ON A PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

S. MUKHIGULASHVILI

Received 26 October 2004 and in revised form 7 March 2005

Unimprovable efficient sufficient conditions are established for the unique solvability of the periodic problem $u''(t) = \ell(u)(t) + q(t)$ for $0 \le t \le \omega$, $u^{(i)}(0) = u^{(i)}(\omega)$ $(i = 0,1)$, where $\omega > 0$, $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear bounded operator, and $q \in L([0,\omega])$.

1. Introduction

Consider the equation

$$u''(t) = \ell(u)(t) + q(t) \quad \text{for } 0 \le t \le \omega \tag{1.1}$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0,1), \tag{1.2}$$

where $\omega > 0$, $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear bounded operator and $q \in L([0,\omega])$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'([0,\omega])$, which satisfies (1.1) almost everywhere on $[0,\omega]$ and satisfies the conditions (1.2).

The periodic boundary value problem for functional differential equations has been studied by many authors (see, for instance, [1, 2, 3, 4, 5, 6, 8, 9] and the references therein). Results obtained in this paper on the one hand generalise the well-known results of Lasota and Opial (see [7, Theorem 6, page 88]) for linear ordinary differential equations, and on the other hand describe some properties which belong only to functional differential equations. In the paper [8], it was proved that the problem (1.1), (1.2) has a unique solution if the inequality

$$\int_0^\omega |\ell(1)(s)|\, ds \le \frac{d}{\omega} \tag{1.3}$$

with $d = 16$ is fulfilled. Moreover, there was also shown that the condition (1.3) is nonimprovable. This paper attempts to find a specific subset of the set of linear monotone operators, in which the condition (1.3) guarantees the unique solvability of the problem

Boundary Value Problems 2005:3 (2005) 247–261
DOI: 10.1155/BVP.2005.247

(1.1), (1.2) even for $d \geq 16$ (see Corollary 2.3). It turned out that if A satisfies some conditions dependent only on the constants d and ω, then $K_{[0,\omega]}(A)$ (see Definition 1.2) is such a subset of the set of linear monotone operators.

The following notation is used throughout.

N is the set of all natural numbers.

R is the set of all real numbers, $R_+ = [0,+\infty[$.

$C([a,b])$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \leq t \leq b\}$.

$\widetilde{C}'([a,b])$ is the set of functions $u : [a,b] \to R$ which are absolutely continuous together with their first derivatives.

$L([a,b])$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

If $x \in R$, then $[x]_+ = (|x|+x)/2$, $[x]_- = (|x|-x)/2$.

Definition 1.1. We will say that an operator $\ell : C([a,b]) \to L([a,b])$ is *nonnegative (nonpositive)*, if for any nonnegative $x \in C([a,b])$ the inequality

$$\ell(x)(t) \geq 0 \quad (\ell(x)(t) \leq 0) \quad \text{for } a \leq t \leq b \tag{1.4}$$

is satisfied.

We will say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

Definition 1.2. Let $A \subset [a,b]$ be a nonempty set. We will say that a linear operator $\ell : C([a,b]) \to L([a,b])$ belongs to the set $K_{[a,b]}(A)$ if for any $x \in C([a,b])$, satisfying

$$x(t) = 0 \quad \text{for } t \in A, \tag{1.5}$$

the equality

$$\ell(x)(t) = 0 \quad \text{for } a \leq t \leq b \tag{1.6}$$

holds.

We will say that $K_{[a,b]}(A)$ is the set of operators *concentrated* on the set $A \subset [a,b]$.

2. Main results

Define, for any nonempty set $A \subseteq R$, the continuous (see Lemma 3.1) functions:

$$\rho_A(t) = \inf\{|t-s| : s \in A\}, \quad \sigma_A(t) = \rho_A(t) + \rho_A\left(t + \frac{\omega}{2}\right) \quad \text{for } t \in R. \tag{2.1}$$

Theorem 2.1. *Let $A \subset [0,\omega]$, $A \neq \emptyset$ and a linear monotone operator $\ell \in K_{[0,\omega]}(A)$ be such that the conditions*

$$\int_0^\omega \ell(1)(s)ds \neq 0, \tag{2.2}$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell(1)(s)|ds \leq \frac{16}{\omega} \tag{2.3}$$

are satisfied,where

$$\delta = \min\left\{\sigma_A(t) : 0 \le t \le \frac{\omega}{2}\right\}. \tag{2.4}$$

Then the problem (1.1), (1.2) has a unique solution.

Example 2.2. The example below shows that condition (2.3) in Theorem 2.1 is optimal and it cannot be replaced by the condition

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right)\int_0^\omega |\ell(1)(s)|\,ds \le \frac{16}{\omega} + \varepsilon, \tag{2.3_ε}$$

no matter how small $\varepsilon \in]0,1]$ would be. Let $\omega = 1$, $\varepsilon_0 \in]0,1/16[$, $\delta_1 \in]0,1/4-2\varepsilon_0[$ and μ_i, ν_i $(i = 1,2)$ be the numbers given by the equalities

$$\mu_i = \frac{1-2\delta_1}{4} + (-1)^i\varepsilon_0, \qquad \nu_i = \frac{3+2\delta_1}{4} + (-1)^i\varepsilon_0 \quad (i = 1,2). \tag{2.5}$$

Let, moreover, the functions $x \in \widetilde{C}'([\mu_1,\mu_2])$, $y \in \widetilde{C}'([\nu_1,\nu_2])$ be such that

$$\begin{gathered} x(\mu_1) = x(\mu_2) = 1, \qquad x'(\mu_1) = \frac{1}{\mu_1}, \qquad x'(\mu_2) = -\frac{1}{\mu_1+\delta_1}, \\ x''(t) \le 0 \quad \text{for } \mu_1 \le t \le \mu_2, \end{gathered} \tag{2.5_1}$$

$$\begin{gathered} y(\nu_1) = y(\nu_2) = -1, \qquad y'(\nu_1) = -\frac{1}{\mu_1+\delta_1}, \qquad y'(\nu_2) = \frac{1}{\mu_1}, \\ y''(t) \ge 0 \quad \text{for } \nu_1 \le t \le \nu_2. \end{gathered} \tag{2.5_2}$$

Define a function

$$u_0(t) = \begin{cases} \dfrac{t}{\mu_1} & \text{for } 0 \le t \le \mu_1 \\ x(t) & \text{for } \mu_1 < t < \mu_2 \\ \dfrac{1-2t}{\nu_1-\mu_2} & \text{for } \mu_2 \le t \le \nu_1 \\ y(t) & \text{for } \nu_1 < t < \nu_2 \\ \dfrac{t-1}{\mu_1} & \text{for } \nu_2 \le t \le 1. \end{cases} \tag{2.6}$$

Obviously, $u_0 \in \widetilde{C}'([0,\omega])$. Now let $A = \{\mu_1,\nu_2\}$, the function $\tau : [0,\omega] \to A$ and the operator $\ell : C([0,\omega]) \to L([0,\omega])$ be given by the equalities:

$$\tau(t) = \begin{cases} \mu_1 & \text{if } u_0''(t) \ge 0 \\ \nu_2 & \text{if } u_0''(t) < 0, \end{cases} \qquad \ell(z)(t) = |u_0''(t)|\,z(\tau(t)). \tag{2.7}$$

It is clear from the definition of the functions τ and σ_A that the nonnegative operator ℓ is concentrated on the set A and the condition (2.4) is satisfied with $\delta = \delta_1 + 2\varepsilon_0$. In view of (2.5_1), (2.5_2), and (2.7) we obtain

$$\int_0^\omega \ell(1)(s)ds = \int_{\nu_1}^{\nu_2} y''(s)ds - \int_{\mu_1}^{\mu_2} x''(s)ds = 2\frac{2\mu_1+\delta_1}{\mu_1(\mu_1+\delta_1)} = 16\frac{1-4\varepsilon_0}{(1-4\varepsilon_0)^2-4\delta_1^2}. \quad (2.8)$$

When ε is small enough, the last equality it implies the existence of ε_0 such that

$$0 < \int_0^\omega \ell(1)(s)ds = \frac{16+\varepsilon}{1-4\delta_1^2}. \quad (2.9)$$

Thus, because $\delta_1 < \delta$, all the assumptions of Theorem 2.1 are satisfied except (2.3), and instead of (2.3) the condition (2.3_ε) is fulfilled with $\omega = 1$. On the other hand, from the definition of the function u_0 and from (2.7), it follows that $\ell(u_0)(t) = |u_0''(t)|u_0(\tau(t)) = |u_0''(t)|\operatorname{sign} u_0''(t)$, that is, u_0 is a nontrivial solution of the homogeneous problem $u''(t) = \ell(u)(t)$, $u^{(i)}(0) = u^{(i)}(1)$ $(i = 1,2)$ which contradicts the conclusion of Theorem 2.1.

Corollary 2.3. *Let the set $A \subset [0,\omega]$, number $d \geq 16$, and a linear monotone operator $\ell \in K_{[0,\omega]}(A)$ be such that the conditions (2.2)*

$$\int_0^\omega |\ell(1)(s)|ds \leq \frac{d}{\omega}, \quad (2.10)$$

are satisfied and

$$\sigma_A(t) \geq \frac{\omega}{2}\sqrt{1-\frac{16}{d}} \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \quad (2.11)$$

Then the problem (1.1), (1.2) has a unique solution.

Corollary 2.4. *Let $\alpha \in [0,\omega]$, $\beta \in [\alpha,\omega]$, and a linear monotone operator $\ell \in K_{[0,\omega]}(A)$ be such that the conditions (2.2) and (2.3) are satisfied, where*

$$A = [\alpha,\beta], \qquad \delta = \left[\frac{\omega}{2} - (\beta-\alpha)\right]_+ \quad (2.11_1)$$

or

$$A = [0,\alpha] \cup [\beta,\omega], \qquad \delta = \left[\frac{\omega}{2} - (\beta-\alpha)\right]_-. \quad (2.11_2)$$

Then the problem (1.1), (1.2) has a unique solution.

Consider the equation with deviating arguments

$$u''(t) = p(t)u(\tau(t)) + q(t) \quad \text{for } 0 \leq t \leq \omega, \quad (2.12)$$

where $p \in L([0,\omega])$ and $\tau : [0,\omega] \to [0,\omega]$ is a measurable function.

Corollary 2.5. *Let there exist $\sigma \in \{-1,1\}$ such that*

$$\sigma p(t) \geq 0 \quad \text{for } 0 \leq t \leq \omega, \quad (2.13)$$

$$\int_0^\omega p(s)ds \neq 0. \quad (2.14)$$

Moreover, let $\delta \in [0,\omega/2]$ *and the function* p *be such that*

$$\left(1-4\left(\frac{\delta}{\omega}\right)^2\right)\int_0^\omega |\,p(s)\,|\,ds \le \frac{16}{\omega}, \tag{2.15}$$

and let at least one of the following items be fulfilled:

(a) *the set* $A \subset [0,\omega]$ *is such that the condition (2.4) holds and*

$$p(t)=0 \quad \text{if } \tau(t) \notin A \tag{2.16}$$

on $[0,\omega]$;

(b) *the constants* $\alpha \in [0,\omega]$, $\beta \in [\alpha,\omega]$ *are such that*

$$\tau(t) \in [\alpha,\beta] \quad \text{for } 0 \le t \le \omega, \tag{2.17}$$

$$\delta = \left[\frac{\omega}{2}-(\beta-\alpha)\right]_+. \tag{2.18}$$

Then the problem (2.12), (1.2) has a unique solution.

Now consider the ordinary differential equation

$$u''(t) = p(t)u(t)+q(t) \quad \text{for } 0 \le t \le \omega, \tag{2.19}$$

where $p,q \in L([0,\omega])$.

Corollary 2.6. *Let*

$$p(t) \le 0 \quad \text{for } 0 \le t \le \omega. \tag{2.20}$$

Moreover, let $\delta \in [0,\omega/2]$ *and the function* p *be such that the conditions (2.14), (2.15) hold, and let at least one of the following items be fulfilled:*

(a) *the set* $A \subset [0,\omega]$ *is such that* $\operatorname{mes} A \neq 0$, *the condition (2.4) holds and*

$$p(t)=0 \quad \text{for } t \notin A; \tag{2.21}$$

(b) *the constants* $\alpha \in [0,\omega]$, $\beta \in [\alpha,\omega]$ *are such that*

$$p(t)=0 \quad \text{for } t \in [0,\alpha[\cup]\beta,\omega], \tag{2.22}$$

and $\delta \in [0,\omega/2]$ *satisfies (2.18). Then the problem (2.19), (1.2) has a unique solution.*

Remark 2.7. As for the case where $p(t) \ge 0$ for $0 \le t \le \omega$, the necessary and sufficient condition for the unique solvability of (2.19), (1.2) is $p(t) \not\equiv 0$ (see [2, Proposition 1.1, page 72]).

3. Auxiliary propositions

Lemma 3.1. *The function $\rho_A : R \to R$ defined by the equalities (2.1), is continuous and*

$$\rho_{\bar{A}}(t) = \rho_A(t) \quad \text{for } t \in R, \tag{3.1}$$

where $\bar{A}$ is the closure of the set A.

Proof. Since $A \subseteq \bar{A}$, it is clear that

$$\rho_{\bar{A}}(t) \le \rho_A(t) \quad \text{for } t \in R. \tag{3.2}$$

Let $t_0 \in R$ be an arbitrary point, $s_0 \in \bar{A}$, and the sequence $s_n \in A$ $(n \in N)$ be such that $\lim_{n\to\infty} s_n = s_0$. Then $\rho_A(t_0) \le \lim_{n\to\infty} |t_0 - s_n| = |t_0 - s_0|$, that is,

$$\rho_{\bar{A}}(t) \ge \rho_A(t) \quad \text{for } t \in R. \tag{3.3}$$

From the last relation and (3.2) we get the equality (3.1).

For arbitrary $s \in A$, $t_1, t_2 \in R$, we have

$$\rho_A(t_i) \le |t_i - s| \le |t_2 - t_1| + |t_{3-i} - s| \quad (i = 1,2). \tag{3.4}$$

Consequently $\rho_A(t_i) - |t_2 - t_1| \le \rho_A(t_{3-i})$ $(i = 1,2)$. Thus the function ρ_A is continuous. □

Lemma 3.2. *Let $A \subseteq [0,\omega]$ be a nonempty set, $A_1 = \{t + \omega : t \in A\}$, $B = A \cup A_1$, and*

$$\min\left\{\sigma_A(t) : 0 \le t \le \frac{\omega}{2}\right\} = \delta. \tag{3.5}$$

Then

$$\min\left\{\sigma_B(t) : 0 \le t \le \frac{3\omega}{2}\right\} = \delta. \tag{3.6}$$

Proof. Let $\alpha = \inf A$, $\beta = \sup A$, and let $t_0 \in [0, 3\omega/2]$ be such that

$$\sigma_B(t_0) = \min\left\{\sigma_B(t) : 0 \le t \le \frac{3\omega}{2}\right\}. \tag{3.7}$$

Assume that $t_1 \in [0, 3\omega/2]$ is such that $t_1 \notin \bar{B}$, $t_1 + \omega/2 \notin \bar{B}$. Then

$$\varepsilon = \min\{\rho_B(t_1), \rho_B(t_1 + \omega/2)\} > 0, \tag{3.8}$$

and either

$$\sigma_B(t_1 - \varepsilon) \le \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 - \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} - \varepsilon\right) = 0 \tag{3.9}$$

or

$$\sigma_B(t_1 + \varepsilon) \le \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 + \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} + \varepsilon\right) = 0. \tag{3.10}$$

In view of this fact, without loss of generality we can assume that

$$t_0 \in \bar{B} \quad \text{or} \quad t_0 + \frac{\omega}{2} \in \bar{B}. \tag{3.11}$$

From (3.5) and the condition $A \subseteq [0,\omega]$, we have

$$\min\left\{\sigma_A(t) : 0 \le t \le \frac{3\omega}{2}\right\} = \delta. \tag{3.12}$$

First suppose that $0 \le t_0 \le \beta - \omega/2$. From this inequality by the inclusion $\beta \in \bar{A}$, we get

$$\inf\left\{\left|t_0 + \frac{\omega i}{2} - s\right| : s \in B\right\} = \inf\left\{\left|t_0 + \frac{\omega i}{2} - s\right| : s \in A\right\} \tag{3.12_i}$$

for $i = 0,1$. Then $\sigma_B(t_0) = \sigma_A(t_0)$ and in view of (3.12)

$$\sigma_B(t_0) \ge \delta. \tag{3.13}$$

Let now

$$\beta - \frac{\omega}{2} < t_0 \le \beta. \tag{3.14}$$

Obviously, either

$$\left(t_0 + \frac{\omega}{2}\right) - \beta \le \alpha + \omega - \left(t_0 + \frac{\omega}{2}\right), \tag{3.14_1}$$

or

$$\left(t_0 + \frac{\omega}{2}\right) - \beta > \alpha + \omega - \left(t_0 + \frac{\omega}{2}\right). \tag{3.14_2}$$

If (3.14_1) is satisfied, then, in view of (3.14) and $\beta \in \bar{A}$, the equalities (3.12_i) ($i = 0,1$) hold. Therefore $\sigma_B(t_0) = \sigma_A(t_0)$ and, in view of (3.12), the inequality (3.13) is fulfilled. Let now (3.14_2) be satisfied. If $\alpha + \omega > t_0 + \omega/2$, then, in view of (3.14), we have $t_0 + \omega/2 \notin \bar{B}$. Consequently, from (3.12) and (3.14_2) by virtue of (3.11) and the inclusions $\alpha,\beta \in \bar{A}$, we get

$$\sigma_B(t_0) = \rho_B\left(t_0 + \frac{\omega}{2}\right) = \alpha + \frac{\omega}{2} - t_0 \ge \rho_A\left(\alpha + \frac{\omega}{2}\right) \ge \delta. \tag{3.15}$$

If $\alpha + \omega \le t_0 + \omega/2$, then $t_0 + \omega/2 \in \bar{A}_1$ and

$$\inf\left\{\left|t_0 + \frac{\omega}{2} - s\right| : s \in B\right\} = \inf\left\{\left|t_0 - \frac{\omega}{2} - s\right| : s \in A\right\}, \tag{3.16}$$

that is, $\rho_B(t_0 + \omega/2) = \rho_A(t_0 - \omega/2)$ and in view of (3.12), (3.14) we get

$$\sigma_B(t_0) = \rho_A(t_0) + \rho_A\left(t_0 - \frac{\omega}{2}\right) = \sigma_A\left(t_0 - \frac{\omega}{2}\right) \ge \delta. \tag{3.17}$$

Consequently the inequality (3.13) is fulfilled as well.

Further, let $\beta \leq t_0 \leq t_0 + \omega/2 \leq \alpha + \omega$. Then $t_0 - \alpha \leq \alpha + \omega - t_0$, and also $t_0 - \beta \leq \alpha + \omega - t_0$. On account of (3.12) and $\beta \in \bar{A}$ we have

$$\sigma_B(t_0) = \alpha + \frac{\omega}{2} - \beta \geq \rho_A\left(\alpha + \frac{\omega}{2}\right) = \sigma_A(\alpha) \geq \delta. \tag{3.18}$$

Thus the inequality (3.13) is fulfilled.

Let now

$$\beta \leq t_0 \leq \alpha + \omega \leq t_0 + \frac{\omega}{2}. \tag{3.19}$$

From (3.19) it follows that

$$\begin{aligned} \inf\left\{\left|t_0 + \frac{\omega}{2} - s\right| : s \in B\right\} &= \inf\left\{\left|t_0 + \frac{\omega}{2} - s\right| : s \in A_1\right\} \\ &= \inf\left\{\left|t_0 - \frac{\omega}{2} - s\right| : s \in A\right\} \geq \inf\left\{\left|t_0 - \frac{\omega}{2} - s\right| : s \in B\right\}, \end{aligned} \tag{3.20}$$

and therefore,

$$\sigma_B(t_0) \geq \rho_B\left(t_0 - \frac{\omega}{2}\right) + \rho_B(t_0) = \sigma_B\left(t_0 - \frac{\omega}{2}\right). \tag{3.21}$$

The inequalities (3.19) imply $t_0 - \omega/2 \leq \alpha + \omega$ and, according to the case considered above, we have $\sigma_B(t_0 - \omega/2) \geq \delta$. Consequently, (3.21) results in (3.13).

Finally, if $\alpha + \omega \leq t_0$, the validity of (3.13) can be proved analogously to the previous cases. Then we have

$$\sigma_B(t) \geq \delta \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \tag{3.22}$$

On the other hand, since $A \subset B$, it is clear that

$$\sigma_B(t) \leq \sigma_A(t) \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \tag{3.23}$$

The last two relations and (3.5) yields the equality (3.6). □

Lemma 3.3. *Let $\sigma \in \{-1, 1\}$, $D \subset [a, b]$, $D \not\equiv \emptyset$, $\ell_1 \in K_{[a,b]}(D)$, and let $\sigma\ell_1$ be nonnegative. Then, for an arbitrary $v \in C([a, b])$,*

$$\begin{aligned} &\min\{v(s) : s \in \bar{D}\}\,|\ell_1(1)(t)| \\ &\quad \leq \sigma\ell_1(v)(t) \leq \max\{v(s) : s \in \bar{D}\}\,|\ell_1(1)(t)| \quad \textit{for } a \leq t \leq b. \end{aligned} \tag{3.24}$$

Proof. Let $\alpha = \inf D$, $\beta = \sup D$,

$$v_0(t) = \begin{cases} v(\alpha) & \text{for } t \in [a, \alpha[\\ v(t) & \text{for } t \in \bar{D} \\ \dfrac{v(\mu(t)) - v(\nu(t))}{\mu(t) - \nu(t)}(t - \nu(t)) + v(\nu(t)) & \text{for } t \in [\alpha, \beta] \setminus \bar{D} \\ v(\beta) & \text{for } t \in]\beta, b], \end{cases} \tag{3.25}$$

where

$$\mu(t) = \min\{s \in \bar{D} : t \le s\}, \quad \nu(t) = \max\{s \in \bar{D} : t \ge s\} \quad \text{for } \alpha \le t \le \beta. \tag{3.26}$$

It is clear that $v_0 \in C([a,b])$ and

$$\begin{gathered} \min\{v(s) : s \in \bar{D}\} \le v_0(t) \le \max\{v(s) : s \in \bar{D}\} \quad \text{for } a \le t \le b, \\ v_0(t) = v(t) \quad \text{for } t \in D. \end{gathered} \tag{3.27}$$

Since $\ell_1 \in K_{[a,b]}(D)$ and the operator $\sigma\ell_1$ is nonnegative, it follows from (3.27) that (3.24) is true. □

Lemma 3.4. *Let $a \in [0,\omega]$, $D \subset [a, a+\omega]$, $c \in [a, a+\omega]$, and $\delta \in [0, \omega/2]$ be such that*

$$\sigma_D(t) \ge \delta \quad \text{for } a \le t \le a + \frac{\omega}{2}, \tag{3.28}$$

$$A_c = \bar{D} \cap [a,c] \ne \varnothing, \qquad B_c = \bar{D} \cap [c, a+\omega] \ne \varnothing. \tag{3.29}$$

Then the estimate

$$\left(\frac{(c - t_1)(t_1 - a)(a + \omega - t_2)(t_2 - c)}{(c-a)(a+\omega-c)}\right)^{1/2} \le \frac{\omega^2 - 4\delta^2}{8\omega} \tag{3.30}$$

for all $t_1 \in A_c$, $t_2 \in B_c$ is satisfied.

Proof. Put $b = a + \omega$ and

$$\sigma_1 = \rho_D\left(\frac{a+c}{2}\right), \qquad \sigma_2 = \rho_D\left(\frac{c+b}{2}\right). \tag{3.31}$$

Then, from the condition (3.28) it is clear

$$\sigma_1 + \sigma_2 \ge \delta. \tag{3.32}$$

Obviously, either

$$\max(\sigma_1, \sigma_2) \ge \delta \tag{3.32$_1$}$$

or

$$\max(\sigma_1, \sigma_2) < \delta. \tag{3.32$_2$}$$

First note that from (3.29) and (3.31) the equalities

$$\begin{aligned} \max\{(c-t_1)(t_1-a) : t_1 \in A_c\} &= (c-t_1')(t_1'-a), \\ \max\{(b-t_2)(t_2-c) : t_2 \in B_c\} &= (b-t_2')(t_2'-c), \end{aligned} \tag{3.33}$$

follow, where $t_1' = (a+c)/2 - \sigma_1$, $t_2' = (c+b)/2 - \sigma_2$. Hence, on account of well-known inequality

$$d_1 d_2 \le \frac{(d_1+d_2)^2}{4}, \tag{3.34}$$

we have

$$\begin{aligned} &\left(\frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)}\right)^{1/2} \\ &\quad \le \left(\frac{c-a}{4} - \frac{\sigma_1^2}{c-a}\right)^{1/2}\left(\frac{b-c}{4} - \frac{\sigma_2^2}{b-c}\right)^{1/2} \le \frac{1}{2}\left(\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c}\right) \end{aligned} \tag{3.35}$$

for all $t_1 \in A_c$, $t_2 \in B_c$. In the case, where inequality (3.32_1) is fulfilled, we have

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} \le \frac{\omega}{4} - \frac{(\max(\sigma_1,\sigma_2))^2}{\omega} \le \frac{\omega^2 - 4\delta^2}{4\omega}. \tag{3.36}$$

This, together with (3.35), yields the estimate (3.30). Suppose now that the condition (3.32_2) is fulfilled. Then in view of Lemma 3.1, we can choose $\alpha,\beta \in \bar{D}$ such that

$$\rho_D\left(\frac{a+c}{2}\right) = \left|\frac{a+c}{2} - \alpha\right|, \qquad \rho_D\left(\frac{c+b}{2}\right) = \left|\frac{c+b}{2} - \beta\right|, \tag{3.37}$$

which together with (3.31) yields

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} = \omega - (\beta-\alpha) - \eta(c), \tag{3.38}$$

where $\eta(t) = (\alpha-a)^2/(t-a) + (b-\beta)^2/(b-t)$. It is not difficult to verify that the function η achieves its minimum at the point $t_0 = ((\alpha-a)b + (b-\beta)a)/(\omega-(\beta-\alpha))$. Thus,

$$\omega - (\beta-\alpha) - \eta(c) \le (\omega - (\beta-\alpha))\frac{\beta-\alpha}{\omega}. \tag{3.39}$$

Put

$$\sigma = \min(\sigma_1,\sigma_2). \tag{3.40}$$

Then it follows from (3.37) that either

$$\alpha \leq \frac{a+c}{2} - \sigma \tag{3.40$_1$}$$

or

$$\alpha \geq \frac{a+c}{2} + \sigma, \tag{3.40$_2$}$$

and either

$$\beta \geq \frac{c+b}{2} + \sigma \tag{3.40$_3$}$$

or

$$\beta \leq \frac{c+b}{2} - \sigma. \tag{3.40$_4$}$$

Consider now the case where α satisfies the inequality (3.40$_1$) and assume that β satisfies the inequality (3.40$_4$). Then from (3.37), (3.40$_1$), and (3.40$_4$) we get

$$\rho_D\left(\frac{a+c}{2} - \sigma\right) = \rho_D\left(\frac{a+c}{2}\right) - \sigma, \qquad \rho_D\left(\frac{c+b}{2} - \sigma\right) = \rho_D\left(\frac{c+b}{2}\right) - \sigma. \tag{3.41}$$

These equalities in view of (3.31) and (3.40) yield

$$\sigma_D\left(\frac{a+c}{2} - \sigma\right) = (\sigma_1 - \sigma) + (\sigma_2 - \sigma) = \max(\sigma_1, \sigma_2) - \sigma, \tag{3.42}$$

but in view of (3.32$_2$) this contradicts the condition (3.28). Consequently, β satisfies the inequality (3.40$_3$). Then from (3.31), (3.37), by (3.40$_1$) and (3.40$_3$), we get $\sigma_1 = (a+c)/2 - \alpha$, $\sigma_2 = \beta - (c+b)/2$, that is,

$$\beta - \alpha = \sigma_1 + \sigma_2 + \frac{\omega}{2}. \tag{3.42$_1$}$$

Now suppose that (3.40$_2$) holds. It can be proved in a similar manner as above that, in this case, the inequality (3.40$_4$) is satisfied. Therefore, from (3.31), (3.37), (3.40$_2$), and (3.40$_4$) we obtain

$$\beta - \alpha = \frac{\omega}{2} - (\sigma_1 + \sigma_2). \tag{3.42$_2$}$$

Then, on account of (3.32), in both (3.42$_1$) and (3.42$_2$) cases we have

$$(\omega - (\beta - \alpha))\frac{\beta - \alpha}{\omega} = \frac{\omega^2 - 4(\sigma_1 + \sigma_2)^2}{4\omega} \leq \frac{\omega^2 - 4\delta^2}{4\omega}. \tag{3.43}$$

Consequently from (3.35), (3.38), (3.39), and (3.43) we obtain the estimate (3.30), also in case where the inequality (3.32$_2$) holds. □

4. Proof of the main results

Proof of Theorem 2.1. Consider the homogeneous problem

$$v''(t) = \ell(v)(t) \quad \text{for } 0 \le t \le \omega, \tag{4.1}$$

$$v^{(i)}(0) = v^{(i)}(\omega) \quad (i = 0,1). \tag{4.2}$$

It is known from the general theory of boundary value problems for functional differential equations that if ℓ is a monotone operator, then problem (1.1), (1.2) has the Fredholm property (see [3, Theorem 1.1, page 345]). Thus, the problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem (4.1), (4.2) has only the trivial solution.

Assume that, on the contrary, the problem (4.1), (4.2) has a nontrivial solution v. If $v \equiv \text{const}$, then, in view of (4.1) we obtain a contradiction with the condition (2.2). Consequently, $v \not\equiv \text{const}$. Then, in view of the conditions (4.2), there exist subsets I_1 and I_2 from $[0,\omega]$ which have positive measure and

$$v''(t) > 0 \quad \text{for } t \in I_1, \qquad v''(t) < 0 \quad \text{for } t \in I_2. \tag{4.3}$$

Assume that v is either nonnegative or nonpositive on the entire set A. Without loss of generality we can suppose $v(t) \ge 0$ for $t \in A$. Then, from Lemma 3.3 with $a = 0$, $b = \omega$, $D = A$, and $\ell_1 \equiv \ell$ we obtain

$$\sigma\ell(v)(t) \ge 0 \quad \text{for } 0 \le t \le \omega. \tag{4.4}$$

In view of (4.1), the inequality (4.4) contradicts one of the inequalities in (4.3). Therefore, the function v changes its sign on the set A, that is, there exist $t_1', t_1 \in \bar{A}$ such that

$$v(t_1') = \min\{v(t) : t \in \bar{A}\}, \qquad v(t_1) = \max\{v(t) : t \in \bar{A}\}, \tag{4.5}$$

and $v(t_1') < 0$, $v(t_1) > 0$. Without loss of generality we can assume that $t_1' < t_1$. Then, in view of the last inequalities, there exists $a \in]t_1', t_1[$ such that $v(a) = 0$.

Let us set $C_\omega([a,a+\omega]) = \{x \in C([a,a+\omega]) : x(a) = x(a+\omega)\}$, and let the continuous operators $\gamma : L([0,\omega]) \to L([a,a+\omega])$, $\ell_1 : C_\omega([a,a+\omega]) \to L([a,a+\omega])$ and the function $v_0 \in C([a,a+\omega])$ be given by the equalities

$$\gamma(x)(t) = \begin{cases} x(t) & \text{for } a \le t \le \omega \\ x(t-\omega) & \text{for } \omega < t \le a+\omega, \end{cases}$$
$$v_0(t) = \gamma(v(t)), \quad \ell_1(x)(t) = \gamma(\ell(\gamma^{-1}(x)))(t) \quad \text{for } a \le t \le a+\omega. \tag{4.6}$$

Let, moreover, $t_2 = t_1' + \omega$ and $D = A \cup \{t+\omega : t \in A\} \cap [a,a+\omega]$. Then (4.1), (4.2) with regard for (4.6) and the definitions of a, t_1', t_1, imply that $v_0 \in \widetilde{C}'([a,a+\omega])$, $t_1, t_2 \in D$,

$$v_0''(t) = \ell_1(v_0)(t) \quad \text{for } a \le t \le a+\omega, \tag{4.7}$$

$$v_0(a) = 0, \qquad v_0(a+\omega) = 0, \tag{4.8}$$

$$v_0(t_1) = \max\{v_0(t) : t \in \bar{D}\}, \qquad v_0(t_2) = \min\{v_0(t) : t \in \bar{D}\}, \tag{4.9}$$

$$v_0(t_1) > 0, \qquad v_0(t_2) < 0, \tag{4.10}$$

and there exists $c \in]t_1, t_2[$ such that

$$v_0(c) = 0. \tag{4.11}$$

It is not difficult to verify that the condition $\ell \in K_{[0,\omega]}(A)$ implies

$$\ell_1 \in K_{[a,a+\omega]}(D). \tag{4.12}$$

Since $D \subset A \cup \{t+\omega : t \in A\}$, it follows from condition (2.4) and Lemma 3.2 that

$$\sigma_D(t) \geq \delta \quad \text{for } a \leq t \leq a + \frac{\omega}{2}. \tag{4.13}$$

Thus, from the general theory of ordinary differential equations (see [6, Theorem 1.1, page 2348]), in view of (4.7), (4.8), (4.9), and (4.11), we obtain the representations

$$v_0(t_1) = -\int_a^c |G_1(t_1,s)|\ell_1(v_0)(s)ds, \tag{4.13$_1$}$$

$$|v_0(t_2)| = \int_c^{a+\omega} |G_2(t_2,s)|\ell_1(v_0)(s)ds, \tag{4.13$_2$}$$

where $G_1(G_2)$ is Green's function of the problem

$$\begin{aligned} z''(t) = 0 \quad &\text{for } a \leq t \leq c \ (c \leq t \leq a+\omega), \\ z(a) = 0, \quad z(c) = 0 \quad &(z(c) = 0,\ z(a+\omega) = 0). \end{aligned} \tag{4.14}$$

If ℓ is a nonnegative operator, then from (4.6) it is clear that ℓ_1 is also nonnegative. Then, from (4.13$_1$) and (4.13$_2$), by Lemma 3.3 and relations (4.9), (4.10), and (4.12), we get the strict estimates

$$\begin{aligned} 0 < \frac{v_0(t_1)}{|v_0(t_2)|} &< \frac{(t_1-a)(c-t_1)}{c-a}\int_a^c \ell_1(1)(s)ds, \\ 0 < \frac{|v_0(t_2)|}{v_0(t_1)} &< \frac{(t_2-c)(a+\omega-t_2)}{a+\omega-c}\int_c^{a+\omega} \ell_1(1)(s)ds, \end{aligned} \tag{4.15}$$

respectively. By multiplying these estimates and applying the numerical inequality (3.34), we obtain

$$1 < \frac{1}{2}\left(\frac{(t_1-a)(c-t_1)(t_2-c)(a+\omega-t_2)}{(c-a)(a+\omega-c)}\right)^{1/2}\int_a^{a+\omega} |\ell_1(1)(s)|ds. \tag{4.16}$$

Reasoning analogously, we can show that the estimate (4.16) is valid also in case where the operator ℓ is nonpositive.

From the definitions of t_1, t_2, c, and (4.13), it follows that all the conditions of Lemma 3.4 are satisfied. In view of the estimate (3.30) and the definition of the operator ℓ_1, the inequality (4.16) contradicts the condition (2.3). $\square$

Proof of Corollary 2.3. Let $\delta = \omega/2(1 - 16/d)^{1/2}$. Then, on account of (2.10) and (2.11), we obtain that the conditions (2.3) and (2.4) of Theorem 2.1 are fulfilled. Consequently, all the assumptions of Theorem 2.1 are satisfied. □

Proof of Corollary 2.4. It is not difficult to verify that if $A = [\alpha,\beta]$ $(A = [0,\alpha] \cup [\beta,\omega])$, then

$$\sigma_A(t) \geq \left[\frac{\omega}{2} - \beta + \alpha\right]_+ \quad \left(\sigma_A(t) \geq \left[\frac{\omega}{2} - \beta + \alpha\right]_-\right) \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \tag{4.17}$$

Consequently, in view of the condition (2.11_1), (2.11_2), all the assumptions of Theorem 2.1 are satisfied. □

Proof of Corollary 2.5. Let $\ell(u)(t) \equiv p(t)u(\tau(t))$. On account of (2.13), (2.14), and (2.15) we see that the operator ℓ is monotone and the conditions (2.2) and (2.3) are satisfied.

(a) It is not difficult to verify that from the condition (2.16) it follows that $\ell \in K_{[0,\omega]}(A)$. Consequently, all the assumptions of Theorem 2.1 are satisfied.

(b) Let $A = [\alpha,\beta]$. Then in view of the condition (2.17) the inclusion $\ell \in K_{[0,\omega]}(A)$ is satisfied. The inequality (4.17) obtained in the proof of Corollary 2.4, by virtue of (2.18), implies the inequality (2.4). Consequently, all the assumptions of Theorem 2.1 are satisfied. □

Proof of Corollary 2.6. The validity of this assertion follows immediately from Corollary 2.5(a). □

Acknowledgment

This paper was prepared during the author's stay at the Mathematical Institute of the Academy of Sciences of the Czech Republic.

References

[1] Sh. Gelashvili and I. Kiguradze, *On multi-point boundary value problems for systems of functional-differential and difference equations*, Mem. Differential Equations Math. Phys. **5** (1995), 1–113, 133.

[2] I. T. Kiguradze and T. Kusano, *On periodic solutions of higher-order nonautonomous ordinary differential equations*, Differ. Uravn. **35** (1999), no. 1, 72–78, 142 (Russian), translation in Differential Equations, **35** (1999), no. 1, 71–77.

[3] I. Kiguradze and B. Půža, *On boundary value problems for systems of linear functional-differential equations*, Czechoslovak Math. J. **47(122)** (1997), no. 2, 341–373.

[4] ———, *On periodic solutions of systems of linear functional-differential equations*, Arch. Math. (Brno) **33** (1997), no. 3, 197–212.

[5] ———, *Boundary Value Problems for Systems of Linear Functional Differential Equations*, Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, vol. 12, Masaryk University, Brno, 2003.

[6] I. T. Kiguradze and B. L. Shekhter, *Singular boundary value problems for second-order ordinary differential equations*, Current Problems in Mathematics. Newest Results, Vol. 30 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987, pp. 105–201, 204, Translated in J. Soviet Math. **43** (1988), no. 2, 2340–2417.

[7] A. Lasota and Z. Opial, *Sur les solutions périodiques des équations différentielles ordinaires*, Ann. Polon. Math. **16** (1964), no. 1, 69–94 (French).
[8] A. Lomtatidze and S. Mukhigulashvili, *On periodic solutions of second order functional differential equations*, Mem. Differential Equations Math. Phys. **5** (1995), 125–126.
[9] Š. Schwabik, M. Tvrdý, and O. Vejvoda, *Differential and Integral Equations. Boundary Value Problems and Adjoints*, D. Reidel, Dordrecht, 1979.

S. Mukhigulashvili: A. Razmadze Mathematical Institute, Georgian Academy of Sciences, M. Aleksidze Street 1, 0193 Tbilisi, Georgia
E-mail address: smukhig@rmi.acnet.ge
Current address: Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic
E-mail address: mukhig@ipm.cz