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TWO–WEIGHT CRITERIA FOR ONE–SIDED STRONG FRACTIONAL MAXIMAL FUNCTIONS

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In this note we present two–weight criteria for one–sided strong fractional maximal operators, provided that the right –hand side weight is a product of one–dimensional weights. As a corollary we have necessary and sufficient condition governing the trace inequality for these operators. Fefferman–Stein – type inequalities are discussed as well.

Two–weight estimates for two–sided strong fractional maximal operators and potentials with multiple kernels were derived in [6] (see also [5]) in terms of integral–type conditions. Necessary and sufficient conditions on a weight function v governing the boundedness of one–sided potential operators with product kernels from L^p to L^q_v were given in the papers [3] and [4]. For Sawyer– type two–weight conditions regarding the two–sided strong Hardy–Littlewood maximal operator we refer to [7], where the two–weight inequality was established provided that the right –hand side weight is a product of single variable weights or belongs to the Muckenhoupt A_p class.

For two–weight criteria for one–dimensional one–sided operators see the monographs [2], [1] and references cited therein.

We denote by D the set of all dyadic intervals in $\mathbb{R}$.

Definition 1. A measure μ on $\mathbb{R}$ satisfies the dyadic doubling condition ($\mu \in DC^{(d)}(\mathbb{R})$) if there exists a constant $a > 1$ such that

$$\mu(I) \leq a\mu(I')$$

for all I', $I \in D$ with $I' \subset I$ and $|I| = 2|I'|$. Further, μ satisfies the doubling condition on $\mathbb{R}$ ($\mu \in DC(\mathbb{R})$) if there is a constant $b > 1$ such that

$$\mu(2I) \leq b\mu(I)$$

for all intervals $I \subset \mathbb{R}$.

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Definition 2. We say that a measure μ defined on $\mathbb{R}$ satisfies the dyadic reverse doubling condition $\big(\mu \in RD^{(d)}(\mathbb{R})\big)$ if there exists a constant $c > 1$ such that

$$\mu(I) \geq c\mu(I'),$$

for all I', $I \in D$ with $I' \subset I$ and $|I| = 2|I'|$.

Remark 1. If $\mu \in DC(\mathbb{R})$, then $\mu \in DC^{(d)}(\mathbb{R})$. Further, $\mu \in RD^{(d)}(\mathbb{R})$ when $\mu \in DC^{(d)}(\mathbb{R})$.

Let $1 < p < \infty$ and let ρ be a weight function on $\mathbb{R}^2$. We denote by $L^p_\rho(\mathbb{R}^2)$ the Lebesgue space with weight ρ. If $\rho \equiv 1$, then $L^p_\rho(\mathbb{R}^2) := L^p(\mathbb{R}^2)$ is the classical Lebesgue space.

Further, we will use the notation $\rho(E) := \int_E \rho(x)dx$ for a weight ρ.

We are interested in the following maximal operators:

$$(M^{+,+}_{\alpha,\beta} f)(x,y) := \sup_{h,s>0} \frac{1}{h^{1-\alpha}s^{1-\beta}} \int\limits_{x}^{x+h}\int\limits_{y}^{y+s} |f(t,\tau)|dtd\tau$$

$$(M^{-,-}_{\alpha,\beta} f)(x,y) := \sup_{h,s>0} \frac{1}{h^{1-\alpha}s^{1-\beta}} \int\limits_{x-h}^{x}\int\limits_{y-s}^{y} |f(t,\tau)|dtd\tau,$$

where $0 < \alpha, \beta < 1$.

Theorem 1. *Let $0 < \alpha,\ \beta < 1,\ 1 < p < q < \infty$. Suppose that $w(x,y) = w_1(x)w_2(y)$ with $w_i^{1-p'} \in RD^{(d)}(\mathbb{R})$, $i = 1,2$. Then the following conditions are equivalent:*

(i) $M^{+,+}_{\alpha,\beta}$ *is bounded from $L^p_w(\mathbb{R})$ to $L^q_v(\mathbb{R})$;*

(iii)

$$\sup_{\substack{a,b\in\mathbb{R}\\ h,s>0}} \frac{1}{h^{1-\alpha}s^{1-\beta}} \Bigg[\int\limits_{a-h}^{a}\int\limits_{b-s}^{b} v(x,y)dxdy\Bigg]^{\frac{1}{q}} \Bigg[\int\limits_{a}^{a+h}\int\limits_{b}^{b+s} w^{1-p'}(x,y)dxdy\Bigg]^{\frac{1}{p'}} < \infty.$$

Theorem 2. *Let $1 < p < q < \infty$ and let $0 < \alpha,\ \beta < 1$. Then $M^{-,-}_{\alpha,\beta}$ is bounded from $L^p_w(\mathbb{R}^2)$ to $L^q_v(\mathbb{R}^2)$ if and only if*

$$\sup_{\substack{a,b\in\mathbb{R}\\ h,s>0}} h^{\alpha-1}s^{\beta-1}\Bigg(\int\limits_{a}^{a+h}\int\limits_{b}^{b+s} v(x,y)dxdy\Bigg)^{1/q}\Bigg(\int\limits_{a-h}^{a}\int\limits_{b-s}^{b} w^{1-p'}(x,y)dxdy\Bigg)^{1/p'} < \infty,$$

provided that $w(x,y) = w_1(x)w_2(y)$, where $w_i^{1-p'} \in RD^{(d)}(\mathbb{R})$, $i = 1,2$.

Next, we formulate a special type of the two–weight inequality for one–sided strong fractional maximal functions.

Theorem 3. *Let $1 < p < q < \infty$ and let $1/p - 1/q < \alpha, \beta < 1/p$. Then the following inequality holds*

$$\left(\iint\limits_{\mathbb{R}^2} (M^{+,+}_{\alpha,\beta} f)^q(x,y) v(x,y)dxdy\right)^{1/q} \leq$$

$$\leq c\left(\iint\limits_{\mathbb{R}^2} |f(x,y)|^p (\mathcal{M}^{-,-}_{\alpha,\beta} v)^{p/q}(x,y)dxdy\right)^{1/p},$$

where

$$(\mathcal{M}^{-,-}_{\alpha,\beta} v)(x,y) := \sup_{\substack{h>0\\ s>0}} h^{(\alpha-1/p)q} s^{(\beta-1/p)q} \int\limits_{x-h}^{x}\int\limits_{y-s}^{y} v(t,\tau)dtd\tau$$

and the positive constant c does not depend on f and v.

Theorem 4. *Let $1 < p < q < \infty$ and let $1/p - 1/q < \alpha, \beta < 1/p$. Then the following inequality holds*

$$\left(\iint\limits_{\mathbb{R}^2} (M^{-,-}_{\alpha,\beta} f)^q(x,y) v(x,y)dxdy\right)^{1/q} \leq$$

$$\leq c\left(\iint\limits_{\mathbb{R}^2} |f(x,y)|^p (\mathcal{M}^{+,+}_{\alpha,\beta} v)^{p/q}(x,y)dxdy\right)^{1/p},$$

where

$$(\mathcal{M}^{+,+}_{\alpha,\beta} v)(x,y) := \sup_{\substack{h>0\\ s>0}} h^{(\alpha-1/p)q} s^{(\beta-1/p)q} \int\limits_{x}^{x+h}\int\limits_{y}^{y+s} v(t,\tau)dtd\tau$$

and the positive constant c does not depend on f and v.

The above presented statements give the criteria guaranteeing the trace inequality for one–sided strong fractional maximal operators.

Theorem 5. *Let $1 < p < q < \infty$ and let $0 < \alpha,\ \beta < 1/p$. The following statements are equivalent:*

(i) $M^{+,+}_{\alpha,\beta}$ *is bounded from $L^p(\mathbb{R}^2)$ to $L^q_v(\mathbb{R}^2)$;*

(ii) $M^{-,-}_{\alpha,\beta}$ *is bounded from $L^p(\mathbb{R}^2)$ to $L^q_v(\mathbb{R}^2)$;*

(iii)

$$\sup \left(\int\limits_I \int\limits_J v(x,y)dxdy\right)|I|^{(\alpha-1/p)q}|J|^{(\beta-1/p)q} < \infty,$$

where the supremum is taken over all one-dimensional intervals I and J.

Finally we mention that the appropriate results for mixed–type operators:

$$(M^{+,-}_{\alpha,\beta} f)(x,y) := \sup_{h,s>0} \frac{1}{h^{1-\alpha}s^{1-\beta}} \int\limits_{x}^{x+h} \int\limits_{y-s}^{y} |f(t,\tau)|dtd\tau,$$

$$(M^{-,+}_{\alpha,\beta} f)(x,y) := \sup_{h,s>0} \frac{1}{h^{1-\alpha}s^{1-\beta}} \int\limits_{x-h}^{x} \int\limits_{y}^{y+s} |f(t,\tau)|dtd\tau.$$

also are derived.

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