite the system as

$$\mathcal{P}_\tau\,f = \chi \ \text{ on } \ \partial\Omega,$$

where

$$\mathcal{P}_\tau := -2^{-1}\,I_5 + \mathcal{K}_\tau + \beta\,\mathcal{H}_\tau. \tag{3.42}$$

Lemma 3.2 *Let* $\mathrm{Re}\,\tau=\sigma>0$. *The operators*

$$\mathcal{P}_\tau : \big[\,H^s_p(\partial\Omega)\,\big]^5 \to \big[\,H^s_p(\partial\Omega)\,\big]^5 \qquad \Big[\big[\,B^s_{p,t}(\partial\Omega)\,\big]^5 \to \big[\,B^s_{p,t}(\partial\Omega)\,\big]^5\Big], \tag{3.43}$$

are invertible for all $1<p<\infty$, $1\le t\le\infty$, *and* $s\in\mathbb{R}$.

As a consequence we have the following representation formula.

Lemma 3.3 *Let* $\mathrm{Re}\,\tau=\sigma>0$ *and* $1<p<\infty$. *An arbitrary solution* $U\in[\,W^1_p(\Omega)\,]^5$ *to the homogeneous equation* (3.40) *can be uniquely represented by the single layer potential* $U(x)=V_\tau\big(\mathcal{P}_\tau^{-1}\chi\big)(x)$, *where* $\chi=\{\mathcal{T}U\}^+ +\beta\,\{U\}^+\in\big[\,B^{-\frac{1}{p}}_{p,p}(\partial\Omega)\,\big]^5$.

Remark 3.4 *By standard arguments it can be shown that Lemmata* 3.1, 3.2 *and* 3.3 *with* $p=2$ *remain true for Lipschitz domains* $\Omega^{(m)}$ *and* Ω.

4 Existence and regularity results for Problem (ICP-A)

4.1 Reduction to boundary equations.

Let us return to the interface crack problem (2.20)-(2.29) and derive the equivalent boundary integral formulation of this problem. Keeping in mind (2.31), let

$$\begin{aligned}
&G:=\begin{cases} Q & \text{on } \ S_N,\\ \widetilde{Q} & \text{on } \ \Gamma^{(m)}_C,\end{cases} \qquad G^{(m)}:=\begin{cases} Q^{(m)} & \text{on } \ S^{(m)}_N,\\ \widetilde{Q}^{(m)} & \text{on } \ \Gamma^{(m)}_C,\end{cases}\\
&G\in[\,B^{-1/p}_{p,p}(S_N\cup\Gamma^{(m)}_C)\,]^5, \qquad G^{(m)}\in[\,B^{-1/p}_{p,p}(S^{(m)}_N\cup\Gamma^{(m)}_C)\,]^4,
\end{aligned} \tag{4.44}$$

and

$$G_0=(G_{01},\cdots,G_{05})^\top\in[\,B^{-1/p}_{p,p}(\partial\Omega)\,]^5, \quad G^{(m)}_0=(G^{(m)}_{01},\cdots,G^{(m)}_{04})^\top\in[\,B^{-1/p}_{p,p}(\partial\Omega^{(m)})]^4$$

be some fixed extensions of the vector-function G and $G^{(m)}$ respectively onto $\partial\Omega$ and $\partial\Omega^{(m)}$ preserving the space. It is evident that arbitrary extensions of the same vector functions can be represented then as

$$G^*=G_0+\psi+h, \quad G^{(m)*}=G^{(m)}_0+h^{(m)}, \tag{4.45}$$

where

$$\begin{aligned}
&\psi:=(\psi_1,\cdots,\psi_5)^\top\in\big[\,\widetilde{B}^{-1/p}_{p,p}(S_D)\,\big]^5, \ \ h:=(h_1,\cdots,h_5)^\top\in\big[\,\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)\,\big]^5,\\
&h^{(m)}:=(h^{(m)}_1,\cdots,h^{(m)}_4)^\top\in\big[\,\widetilde{B}^{-1/p}_{p,p}(\Gamma^{(m)}_T)\,\big]^4
\end{aligned} \tag{4.46}$$

are arbitrary vector-functions.

We develop here the so-called indirect boundary integral equations method. In accordance with Lemmas 3.1 and 3.3 we look for a solution pair $(U^{(m)},U)$ of the interface crack problem (2.20)-(2.29) in the form of single layer potentials,

$$U^{(m)}=(u^{(m)},\cdots,u^{(m)}_4)^\top=V^{(m)}_\tau\big(\big[\,\mathcal{P}^{(m)}_\tau\,\big]^{-1}\big[\,G^{(m)}_0+h^{(m)}\big]\big) \quad \text{in} \quad \Omega^{(m)}, \tag{4.47}$$

$$U=(u_1,\cdots,u_5)^\top=V_\tau\,\big(\mathcal{P}_\tau^{-1}\,\big[\,G_0+\psi+h\big]\big) \quad \text{in} \quad \Omega, \tag{4.48}$$

where $\mathcal{P}_\tau^{(m)}$ and $\mathcal{P}_\tau$ are given by (3.39) and (3.42), and $h^{(m)}$, h and ψ are unknown vector-functions satisfying the inclusions (4.46).

By Lemmas 3.1, 3.3 and the property (3.41), we see that the homogeneous differential equations (2.20)-(2.21), boundary conditions (2.22)-(2.23) and crack conditions (2.28)-(2.29) are satisfied automatically.

The remaining boundary and transmission conditions (2.24)-(2.27) lead to the equations

$$r_{S_D}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(G_0+\psi+h\big)\big]_k=f_k \text{ on } S_D,\quad k=\overline{1,5}, \tag{4.49}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(G_0+\psi+h\big)\big]_5=f_5^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{4.50}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(G_0+\psi+h\big)\big]_j-r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\big[G_0^{(m)}+h^{(m)}\big]\big]_j$$
$$=f_j^{(m)} \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{4.51}$$

$$r_{\Gamma_T^{(m)}}\big[G_0+\psi+h\big]_j+r_{\Gamma_T^{(m)}}\big[G_0^{(m)}+h^{(m)}\big]_j=F_j^{(m)} \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}. \tag{4.52}$$

After some evident simplification we arrive at the simultaneous pseudodifferential equations with respect to the unknown vector-functions ψ, h and $h^{(m)}$

$$r_{S_D}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(\psi+h\big)\big]_k=\widetilde{f}_k \text{ on } S_D,\quad k=\overline{1,5}, \tag{4.53}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(\psi+h\big)\big]_5=\widetilde{f}_5^{(m)} \text{ on } \Gamma_T^{(m)}, \tag{4.54}$$

$$r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\big(\psi+h\big)\big]_j-r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\,h^{(m)}\big]_j$$
$$=\widetilde{f}_j^{(m)} \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{4.55}$$

$$r_{\Gamma_T^{(m)}}\,h_j^{(m)}+r_{\Gamma_T^{(m)}}\,h_j=\widetilde{F}_j^{(m)} \text{ on } \Gamma_T^{(m)},\ j=\overline{1,4}. \tag{4.56}$$

where

$$\widetilde{f}_k:=f_k-r_{S_D}\,[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\,G_0]_k\in B_{p,p}^{1-1/p}(S_D),\quad k=\overline{1,5}, \tag{4.57}$$

$$\widetilde{f}_5^{(m)}:=f_5^{(m)}-r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\,G_0\big]_5\in B_{p,p}^{1-1/p}(\Gamma_T^{(m)}), \tag{4.58}$$

$$\widetilde{f}_j^{(m)}:=f_j^{(m)}+r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\,G_0^{(m)}\big]_j$$
$$-r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1}\,G_0\big]_j\in B_{p,p}^{1-1/p}(\Gamma_T^{(m)}),\quad j=\overline{1,4}, \tag{4.59}$$

$$\widetilde{F}_j^{(m)}:=F_j^{(m)}-r_{\Gamma_T^{(m)}}\,G_{0j}-r_{\Gamma_T^{(m)}}G_{0j}^{(m)}\in r_{\Gamma_T^{(m)}}\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)}),\quad j=\overline{1,4}. \tag{4.60}$$

The last inclusions are the *compatibility conditions* for Problem (ICP-A). Therefore, in what follows we assume that $\widetilde{F}_j^{(m)}$ are extended from $\Gamma_T^{(m)}$ to $\partial\Omega^{(m)}\cup\partial\Omega$ by zero, i.e., $\widetilde{F}_j^{(m)}\in\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)})$, $j=\overline{1,3}$.

Let us introduce the Steklov-Poincaré type 5×5 matrix pseudodifferential operators

$$\mathcal{A}_\tau:=\mathcal{H}_\tau\,\mathcal{P}_\tau^{-1},\qquad \mathcal{B}_\tau^{(m)}:=\begin{bmatrix}[\mathcal{H}_\tau^{(m)}\,[\mathcal{P}_\tau^{(m)}]^{-1}]_{4\times4} & [\,0\,]_{4\times1}\\ [\,0\,]_{1\times4} & [\,0\,]_{1\times1}\end{bmatrix}_{5\times5}, \tag{4.61}$$

and rewrite equations (4.53)-(4.56) as

$$r_{S_D}\,\mathcal{A}_\tau\,\big(\,\psi+h\big)=\widetilde{f}\ \text{on}\ S_D, \tag{4.62}$$

$$r_{\Gamma_T^{(m)}}\,\mathcal{A}_\tau\,\big(\,\psi+h\big)+r_{\Gamma_T^{(m)}}\,\mathcal{B}_\tau^{(m)}h=\widetilde{g}^{\,(m)}\ \text{on}\ \Gamma_T^{(m)}, \tag{4.63}$$

$$r_{\Gamma_T^{(m)}}\,h_j+r_{\Gamma_T^{(m)}}\,h_j^{(m)}=\widetilde{F}_j^{(m)}\ \text{on}\ \Gamma_T^{(m)},\ j=\overline{1,4}, \tag{4.64}$$

where

$$\widetilde{f}:=(\widetilde{f}_1,\cdots,\widetilde{f}_5)^\top\in[B_{p,p}^{1-1/p}(S_D)]^5, \tag{4.65}$$

$$\widetilde{g}^{\,(m)}:=(\widetilde{g}_1^{(m)},\cdots,\widetilde{g}_5^{(m)})^\top\in[B_{p,p}^{1-1/p}(\Gamma_T^{(m)})]^5, \tag{4.66}$$

$$\widetilde{g}_j^{(m)}:=\widetilde{f}_j^{(m)}+r_{\Gamma_T^{(m)}}\big[\mathcal{H}_\tau^{(m)}\,\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\,\widetilde{F}^{\,(m)}\,\big]_j,\ j=\overline{1,4},\quad \widetilde{g}_5^{(m)}=\widetilde{f}_5^{(m)}, \tag{4.67}$$

$$\widetilde{F}^{\,(m)}:=(\widetilde{F}_1^{(m)},\cdots,\widetilde{F}_4^{(m)})^\top\in[\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)})]^4. \tag{4.68}$$

We note here that since the unknown vector function h is supported on $\Gamma_T^{(m)}$, the operator $\mathcal{B}_\tau^{(m)}h$ is defined correctly provided h is extended by zero on $S_N^{(m)}\cup\Gamma_C^{(m)}$ (see Figure 1). For this extended vector function we will keep the same notation h. So, actually, in what follows we can assume that h is a vector function defined on $\partial\Omega\cup\partial\Omega^{(m)}$ and is supported on $\Gamma_T^{(m)}$.

It is easy to see that the simultaneous equations (4.49)-(4.52) and (4.62)-(4.64), where the right hand sides are related by the equalities (4.57)-(4.60) and (4.65)-(4.67), are equivalent in the following sense: if the triplet $(\psi,h,h^{(m)})\in[\widetilde{B}_{p,p}^{-1/p}(S_D)]^5\times[\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)})]^5\times[\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)})]^4$ solves the system (4.62)-(4.64), then the pair $(G_0+\psi+h,G_0^{(m)}+h^{(m)})$ solves the system (4.49)-(4.52), and vice versa.

4.2 Existence theorems and regularity of solutions.

Here we show that the system of pseudodifferential equations (4.62)-(4.64) is uniquely solvable in appropriate function spaces. To this end, let us put

$$\mathcal{N}_\tau^{(A)}:=\begin{bmatrix} r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,\mathcal{A}_\tau & r_{S_D}\,[\,0\,]_{5\times4}\\ r_{\Gamma_T^{(m)}}\,\mathcal{A}_\tau & r_{\Gamma_T^{(m)}}[\mathcal{A}_\tau+\mathcal{B}_\tau^{(m)}] & r_{\Gamma_T^{(m)}}\,[\,0\,]_{5\times4}\\ r_{\Gamma_T^{(m)}}\,[\,0\,]_{4\times5} & r_{\Gamma_T^{(m)}}\,I_{4\times5} & r_{\Gamma_T^{(m)}}\,I_4\end{bmatrix}_{14\times14}\qquad I_{4\times5}:=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}. \tag{4.69}$$

Further, let

$$\begin{aligned}&\Phi:=(\psi,\ h,\ h^{(m)})^\top,\quad Y:=(\widetilde{f},\ \widetilde{g}^{\,(m)},\ \widetilde{F}^{\,(m)})^\top,\\ &\mathbf{X}_p^s:=[\widetilde{B}_{p,p}^s(S_D)]^5\times[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^5\times[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^4,\\ &\mathbf{Y}_p^s:=[B_{p,p}^{s+1}(S_D)]^5\times[B_{p,p}^{s+1}(\Gamma_T^{(m)})]^5\times[\widetilde{B}_{p,p}^s(\Gamma_T^{(m)})]^4,\\ &\mathbf{X}_{p,q}^s:=[\widetilde{B}_{p,q}^s(S_D)]^5\times[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^5\times[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4,\\ &\mathbf{Y}_{p,q}^s:=[B_{p,q}^{s+1}(S_D)]^5\times[B_{p,q}^{s+1}(\Gamma_T^{(m)})]^5\times[\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4.\end{aligned}$$

Due to Theorems 7.3 and 7.4 in Appendix B, the operator $\mathcal{N}_\tau^{(A)}$ has the following mapping properties

$$\mathcal{N}_\tau^{(A)} \,:\, \mathbf{X}_p^s \to \mathbf{Y}_p^s \qquad \Big[\mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s\Big], \tag{4.70}$$

for all $s \in \mathbb{R}$, $1 < p < \infty$ and all $1 \le q \le \infty$.

Clearly, we can rewrite the system (4.62)-(4.64) as

$$\mathcal{N}_\tau^{(A)}\, \Phi = Y, \tag{4.71}$$

where $\Phi \in \mathbf{X}_p^s$ is the unknown vector introduced above and $Y \in \mathbf{Y}_p^s$ is a given vector.

As it will become clear later the operator (4.70) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau$ and $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ (see (4.61) and (4.75)). Note that the numbers γ' and γ'' define also Hölder smoothness exponents for the solutions to the original interface crack problem in a neighbourhood of the exceptional curves ∂S_D, $\partial\Gamma_C^{(m)}$ and $\partial\Gamma^{(m)}$.

Let

$$\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau)(x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau$ and $\lambda_j^{(1)}(x)$ ($j = \overline{1,5}$) be the eigenvalues of the matrix $\mathcal{D}_1(x) := [\mathfrak{S}_1(x,0,+1)]^{-1}\, \mathfrak{S}_1(x,0,-1)$ for $x \in \partial S_D$.

Similarly, let $\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x,\xi_1,\xi_2)$ be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ and $\lambda_j^{(2)}(x)$ ($j = \overline{1,5}$) be the eigenvalues of the corresponding matrix $\mathcal{D}_2(x) := [\mathfrak{S}_2(x,0,+1)]^{-1}\, \mathfrak{S}_2(x,0,-1)$ for $x \in \partial\Gamma_T^{(m)}$. Note that the curve $\partial\Gamma_T^{(m)}$ is the union of the curves where the interface intersects the exterior boundary, $\partial\Gamma^{(m)}$, and the crack edge, $\partial\Gamma_C^{(m)}$.

Further, we set

$$\gamma_1' := \inf_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi}\, \arg\lambda_j^{(1)}(x), \quad \gamma_1'' := \sup_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi}\, \arg\lambda_j^{(1)}(x), \tag{4.72}$$

$$\gamma_2' := \inf_{x\in\partial\Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi}\, \arg\lambda_j^{(2)}(x), \quad \gamma_2'' := \sup_{x\in\partial\Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi}\, \arg\lambda_j^{(2)}(x). \tag{4.73}$$

It can be shown that one of the eigenvalues equals to one, namely, $\lambda_5^{(1)} = 1$. Therefore

$$\gamma_1' \le 0, \qquad \gamma_1'' \ge 0. \tag{4.74}$$

Note that γ_j' and γ_j'' ($j = 1,2$) depend on the material parameters, in general, and belong to the interval $(-\frac{1}{2},\, \frac{1}{2})$. We put

$$\gamma' := \min\{\gamma_1',\, \gamma_2'\}, \qquad \gamma'' := \max\{\gamma_1'',\, \gamma_2''\}. \tag{4.75}$$

In view of (4.74) we have

$$-\frac{1}{2} < \gamma' \le 0 \le \gamma'' < \frac{1}{2}\,. \tag{4.76}$$

Theorem 4.1 *Let the conditions*

$$1 < p < \infty, \quad 1 \leq q \leq \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{4.77}$$

be satisfied with γ' and γ'' given by (4.72), (4.73)*, and* (4.75)*. Then the operators in* (4.70) *are invertible.*

Now we formulate the basic existence and uniqueness results for the interface crack problem under consideration.

Theorem 4.2 *Let the inclusions* (2.30) *and compatibility conditions* (4.60) *hold and let*

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'} \tag{4.78}$$

with γ' and γ'' defined in (4.75)*. Then the interface crack problem* (2.20)-(2.29) *has a unique solution $(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5$, which can be represented by formulas*

$$U^{(m)} = V_\tau^{(m)}\big(\big[\mathcal{P}_\tau^{(m)} \big]^{-1} \big[G_0^{(m)} + h^{(m)} \big] \big) \quad \text{in} \quad \Omega^{(m)}, \tag{4.79}$$

$$U = V_\tau\big(\mathcal{P}_\tau^{-1} \big[G_0 + \psi + h \big] \big) \quad \text{in} \quad \Omega, \tag{4.80}$$

where the densities ψ, h, and $h^{(m)}$ are to be determined from the system (4.53)-(4.56) $($*or from the system* (4.62)-(4.64)$)$*. Moreover, the vector functions $G_0 + \psi + h$ and $G_0^{(m)} + h^{(m)}$ are defined uniquely by the above systems and are independent of the extension operators.*

Remark 4.3 *Theorems* 4.1 *and* 4.2 *remain valid with $p = 2$ and $s = -1/2$ for Lipschitz domains $\Omega^{(m)}$ and Ω. Indeed, one can easily verify that the arguments, applied in the first two steps of the proof of Theorem* 4.1 *and in the proof of Theorem* 4.2*, hold true in the case of Lipschitz domains.*

Finally, we formulate the following regularity result for the solution of Problem (ICP).

Theorem 4.4 *Let the inclusions* (2.30) *and compatibility conditions* (4.60) *hold. Further, let $U^{(m)} \in [W_p^1(\Omega^{(m)})]^4$ and $U \in [W_p^1(\Omega)]^5$ be a unique solution pair to the interface crack problem* (2.20)-(2.29)*. If $\alpha > 0$ is not integer, $\kappa = \min\{\alpha, \ \gamma' + \frac{1}{2}\} > 0$ and*

$$\begin{aligned} &Q_k \in B_{\infty,\infty}^{\alpha-1}(S_N), \ Q_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(m)}), \ f_k \in C^\alpha(\overline{S_D}), \ f_k^{(m)} \in C^\alpha(\overline{\Gamma_T^{(m)}}), \\ &F_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_T^{(m)}), \ \widetilde{Q}_j^{(m)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C^{(m)}), \ \widetilde{Q}_k \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C^{(m)}), \ k = \overline{1,5}, \ j = \overline{1,4}, \end{aligned} \tag{4.81}$$

and the compatibility conditions

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{\Gamma_T^{(m)}} G_{0j} - r_{\Gamma_T^{(m)}} G_{0j}^{(m)} \in r_{\Gamma_T^{(m)}} \widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T^{(m)}), \ j = \overline{1,4},$$

are satisfied, then $U^{(m)} \in \bigcap\limits_{\alpha' < \kappa} [\, C^{\alpha'}(\, \overline{\Omega^{(m)}}\,)]^4, \qquad U \in \bigcap\limits_{\alpha' < \kappa} [\, C^{\alpha'}(\, \overline{\Omega}\,)\,]^5.$

4.3 Numerical results for stress singularity exponents.

On the basis of the asymptotic expansions of solutions (see [4], [5]) we can shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the principal dominant singular terms of the solution vectors $U^{(m)}$ and U near the exceptional curves ∂S_D and $\partial\Gamma_T^{(m)}$ can be represented as a product of a "good" vector-function and a singular factor of the form $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{\gamma_k+i\,\delta_k}$. Note that the crack edge $\partial\Gamma_C^{(m)}$ is a proper connected part of the curve $\partial\Gamma_T^{(m)}$. Here $\varrho(x)$ is the distance from a reference point x to the exceptional curves. Therefore, near these curves the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(m)}u^{(m)}$ and $\mathcal{T}U$ are represented as a product of a "good" vector-function and the factor $[\ln\varrho(x)]^{m_k-1}[\varrho(x)]^{-1+\gamma_k+i\,\delta_k}$. The numbers δ_k are different from zero, in general, and display the oscillating character of the stress singularities.

The exponents $\gamma_k + i\,\delta_k$ and the eigenvalues of the corresponding matrices are related by the equalities

$$\gamma_k = \frac{1}{2} + \frac{\arg\lambda_k}{2\pi}, \quad \delta_k = -\frac{\ln|\lambda_k|}{2\pi}.$$

Here $\lambda_k \in \{\lambda_1^{(1)}(x),\cdots,\lambda_5^{(1)}(x)\}$ for $x\in\partial S_D$, and $\lambda_k \in \{\lambda_1^{(2)}(x),\cdots,\lambda_5^{(2)}(x)\}$ for $x\in\partial\Gamma_T^{(m)}$. In the above expressions the parameter m_k denotes the multiplicity of the eigenvalue λ_k. It is evident that at the exceptional curves the components of the generalized stress vector behave like $\mathcal{O}\big([\ln\varrho(x)]^{m_0-1}[\varrho(x)]^{-\frac{1}{2}+\gamma'}\big)$, where m_0 denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(m)}$ and U. Note that γ_k, δ_k and γ' depend on the material parameters (see (4.72)-(4.75)). Moreover, γ' is non-positive and $\delta_k\neq 0$, in general. This is related to the fact that the eigenvalues λ_k are complex and $|\lambda_k|\neq 1$, in general (see Appendix B).

For numerical calculations, we have considered particular cases when the domain $\Omega^{(m)}$ is occupied by the isotropic metallic material *silver-palladium alloy* whereas the domain Ω is occupied by one of the following piezoelectric materials $BaTiO_3$ (with the crystal symmetry of the class **4mm**), PZT-4 and PZT-5A (with the crystal symmetry of the class **6mm**). Calculations have shown that the parameters γ_k' and γ_k'' depend on the material parameters. In particular, $\gamma_k' = -\gamma_k''$ and we have the following values for them

	$BaTiO_3$	PZT-4	PZT-5A
γ_1'	-0.12	-0.12	-0.13
γ_2'	-0.06	-0.08	-0.09.

(4.82)

Therefore, for $\gamma' := \min\{\gamma_1', \gamma_2'\}$ we have (see (4.72)-(4.75))

	$BaTiO_3$	PZT-4	PZT-5A
γ'	-0.12	-0.12	-0.13.

Consequently, if the boundary data of the transmission problem under consideration are sufficiently smooth (e.g., satisfy the conditions of Theorem 4.4.iii with $\alpha > 0.5$), then for the Hölder smoothness exponent κ, involved in Theorem 4.4.iii, we derive

	$\mathrm{BaTiO_3}$	PZT-4	PZT-5A
κ	0.38	0.38	0.37 .

Thus, in the closed domains the solution vectors have $C^{\kappa-\delta}$-smoothness , where $\delta > 0$ is an arbitrarily small number. This shows that the Hölder smoothness exponents depend on the material parameters. Moreover, for these particular cases, from the table (4.82) it follows that $\gamma_1' < \gamma_2'$, which yields that the stress singularities at the curve ∂S_D are higher than the singularities near the curve $\partial\Gamma_T^{(m)}$.

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