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ON A TWO–POINT BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS *

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Abstract. The effective sufficient conditions are found for the existence and uniqueness of a solution of the problem

$$u''(t) = \ell(u)(t) + q(t), \qquad u(a) = c_1, \quad u(b) = c_2,$$

where $\ell : C([a,b]; R) \to L([a,b]; R)$ is a monotone linear bounded operator, $q \in L([a,b]; R)$, and $c_1, c_2 \in R$.

Key Words. Functional differential equation, monotone operator, boundary value problem, unique solvability

AMS(MOS) subject classification. 34K06, 34K10

1. Statement of the Problem and the Main Notation. On the interval $[a, b]$, we consider the functional differential equation

$$u''(t) = \ell(u)(t) + q(t) \tag{1}$$

with the boundary conditions

$$u(a) = c_1, \qquad u(b) = c_2, \tag{2}$$

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where $\ell : C([a,b];R) \to L([a,b];R)$ is a linear bounded operator, $q \in L([a,b];R)$, and $c_1, c_2 \in R$.

By a solution of the equation (1) we understand an absolutely continuous function $u : [a,b] \to R$ satisfying the equality (1) almost everywhere on the interval $[a,b]$. A solution of the equation (1) satisfying (2) is said to be a solution of the problem (1), (2).

As for ordinary differential equations, the problem on the solvability of the Dirichlet problem is studied in detail (see, e.g., [2]), while this problem for functional differential equations is still not sufficiently investigated. It should be mentioned that the Dirichlet problem for ordinary and functional differential equations has been studied, among others, in [6, 7, 8, 10, 11, 5, 9].

In the present paper, we consider the problem on the unique solvability of (1), (2). The results obtained, on one hand generalize well-known results of Lyapunov (see, e.g., [2, p. 346]) and, on the other hand, describe some properties belonging only to the functional differential equations.

It follows from [10, Theorem 1.1.2$_1$] and [8, Theorem 1.4] that the problem (1), (2) is uniquely solvable if

$$\int_a^b \ell(1)(s)ds \leq \frac{16}{b-a} \tag{3}$$

provided that the operator ℓ is nondecreasing. There is also shown that the condition (3) is (in general) nonimprovable. In this paper, there is proved that if the nondecreasing operator ℓ is concentrated on the set $A \subseteq [a,b]$ (see Definition 1 below), then the constant 16 on the right hand side of the inequality (3) can be replaced by a constant $d \geq 16$ depending only on the set A. Further, a similar result is established for nonincreasing operators.

The main results are concretized for the equation with a deviating argument

$$u''(t) = p(t)u(\tau(t)) + q(t), \tag{4}$$

where $p, q \in L([a,b];R)$ and $\tau : [a,b] \to [a,b]$ is a measurable function. The optimality of the results obtained is verified by counter–examples.

If ℓ is a nondecreasing operator concentrated on the set $A = [\alpha, \beta]$ (or $A = [a, \alpha] \cup [\beta, b]$), the results presented in this paper correspond to the results in [11].

The following notation is used throughout the paper:

N is the set of all natural numbers.
R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm

$$\|u\|_C = \max\{|u(t)| : t \in [a,b]\}.$$

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}'([a,b];R)$ is the set of functions $u : [a,b] \to R$, which are absolutely continuous together with their first derivatives.

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)|ds.$$

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \geq 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];R) \to L([a,b];R)$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

$\bar{A}$ is the closure of the set A.

If $x \in R$, then

$$[x]_+ = \frac{1}{2}(|x| + x), \qquad [x]_- = \frac{1}{2}(|x| - x).$$

DEFINITION 1. *Let $A \subseteq [a,b]$ be a nonempty set. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $K_{ab}(A)$ if*

$$\ell(v)(t) = 0 \quad \textit{for almost all} \quad t \in [a,b],$$

whenever $v \in C([a,b];R)$ is such that

$$v(t) = 0 \quad \textit{for} \qquad t \in A.$$

We say in this case that the operator ℓ is concentrated on the set A.

Note also that throughout the paper the equalities and inequalities between the integrable functions are understood to hold almost everywhere.

2. Formulation of the Main Results. Let $A \subseteq [a,b]$ be a nonempty set. Put

(5) $$\rho_A(t) = \inf\{|t-s| : s \in A\} \quad \text{for} \quad t \in [a,b]$$

and

$$\sigma_A(t) = \rho_A(t) + \rho_A\Big(t + \frac{b-a}{2}\Big) \quad \text{for} \quad t \in \Big[a, \frac{a+b}{2}\Big]. \tag{6}$$

It is not difficult to verify that the function σ_A is continuous on the interval $[a, \frac{a+b}{2}]$ (see Lemma 1 below). It allows us to denote

$$\delta = \min\Big\{\sigma_A(t) : a \le t \le \frac{a+b}{2}\Big\}. \tag{7}$$

It is clear that

$$\delta \le \frac{b-a}{2}.$$

Further, put

$$\gamma = \rho_A\Big(\frac{a+b}{2}\Big). \tag{8}$$

THEOREM 1. *Let $A \subseteq [a,b]$ be a nonempty set and let $\ell \in K_{ab}(A) \cap P_{ab}$ be such that*

$$\bigg(1 - 4\Big[\frac{\delta}{b-a}\Big]^2\bigg) \int_a^b \ell(1)(s)ds \le \frac{16}{b-a}, \tag{9}$$

where δ is given by (7). Then the problem (1), (2) has a unique solution.

REMARK 1. *If the set $A \subseteq [a,b]$ is such that $\delta = \frac{b-a}{2}$, then the condition (9) holds for any $\ell \in K_{ab}(A) \cap P_{ab}$.*

On the other hand, if $\delta < \frac{b-a}{2}$, then the condition (9) in Theorem 1 cannot be replaced by the condition

$$\bigg(1 - 4\Big[\frac{\delta}{b-a}\Big]^2\bigg) \int_a^b \ell(1)(s)ds \le \frac{16}{b-a} + \varepsilon, \tag{10}$$

no matter how small $\varepsilon > 0$ would be (see Example 1).

THEOREM 2. *Let $A \subseteq [a,b]$ be a nonempty set and let $-\ell \in K_{ab}(A) \cap P_{ab}$ be such that*

$$\bigg(1 - 4\Big[\frac{\gamma}{b-a}\Big]^2\bigg) \int_a^b |\ell(1)(s)|ds \le \frac{4}{b-a}, \tag{11}$$

where γ is given by (8). *Then the problem* (1), (2) *has a unique solution.*

REMARK 2. *If the set $A \subseteq [a,b]$ is such that $\gamma = \frac{b-a}{2}$ (i.e., $A \subseteq \{a,b\}$), then the condition* (11) *holds for any $-\ell \in K_{ab}(A) \cap P_{ab}$.*

On the other hand, if $\gamma < \frac{b-a}{2}$, then the condition (11) *in Theorem 2 cannot be replaced by the condition*

$$(12)\qquad \left(1 - 4\left[\frac{\gamma}{b-a}\right]^2\right) \int_a^b |\ell(1)(s)|ds \le \frac{4}{b-a} + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see Example 2).

Now we will formulate two consenquences of Theorems 1 and 2 for the equation (4).

COROLLARY 1. *Let $p \in L([a,b]; R_+)$ and let at least one of the following items be fulfilled:*

a) there exists a nonempty set $A \subseteq [a,b]$ such that the condition

$$(13)\qquad p(t) = 0 \quad \text{if} \quad \tau(t) \notin A$$

holds for $t \in [a,b]$ and

$$(14)\qquad \left(1 - 4\left[\frac{\delta}{b-a}\right]^2\right) \int_a^b p(s)ds \le \frac{16}{b-a},$$

where δ is given by (7);

b) there exist $\alpha \in [a,b]$ and $\beta \in [\alpha,b]$ such that

$$(15)\qquad \tau(t) \in [\alpha,\beta] \quad \text{for} \quad t \in [a,b]$$

and

$$(16)\qquad \left(1 - 4\left[\frac{1}{2} - \frac{\beta-\alpha}{b-a}\right]_+^2\right) \int_a^b p(s)ds \le \frac{16}{b-a};$$

c) there exist $\alpha \in [a,b]$ and $\beta \in [\alpha,b]$ such that

$$(17)\qquad \tau(t) \in [a,\alpha] \cup [\beta,b] \quad \text{for} \quad t \in [a,b]$$

and

$$(18)\qquad \left(1 - 4\left[\frac{1}{2} - \frac{\beta-\alpha}{b-a}\right]_-^2\right) \int_a^b p(s)ds \le \frac{16}{b-a}.$$

Then the problem (4), (2) *has a unique solution.*

COROLLARY 2. *Let* $-p \in L([a,b];R_+)$ *and let at least one of the following items be fulfilled:*

a) there exists a nonempty set $A \subseteq [a,b]$ *such that* (13) *holds for* $t \in [a,b]$ *and*

$$\left(1 - 4\left[\frac{\gamma}{b-a}\right]^2\right) \int_a^b |p(s)|ds \leq \frac{4}{b-a}, \tag{19}$$

where γ *is given by* (8)*;*

b) there exist $\alpha \in [a,b]$ *and* $\beta \in [\alpha,b]$ *such that* (15) *holds and*

$$\gamma_0 \int_a^b |p(s)|ds \leq \frac{4}{b-a}, \tag{20}$$

where

$$\gamma_0 = 1 - 4\max\left\{\left[\frac{1}{2} - \frac{\beta-a}{b-a}\right]_+^2, \left[\frac{\alpha-a}{b-a} - \frac{1}{2}\right]_+^2\right\}; \tag{21}$$

c) there exist $\alpha \in [a,b]$ *and* $\beta \in [\alpha,b]$ *such that* (17) *and* (20) *hold, where*

$$\gamma_0 = 1 - 4\min\left\{\left[\frac{1}{2} - \frac{\beta-a}{b-a}\right]_-^2, \left[\frac{\alpha-a}{b-a} - \frac{1}{2}\right]_-^2\right\}. \tag{22}$$

Then the problem (4), (2) *has a unique solution.*

Finally, we consider the ordinary differential equation

$$u''(t) = p(t)u(t) + q(t), \tag{23}$$

where $p, q \in L([a,b];R)$. If $p(t) \geq 0$ for $t \in [a,b]$, then it is clear that the problem (23), (2) is uniquely solvable. In the opposite case, the following statement is true.

COROLLARY 3. *Let* $-p \in L([a,b];R_+)$ *and let at least one of the following items be fulfilled:*

a) there exists a nonempty set $A \subseteq [a,b]$ *such that*

$$p(t) = 0 \quad \textit{for} \quad t \in [a,b] \setminus A \tag{24}$$

and (19) *is satisfied, where* γ *is given by* (8)*;*

b) there exist $\alpha \in [a,b]$ *and* $\beta \in [\alpha,b]$ *such that*

$$p(t) = 0 \quad \textit{for} \quad t \in [a,\alpha] \cup [\beta,b], \tag{25}$$

and (20) *is satisfied, where* γ_0 *is defined by* (21).

Then the problem (23), (2) *has a unique solution.*

3. Auxiliary Lemmas. To prove main results we need the following auxiliary assertions.

LEMMA 1. *Let $A \subseteq [a,b]$ be a nonempty set. Then the function ρ_A, defined by* (5), *is continuous on the interval* $[a,b]$ *and, moreover,*

$$\rho_A(t) = \rho_{\bar{A}}(t) \quad \textit{for} \quad t \in [a,b]. \tag{26}$$

Proof. For any $t_1, t_2, s \in [a,b]$ we have $||t_2 - s| - |t_1 - s|| \le |t_2 - t_1|$. Therefore, by virtue of (5), we get

$$\rho_A(t_i) \le |t_i - s| \le |t_2 - t_1| + |t_{3-i} - s| \quad \text{for} \quad s \in A \qquad (i = 1,2).$$

Consequently,

$$\rho_A(t_i) - |t_2 - t_1| \le \rho_A(t_{3-i}) \qquad (i = 1,2),$$

whence,

$$|\rho_A(t_2) - \rho_A(t_1)| \le |t_2 - t_1|,$$

i.e., the function ρ_A is continuous on the interval $[a,b]$.

Now we will show that (26) is satisfied. Since $A \subseteq \bar{A}$, it is clear that

$$\rho_A(t) \ge \rho_{\bar{A}}(t) \quad \text{for} \quad t \in [a,b]. \tag{27}$$

Let $t_0 \in [a,b]$ be an arbitrary point. Then

$$\rho_A(t_0) \le |t_0 - s| \quad \text{for} \quad s \in A.$$

For any $s_0 \in \bar{A}$ there exist $s_n \in A$ $(n \in N)$ such that $\lim\limits_{n \to +\infty} s_n = s_0$. Therefore,

$$\rho_A(t_0) \le \lim_{n \to +\infty} |t_0 - s_n| = |t_0 - s_0|.$$

Consequently,

$$\rho_A(t_0) \le \rho_{\bar{A}}(t_0),$$

which, according to the arbitrariness of t_0, guarantees

$$\rho_A(t) \le \rho_{\bar{A}}(t) \quad \text{for} \quad t \in [a,b].$$

The last relation, together with (27), implies the equality (26). □

LEMMA 2. *Let $i \in \{0,1\}$, $A \subseteq [a,b]$ be a nonempty set, and let $\ell \in \mathcal{L}_{ab}$ be such that $(-1)^i \ell \in P_{ab}$. If $\ell \in K_{ab}(A)$, then the estimate*

$$\begin{aligned}(28)\qquad \min\{v(s) : s \in \bar{A}\}|\ell(1)(t)| \leq (-1)^i \ell(v)(t) \leq \\ \leq \max\{v(s) : s \in \bar{A}\}|\ell(1)(t)| \qquad \textit{for} \quad t \in [a,b]\end{aligned}$$

holds for every $v \in C([a,b];R)$.

Proof. Let $v \in C([a,b];R)$ be an arbitrary function. Put $a_0 = \inf A$, $b_0 = \sup A$,

$$\mu(t) = \min\{s \in \bar{A} : t \leq s\}, \quad \nu(t) = \max\{s \in \bar{A} : t \geq s\} \quad \text{for} \quad t \in [a_0, b_0],$$

and

$$v_0(t) = \begin{cases} v(a_0) & \text{for} \quad t \in [a, a_0[\\ v(t) & \text{for} \quad t \in \bar{A} \\ \frac{v(\mu(t)) - v(\nu(t))}{\mu(t) - \nu(t)}(t - \nu(t)) + v(\nu(t)) & \text{for} \quad t \in [a_0, b_0] \setminus \bar{A} \\ v(b_0) & \text{for} \quad t \in]b_0, b] \end{cases}$$

Clearly, $v_0 \in C([a,b];R)$,

$$(29)\qquad \min\{v(s) : s \in \bar{A}\} \leq v_0(t) \leq \max\{v(s) : s \in \bar{A}\} \quad \text{for} \quad t \in [a,b],$$

and

$$(30)\qquad v_0(t) = v(t) \quad \text{for} \quad t \in A.$$

It follows from (29) and the assumption $(-1)^i \ell \in P_{ab}$ that

$$\begin{aligned}(31)\qquad \min\{v(s) : s \in \bar{A}\}|\ell(1)(t)| \leq (-1)^i \ell(v_0)(t) \leq \\ \leq \max\{v(s) : s \in \bar{A}\}|\ell(1)(t)| \qquad \text{for} \quad t \in [a,b].\end{aligned}$$

On the other hand, by virtue of (30) and the assumption $\ell \in K_{ab}(A)$, we get

$$(32)\qquad \ell(v_0)(t) = \ell(v)(t) \quad \text{for} \quad t \in [a,b].$$

Consequently, (31) and (32) guarantee the estimate (28). □

LEMMA 3. *Let $A \subseteq [a,b]$ be a nonempty set and let $c \in]a,b[$ be such that*

$$(33)\qquad \bar{A} \cap [a,c] \neq \emptyset, \qquad \bar{A} \cap [c,b] \neq \emptyset.$$

Then the estimate

$$\left(\frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)}\right)^{\frac{1}{2}} \le \frac{b-a}{8} - \frac{\delta^2}{2(b-a)} \tag{34}$$

holds for $t_1 \in A_c$ and $t_2 \in B_c$, where δ is given by (7) *and*

$$A_c = \bar{A} \cap [a,c], \qquad B_c = \bar{A} \cap [c,b]. \tag{35}$$

Proof. Put

$$\sigma_1 = \rho_A\left(\frac{a+c}{2}\right), \qquad \sigma_2 = \rho_A\left(\frac{c+b}{2}\right). \tag{36}$$

It is clear that

$$\sigma_1 + \sigma_2 \ge \delta, \tag{37}$$

where δ is given by (7). Obviously, either

$$\max\{\sigma_1, \sigma_2\} \ge \delta \tag{38}$$

or

$$\max\{\sigma_1, \sigma_2\} < \delta. \tag{39}$$

By virtue of (33), (35), and (36), it is not difficult to verify that

$$\max\{(c-t_1)(t_1-a) : t_1 \in A_c\} = (c-t_1^*)(t_1^*-a)$$

and

$$\max\{(b-t_2)(t_2-c) : t_2 \in B_c\} = (b-t_2^*)(t_2^*-c),$$

where

$$t_1^* = \frac{a+c}{2} - \sigma_1, \qquad t_2^* = \frac{c+b}{2} - \sigma_2.$$

Therefore, in view of the inequality $4xy \le (x+y)^2$, we get

$$\begin{aligned} &\left(\frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)}\right)^{\frac{1}{2}} \le \\ &\le \left(\frac{c-a}{4} - \frac{\sigma_1^2}{c-a}\right)^{\frac{1}{2}} \left(\frac{b-c}{4} - \frac{\sigma_2^2}{b-c}\right)^{\frac{1}{2}} \le \\ &\le \frac{1}{2}\left(\frac{b-a}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c}\right) \quad \text{for} \quad t_1 \in A_c,\ t_2 \in B_c. \end{aligned} \tag{40}$$

First suppose that (38) holds. Obviously,

$$\frac{\sigma_1^2}{c-a}+\frac{\sigma_2^2}{b-c}\geq\frac{(\max\{\sigma_1,\sigma_2\})^2}{b-a}\geq\frac{\delta^2}{b-a},$$

which, together with (40), guarantees the estimate (34) for $t_1\in A_c$ and $t_2\in B_c$.

Now suppose that (39) is satisfied. By virtue of (36) and Lemma 1, there exist $\alpha,\beta\in\bar{A}$ such that

$$\sigma_1=\left|\frac{a+c}{2}-\alpha\right|,\qquad\sigma_2=\left|\frac{c+b}{2}-\beta\right|. \tag{41}$$

Further, it is clear that

$$\frac{b-a}{4}-\frac{\sigma_1^2}{c-a}-\frac{\sigma_2^2}{b-c}=b-a-(\beta-\alpha)-\eta(c), \tag{42}$$

where

$$\eta(t)=\frac{(\alpha-a)^2}{t-a}+\frac{(b-\beta)^2}{b-t}\quad\text{for}\quad t\in]a,b[\,.$$

It is easy to verify that the function η achieves its minimum at the point

$$t_0=\frac{(\alpha-a)b+(b-\beta)a}{b-a-(\beta-\alpha)}.$$

Hence, (42) yields

$$\frac{b-a}{4}-\frac{\sigma_1^2}{c-a}-\frac{\sigma_2^2}{b-c}\leq(b-a-(\beta-\alpha))\frac{\beta-\alpha}{b-a}. \tag{43}$$

On the other hand, put

$$\sigma=\min\{\sigma_1,\sigma_2\}. \tag{44}$$

It is clear from (41) that either

$$\alpha\leq\frac{a+c}{2}-\sigma \tag{45}$$

or

$$\alpha\geq\frac{a+c}{2}+\sigma, \tag{46}$$

and either

$$\beta \geq \frac{c+b}{2} + \sigma \tag{47}$$

or

$$\beta \leq \frac{c+b}{2} - \sigma. \tag{48}$$

First suppose that (45) holds. We will show that in this case the inequality (47) is satisfied. Indeed, if (48) is fulfilled, then from (41), (45), and (48) we get

$$\rho_A\Big(\frac{a+c}{2} - \sigma\Big) \leq \frac{a+c}{2} - \sigma - \alpha = \sigma_1 - \sigma,$$
$$\rho_A\Big(\frac{c+b}{2} - \sigma\Big) \leq \frac{c+b}{2} - \sigma - \beta = \sigma_2 - \sigma.$$

These inequalities, in view of (44), imply

$$\sigma_A\Big(\frac{a+c}{2} - \sigma\Big) \leq (\sigma_1 - \sigma) + (\sigma_2 - \sigma) = \max\{\sigma_1, \sigma_2\} - \sigma,$$

which, by virtue of (39), contradicts (7). The contradiction obtained proves the validity of (47). Consequently, from (41), (45), and (47) we obtain

$$\sigma_1 = \frac{a+c}{2} - \alpha, \qquad \sigma_2 = \beta - \frac{c+b}{2}.$$

Hence,

$$(b - a - (\beta - \alpha))\frac{\beta - \alpha}{b - a} = \frac{b-a}{4} - \frac{(\sigma_1 + \sigma_2)^2}{b-a}. \tag{49}$$

Now suppose that (46) holds. It can be proved in a similar manner as above that in this case the inequality (48) is satisfied. Consequently, from (41), (46), and (48) we obtain

$$\sigma_1 = \alpha - \frac{a+c}{2}, \qquad \sigma_2 = \frac{c+b}{2} - \beta,$$

and thus the equality (49) holds.

We have proved that, in both cases (45) and (46), the equality (49) is satisfied. However, in view of (37), the equality (49) yields

$$(b - a - (\beta - \alpha))\frac{\beta - \alpha}{b - a} \leq \frac{b-a}{4} - \frac{\delta^2}{b-a},$$

which, together with (40) and (43), guarantees the estimate (34) for $t_1 \in A_c$ and $t_2 \in B_c$. $\square$

LEMMA 4. *Let $A \subseteq [a, b]$ be a nonempty set. Then the estimate*

$$\frac{(b-t)(t-a)}{b-a} \le \frac{b-a}{4} - \frac{\gamma^2}{b-a} \quad \textit{for} \quad t \in \bar{A} \tag{50}$$

holds, where γ is given by (8).

Proof. According to (8), it is clear that

$$\max\{(t-a)(b-t) : t \in \bar{A}\} = (b-t^*)(t^*-a),$$

where

$$t^* = \frac{a+b}{2} - \gamma.$$

Therefore, the estimate (50) is valid. $\square$

4. Proofs of the Main Results. Along with the problem (1), (2), we consider the corresponding homogeneous problem

$$u''(t) = \ell(u)(t), \tag{51}$$

$$u(a) = 0, \quad u(b) = 0. \tag{52}$$

The following result is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [1, 3, 12, 4]).

PROPOSITION 1. *The problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem* (51), (52) *has only the trivial solution.*

Proof of Theorem 1. According to Proposition 1, it is sufficient to show that the homogeneous problem (51), (52) has only the trivial solution. Assume the contrary that the problem (51), (52) has a nontrivial solution u.

First suppose that u does not change its sign on the set $\bar{A}$. Then there exists $\sigma \in \{-1, 1\}$ such that

$$\sigma u(t) \ge 0 \quad \text{for} \quad t \in \bar{A}.$$

By virtue of Lemma 2 (with $i = 0$), the equation (51) yields

$$\sigma u''(t) = \ell(\sigma u)(t) \ge 0 \quad \text{for} \quad t \in [a, b], \tag{53}$$

which, together with (52), guarantees $\sigma u(t) \le 0$ for $t \in [a, b]$. Therefore, the assumption $\ell \in P_{ab}$ implies

$$\ell(\sigma u)(t) \le 0 \quad \text{for} \quad t \in [a, b].$$

Consequently, from (53) we get

$$u''(t) = 0 \quad \text{for} \quad t \in [a, b]. \tag{54}$$

However, u satisfies (52) and thus, (54) yields $u \equiv 0$, a contradiction.

Now suppose that u changes its sign on the set $\bar{A}$. Then there exist $t_1, t_2 \in \bar{A}$ such that

$$u(t_1) = \min\{u(s) : s \in \bar{A}\}, \qquad u(t_2) = \max\{u(s) : s \in \bar{A}\}. \tag{55}$$

Obviously,

$$u(t_1) < 0, \qquad u(t_2) > 0, \tag{56}$$

and without loss of generality we can assume that $t_1 < t_2$. Consequently, there exists $c \in]t_1, t_2[$ such that

$$u(c) = 0. \tag{57}$$

It is not difficult to verify that, in view of (51), (52), and (57), the function u satisfies

$$u(t) = -\frac{c-t}{c-a}\int_a^t (s-a)\ell(u)(s)ds - \frac{t-a}{c-a}\int_t^c (c-s)\ell(u)(s)ds \quad \text{for} \quad t \in [a, c], \tag{58}$$

$$u(t) = -\frac{b-t}{b-c}\int_c^t (s-c)\ell(u)(s)ds - \frac{t-c}{b-c}\int_t^b (b-s)\ell(u)(s)ds \quad \text{for} \quad t \in [c, b]. \tag{59}$$

By virtue of (55), (56), and Lemma 2 (with $i = 0$), from (58) and (59) we get

$$0 < \frac{|u(t_1)|}{u(t_2)} \le \frac{c-t_1}{c-a}\int_a^{t_1} (s-a)\ell(1)(s)ds + \frac{t_1-a}{c-a}\int_{t_1}^c (c-s)\ell(1)(s)ds,$$

$$0 < \frac{u(t_2)}{|u(t_1)|} \le \frac{b-t_2}{b-c}\int_c^{t_2} (s-c)\ell(1)(s)ds + \frac{t_2-c}{b-c}\int_{t_2}^b (b-s)\ell(1)(s)ds.$$

These conditions, in view of the inequalities $a < t_1 < c < t_2 < b$, result in

$$\frac{|u(t_1)|}{u(t_2)} < \frac{(c-t_1)(t_1-a)}{c-a} \int_a^c \ell(1)(s)ds, \tag{60}$$

$$\frac{u(t_2)}{|u(t_1)|} < \frac{(b-t_2)(t_2-c)}{b-c} \int_c^b \ell(1)(s)ds. \tag{61}$$

Therefore, by virtue of the inequality $4xy \le (x+y)^2$, from (60) and (61) we obtain

$$1 < \frac{1}{2}\left(\frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)}\right)^{\frac{1}{2}} \int_a^b \ell(1)(s)ds. \tag{62}$$

On the other hand, according to Lemma 3, the inequality (62) implies

$$1 < \left(\frac{b-a}{16} - \frac{\delta^2}{4(b-a)}\right) \int_a^b \ell(1)(s)ds,$$

where δ is given by (7). However, it contradicts (9). □

Proof of Theorem 2. According to Proposition 1, it is sufficient to show that the homogeneous problem (51), (52) has only the trivial solution. Assume the contrary that the problem (51), (52) has a nontrivial solution u.

If $u(t) = 0$ for $t \in \bar{A}$, then, according to Lemma 2 (with $i = 1$), we get $\ell(u)(t) = 0$ for $t \in [a, b]$. Consequently, (51) implies (54). However, u satisfies (52) and thus, (54) yields $u \equiv 0$, a contradiction.

Therefore,

$$\max\{|u(s)| : s \in \bar{A}\} > 0.$$

Obviously, there exists $t_0 \in \bar{A}$ such that

$$|u(t_0)| = \max\{|u(s)| : s \in \bar{A}\} \tag{63}$$

Without loss of generality we can assume that

$$u(t_0) > 0. \tag{64}$$

It is not difficult to verify that, in view of (51) and (52), the function u satisfies

$$\begin{aligned} u(t) = &- \frac{b-t}{b-a}\int_a^t (s-a)\ell(u)(s)ds - \\ &- \frac{t-a}{b-a}\int_t^b (b-s)\ell(u)(s)ds \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{65}$$

By virtue of (63), (64), and Lemma 2 (with $i = 1$), from (65) we get

$$1 \leq \frac{b-t_0}{b-a} \int\limits_a^{t_0} (s-a)|\ell(1)(s)|ds + \frac{t_0-a}{b-a} \int\limits_{t_0}^{b} (b-s)|\ell(1)(s)|ds.$$

Since $a < t_0 < b$, the latter inequality results in

$$1 < \frac{(b-t_0)(t_0-a)}{b-a} \int\limits_a^b |\ell(1)(s)|ds. \tag{66}$$

On the other hand, according to Lemma 4, the inequality (66) implies

$$1 < \left(\frac{b-a}{4} - \frac{\gamma^2}{b-a}\right) \int\limits_a^b |\ell(1)(s)|ds,$$

where γ is given by (8). However, it contradicts (11). □

Proof of Corollary 1. Put

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) \quad \text{for} \quad t \in [a,b]. \tag{67}$$

It is clear that $\ell \in P_{ab}$.

a) According to (13), we have $\ell \in K_{ab}(A)$. On the other hand, the inequality (14) yields (9). Therefore, the assumptions of Theorem 1 are satisfied.

b) Put $A = [\alpha, \beta]$. According to (15), we have $\ell \in K_{ab}(A)$. On the other hand, it is not difficult to verify that

$$\sigma_A(t) \geq \left[\frac{b-a}{2} - (\beta - \alpha)\right]_+ \quad \text{for} \quad t \in \left[a, \frac{a+b}{2}\right],$$

i.e.,

$$\delta \geq \left[\frac{b-a}{2} - (\beta - \alpha)\right]_+,$$

where δ is given by (7). Consequently, the inequality (16) yields (9). Therefore, the assumptions of Theorem 1 are satisfied.

c) Put $A = [a, \alpha] \cup [\beta, b]$. According to (17), we have $\ell \in K_{ab}(A)$. On the other hand, it is not difficult to verify that

$$\sigma_A(t) \geq \left[\frac{b-a}{2} - (\beta - \alpha)\right]_- \quad \text{for} \quad t \in \left[a, \frac{a+b}{2}\right],$$

i.e.,

$$\delta \geq \left[\frac{b-a}{2} - (\beta - \alpha)\right]_-,$$

where δ is given by (7). Consequently, the inequality (18) yields (9). Therefore, the assumptions of Theorem 1 are satisfied. □

Proof of Corollary 2. Let ℓ be defined by (67). It is clear that $-\ell \in P_{ab}$.

a) According to (13), we have $\ell \in K_{ab}(A)$. On the other hand, the inequality (19) yields (11). Therefore, the assumptions of Theorem 2 are satisfied.

b) Put $A = [\alpha, \beta]$. According to (15), we have $\ell \in K_{ab}(A)$. On the other hand, it is not difficult to verify that

$$\gamma = \max\left\{\left[\frac{b-a}{2} - (\beta - a)\right]_+, \left[\alpha - a - \frac{b-a}{2}\right]_+\right\},$$

where γ is given by (8). Consequently, the inequality (20), with γ_0 defined by (21), yields (11). Therefore, the assumptions of Theorem 2 are satisfied.

c) Put $A = [a, \alpha] \cup [\beta, b]$. According to (17), we have $\ell \in K_{ab}(A)$. On the other hand, it is not difficult to verify that

$$\gamma = \min\left\{\left[\frac{b-a}{2} - (\beta - a)\right]_-, \left[\alpha - a - \frac{b-a}{2}\right]_-\right\},$$

where γ is given by (8). Consequently, the inequality (20), with γ_0 defined by (22), yields (11). Therefore, the assumptions of Theorem 2 are satisfied. □

Proof of Corollary 3. The validity of corollary immediately follows from Corollary 2. □

5. Examples. Below two examples are constructed verifying the optimality of the main results established above.

EXAMPLE 1. *Let $a < b$, $\varepsilon > 0$, and $\delta = \left[0, \frac{b-a}{2}\right[$. It is clear that there exists $\varepsilon_0 \in \left]0, \frac{b-a}{2} - \delta\right[$ such that*

$$\frac{16(b-a-\varepsilon_0)}{(b-a-\varepsilon_0)^2 - (2\delta + \varepsilon_0)^2} = \frac{16(b-a) + \varepsilon(b-a)^2}{(b-a)^2 - 4\delta^2}. \tag{68}$$

Put $c_1 = \frac{3a+b}{4}$, $c_2 = \frac{a+3b}{4}$, and

$$\mu_i = c_1 - \frac{\delta}{2} - \frac{2-i}{2}\varepsilon_0, \qquad \nu_i = c_2 + \frac{\delta}{2} + \frac{i-1}{2}\varepsilon_0 \qquad (i = 1, 2).$$

Let $x \in \widetilde{C}'([\mu_1, \mu_2]; R)$ *and* $y \in \widetilde{C}'([\nu_1, \nu_2]; R)$ *be such that*

$$x(\mu_1) = x(\mu_2) = 1, \qquad x'(\mu_1) = \frac{1}{\mu_1 - a}, \qquad x'(\mu_2) = -\frac{2}{\nu_1 - \mu_2},$$

$$y(\nu_1) = y(\nu_2) = -1, \qquad y'(\nu_1) = -\frac{2}{\nu_1 - \mu_2}, \qquad y'(\nu_2) = \frac{1}{b - \nu_2},$$

and

$$x''(t) \le 0 \quad \textit{for} \quad t \in [\mu_1, \mu_2], \qquad y''(t) \ge 0 \quad \textit{for} \quad t \in [\nu_1, \nu_2].$$

Further, put $A = \{\nu_1, \mu_2\}$,

$$u(t) = \begin{cases} \frac{t-a}{\mu_1 - a} & \textit{for} \quad t \in [a, \mu_1[\\ x(t) & \textit{for} \quad t \in [\mu_1, \mu_2[\\ \frac{b-a-2t}{\nu_1 - \mu_2} & \textit{for} \quad t \in [\mu_2, \nu_1[\\ y(t) & \textit{for} \quad t \in [\nu_1, \nu_2[\\ \frac{t-b}{b-\nu_2} & \textit{for} \quad t \in [\nu_2, b] \end{cases},$$

$p(t) = |u''(t)|$ *for* $t \in [a, b]$,

$$\tau(t) = \begin{cases} \nu_1 & \textit{for} \quad t \in [a, c_2[\\ \mu_2 & \textit{for} \quad t \in [c_2, b] \end{cases},$$

and define ℓ *by* (67). *It is clear that* $\ell \in K_{ab}(A) \cap P_{ab}$, *the condition* (7) *holds, and*

$$\int_a^b \ell(1)(s)ds = \int_{\nu_1}^{\nu_2} y''(s)ds - \int_{\mu_1}^{\mu_2} x''(s)ds = \frac{16(b - a - \varepsilon_0)}{(b - a - \varepsilon_0)^2 - (2\delta + \varepsilon_0)^2}.$$

Therefore, in view of (68), *the inequality* (10) *is fulfilled. On the other hand,* u *is a nontrivial solution of the problem* (51), (52).

EXAMPLE 2. *Let* $a < b$, $\varepsilon > 0$, *and* $\gamma = \left[0, \frac{b-a}{2}\right[$. *It is clear that there exists* $\varepsilon_0 \in \left]0, \frac{b-a}{2} - \gamma\right[$ *such that*

$$\frac{4(b - a - \varepsilon_0)}{(b - a - \varepsilon_0)^2 - (2\gamma + \varepsilon_0)^2} = \frac{4(b - a) + \varepsilon(b - a)^2}{(b - a)^2 - 4\gamma^2}. \tag{69}$$

Put $c = \frac{a+b}{2}$ *and*

$$\mu_i = c - \gamma - (2 - i)\varepsilon_0 \qquad (i = 1, 2).$$

Let $x \in \widetilde{C}'([\mu_1, \mu_2]; R)$ *be such that*

$$x(\mu_1) = x(\mu_2) = 1, \qquad x'(\mu_1) = \frac{1}{\mu_1 - a}, \qquad x'(\mu_2) = -\frac{1}{b - \mu_2},$$

and

$$x''(t) \le 0 \quad \text{for} \quad t \in [\mu_1, \mu_2].$$

Further, put $A = \{\mu_2\}$,

$$u(t) = \begin{cases} \frac{t-a}{\mu_1 - a} & \text{for} \quad t \in [a, \mu_1[\\ x(t) & \text{for} \quad t \in [\mu_1, \mu_2[\\ \frac{b-t}{b-\mu_2} & \text{for} \quad t \in [\mu_2, b] \end{cases},$$

$p(t) = u''(t)$ *for* $t \in [a, b]$, $\tau \equiv \mu_2$, *and define* ℓ *by* (67). *It is clear that* $-\ell \in K_{ab}(A) \cap P_{ab}$, *the condition* (8) *holds, and*

$$\int_a^b |\ell(1)(s)| ds = -\int_{\mu_1}^{\mu_2} x''(s) ds = \frac{4(b - a - \varepsilon_0)}{(b - a - \varepsilon_0)^2 - (2\gamma + \varepsilon_0)^2}.$$

Therefore, in view of (69), *the inequality* (12) *is fulfilled. On the other hand,* u *is a nontrivial solution of the problem* (51), (52).

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