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TWO–WEIGHT NORM ESTIMATES FOR MAXIMAL, POTENTIAL AND SINGULAR OPERATORS IN $L^{p(x)}$ SPACES

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Let q be a measurable function on $\mathbb{R}^n$ such that $1 \le q \le \sup\limits_{\mathbb{R}^n} q < \infty$. We denote by $L^{q(\cdot)}(\mathbb{R}^n)$ the variable exponent Lebesgue space which is the class of all measurable functions f on $\mathbb{R}^n$ such that

$$S_{q(\cdot)}(f) := \int\limits_{\mathbb{R}^n} |f(x)|^{q(x)} dx < \infty.$$

This is a Banach space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n} := \inf \Big\{ \lambda > 0 : S_{q(\cdot)}\big(f/\lambda\big) \le 1 \Big\}.$$

In the sequel we will use the following notation:

$$p_-(E) := \inf_{x \in E} p(x), \; p_+(E) := \sup_{x \in E} p(x), \; p_+ := p_+(\mathbb{R}^n), \; p_- := p_-(\mathbb{R}^n),$$

where p is a measurable function on $\mathbb{R}^n$ and E is a measurable set in $\mathbb{R}^n$.

Definition 1. We say that a measurable function p on $\mathbb{R}^n$ belongs to $\mathcal{P}$ ($p \in \mathcal{P}$) if

(i) $1 < p_- \le p_+ < \infty$;

(ii) there is a positive constant A such that

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}; \;\; 0 < |x-y| \le 1/2; \;\; x, y \in \mathbb{R}^n;$$

(iii) p is constant outside some ball $B(0,R) := \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$.

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Let

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n,$$

be the Hardy-Littlewood maximal function.

The boundedness of the operator M in $L^{p(\cdot)}(\mathbb{R}^n)$ under the condition $p \in \mathcal{P}$ was established in [4].

Further, a kernel k on $\mathbb{R}^n \times \mathbb{R}^n$, which is a locally integrable complex-valued function defined off the diagonal, is said to satisfy the standard estimates if there exist $\delta > 0$ and $A > 0$, such that for all distinct $x, y \in \mathbb{R}^n$ and all $z \in \mathbb{R}^n$ with $|x - z| < \frac{1}{2}|x - y|$ there holds:

$$(a) \qquad |k(x,y)| \le A|x-y|^{-n};$$

$$(b) \qquad |k(x,y) - k(z,y)| \le A|x-z|^{\delta}|x-y|^{-n-\delta};$$

$$(c) \qquad |k(y,x) - k(y,z)| \le A|x-z|^{\delta}|x-y|^{-n-\delta}.$$

We say that a continuous linear operator $K : C_0^{\infty}(\mathbb{R}^n) \to \mathbb{D}'(\mathbb{R}^n)$, where $\mathbb{D}(\mathbb{R}^n)$ is the space of distributions, is associated with a kernel k if

$$< Kf, g >= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} k(x,y)f(y)g(x)dxdy,$$

whenever $f, g \in C_0^{\infty}(\mathbb{R}^n)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$. K is said to be a singular integral operator if K is associated to a standard kernel. If, in addition, K extends to a bounded, linear operator on $L^2(\mathbb{R}^n)$, then we call K a Calderón-Zygmund operator.

Together with M and K we are interested in the Riesz potential operator

$$I_{\alpha}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \;\; x \in \mathbb{R}^n, \;\; 0 < \alpha < n.$$

Now we formulate the main statements of this note.

Theorem 1. *Let $p \in \mathcal{P}$ and let v and w be positive increasing functions on $\mathbb{R}_+$. Then the two–weight inequality*

$$\|v(|\cdot|)Nf(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le c\|w(|\cdot|)f(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{1}$$

where N is M, holds if and only if

$$\sup_{t>0} \Big\| \frac{v(|\cdot|)}{|\cdot|^n} \chi_{\{|\cdot|>t\}}(\cdot) \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1}(|\cdot|)\chi_{\{|\cdot|<t\}}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty. \tag{2}$$

Theorem 2. *Let $p \in \mathcal{P}$. Suppose that v and w are positive increasing functions on $\mathbb{R}_+$. Then inequality (1) for $N = K$ holds if condition (2) is satisfied. Conversely, if (1) holds for $N = H$, where H is the the Hilbert transform $Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(t)}{(x-t)} dt$, then condition (2) is satisfied for $n = 1$.*

Theorem 3. *Let $p \in \mathcal{P}$. Suppose that v and w are positive decreasing functions on $\mathbb{R}_+$. If*

$$\sup_{t>0} \Big\| v(|\cdot|)\chi_{\{|\cdot|<t\}}(\cdot)\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big\| \frac{w^{-1}(|\cdot|)}{|\cdot|^n}\chi_{\{|\cdot|>t\}}(\cdot)\Big\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty, \qquad (3)$$

then inequality (1) *for $N = K$ holds. Conversely, if we have* (1) *for $N = H$, where H is the Hilbert transform, then* (3) *is satisfied for $n = 1$.*

For fractional integrals we have the next two statements:

Theorem 4. *Let $p \in \mathcal{P}$. We set $q(x) = \frac{np(x)}{n-\alpha p(x)}$, where α is the constant satisfying the condition $0 < \alpha < n/p_+$. Let v and w be positive increasing functions on $\mathbb{R}_+$. Then the two–weight inequality*

$$\|v(|\cdot|)I_\alpha f(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c\|w(|\cdot|)f(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \qquad (4)$$

holds if and only if

$$\sup_{t>0} \Big\| \frac{v(|\cdot|)}{|\cdot|^{n-\alpha}}\chi_{\{|\cdot|>t\}}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\| w^{-1}(|\cdot|)\chi_{\{|\cdot|<t\}}(\cdot)\Big\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty.$$

Theorem 5. *Let $p \in \mathcal{P}$. We set $q(x) = \frac{np(x)}{n-\alpha p(x)}$, where α is the constant satisfying the condition $0 < \alpha < n/p_+$. Let v and w be positive decreasing functions on $\mathbb{R}_+$. Then the inequality* (4) *holds if and only if*

$$\sup_{t>0} \Big\| v(|\cdot|)\chi_{\{|\cdot|<t\}}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\| \frac{w^{-1}(|\cdot|)}{|\cdot|^{n-\alpha}}\chi_{\{|\cdot|>t\}}(\cdot)\Big\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty.$$

Weighted inequalities with power–type weights for classical integral operators were established in the papers [11]–[14], [7], [16], while the two–weight problem for general weights was studied in [2], [3], [9], [13], [15], [5]. In the paper [6] a solution of the one–weight problem in terms of Muckenhoupt–type conditions is given. Sawyer–type two–weight criteria for maximal operators were presented in [10]. For two–weight estimates regarding potentials and singular integrals we refer, e.g., to the monographs [8], [1] and references cited therein.

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