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Morrey spaces and fractional integral operators

A. Eridani[a, b], Vakhtang Kokilashvili[c], Alexander Meskhi[b, c, *]

[a] *Department of Mathematics, Airlangga University, Campus C, Mulyorejo, Surabaya 60115, Indonesia*
[b] *School of Mathematical Sciences, Government College University, 68-B New Muslim Town, Lahore, Pakistan*
[c] *A. Razmadze Mathematical Institute, M. Aleksidze St., Tbilisi 0193, Georgia*

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Abstract

The present paper is devoted to the boundedness of fractional integral operators in Morrey spaces defined on quasimetric measure spaces. In particular, Sobolev, trace and weighted inequalities with power weights for potential operators are established. In the case when measure satisfies the doubling condition the derived conditions are simultaneously necessary and sufficient for appropriate inequalities.

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1. Introduction

The main purpose of this paper is to establish the boundedness of fractional integral operators in (weighted) Morrey spaces defined on quasimetric measure spaces. We derive Sobolev, trace and two-weight inequalities for fractional integrals. In particular, we

* Corresponding author at: School of Mathematical Sciences, Government College University, 68-B New Muslim Town, Lahore, Pakistan. Tel.: +995 32 32 62 47; fax: +995 32 36 40 86.

E-mail addresses: keumala_ikhsanti@yahoo.com (A. Eridani), kokil@rmi.acnet.ge (V. Kokilashvili), meskhi@rmi.acnet.ge (A. Meskhi).

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generalize: (a) Adams [1] trace inequality; (b) the theorem by Stein and Weiss [18] regarding the two-weight inequality for the Riesz potentials; (c) Sobolev-type inequality. We emphasize that in the most cases the derived conditions are necessary and sufficient for appropriate inequalities.

In the paper [9] (see also [10, Chapter 2]) integral-type sufficient condition guaranteeing the two-weight weak-type inequality for integral operator with positive kernel defined on non-homogeneous spaces was established. In the same paper (see also [10, Chapter 2]) the authors solved the two-weight problem for kernel operators on spaces of homogeneous type.

In [12] (see also [5, Chapter 6]) a complete description of non-doubling measure μ guaranteeing the boundedness of fractional integral operator I_α (see the next section for the definition) from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1 < p < q < \infty$, was given. We notice that this result was derived in [11] for potentials on Euclidean spaces. In [12], theorems of Sobolev and Adams type for fractional integrals defined on quasimetric measure spaces were established. For the boundedness of fractional integrals on metric measure spaces we refer also to [7]. Some two-weight norm inequalities for fractional operators on $\mathbf{R}^n$ with non-doubling measure were studied in [8]. Further, in the paper [13] necessary and sufficient conditions on measure μ governing the inequality of Stein–Weiss type on non-homogeneous spaces were established. For some properties of fractional integrals defined on $\mathbf{R}^n$ in weighted Lebesgue spaces with power type weights see e.g., [16, Chapter 5].

The boundedness of the Riesz potential in Morrey spaces defined on Euclidean spaces was studied in [15,2]. The same problem for fractional integrals on $\mathbb{R}^n$ with non-doubling measure was investigated in [17].

Finally, we mention that necessary and sufficient conditions for the boundedness of maximal operators and Riesz potentials in the local Morrey-type spaces were derived in [3,4].

The main results of this paper were presented in [6].

It should be emphasized that the results of this work are new even for Euclidean spaces.

Constants (often different constants in the same series of inequalities) will generally be denoted by c or C.

2. Preliminaries

Throughout the paper we assume that $X := (X, \rho, \mu)$ is a topological space, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (quasimetric) $\rho : X \times X \longrightarrow [0, \infty)$ satisfying the conditions:

(1) $\rho(x, y) > 0$ for all $x \neq y$, and $\rho(x, x) = 0$ for all $x \in X$;
(2) there exists a constant $a_0 \geqslant 1$, such that $\rho(x, y) \leqslant a_0 \rho(y, x)$ for all $x, y \in X$;
(3) there exists a constant $a_1 \geqslant 1$, such that $\rho(x, y) \leqslant a_1(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$.

We assume that the balls $B(a, r) := \{x \in X : \rho(a, x) < r\}$ are μ-measurable and $0 < \mu(B(a, r)) < \infty$ for $a \in X$, $r > 0$. For every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty$, $\mu\{a\} = 0$, and $B(a, r_2) \setminus B(a, r_1) \neq \emptyset$, for all $a \in X$, $0 < r_1 < r_2 < \infty$.

The triple (X, ρ, μ) will be called quasimetric measure space.

Let $0 < \alpha < 1$. We consider the fractional integral operators I_α, and K_α given by

$$I_\alpha f(x) := \int_X f(y)\rho(x, y)^{\alpha-1}\, d\mu(y),$$

$$K_\alpha f(x) := \int_X f(y)(\mu B(x, \rho(x, y)))^{\alpha-1}\, d\mu(y),$$

for suitable f on X.

Suppose that ν is another measure on X, $\lambda \geqslant 0$ and $1 \leqslant p < \infty$. We deal with the Morrey space $L^{p,\lambda}(X, \nu, \mu)$, which is the set of all functions $f \in L^p_{\mathrm{loc}}(X, \nu)$ such that

$$\|f\|_{L^{p,\lambda}(X,\nu,\mu)} := \sup_B \left(\frac{1}{\mu(B)^\lambda} \int_B |f(y)|^p\, d\nu(y) \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B.

If $\nu = \mu$, then we have the classical Morrey space $L^{p,\lambda}(X, \mu)$ with measure μ. When $\lambda = 0$, then $L^{p,\lambda}(X, \nu, \mu) = L^p(X, \nu)$ is the Lebesgue space with measure ν.

Further, suppose that $\beta \in \mathbf{R}$. We are also interested in weighted Morrey space $M^{p,\lambda}_\beta(X, \mu)$ which is the set of all μ-measurable functions f such that

$$\|f\|_{M^{p,\lambda}_\beta(X,\mu)} := \sup_{a\in X; r>0} \left(\frac{1}{r^\lambda} \int_{B(a,r)} |f(y)|^p \rho(a, y)^\beta\, d\mu(y) \right)^{1/p} < \infty.$$

If $\beta = 0$, then we denote $M^{p,\lambda}_\beta(X, \mu) := M^{p,\lambda}(X, \mu)$.

We say that a measure μ satisfies the growth condition ($\mu \in (GC)$), if there exists $C_0 > 0$ such that $\mu(B(a, r)) \leqslant C_0 r$; further, μ satisfies the doubling condition ($\mu \in (DC)$) if $\mu(B(a, 2r)) \leqslant C_1\, \mu(B(a, r))$ for some $C_1 > 1$. If $\mu \in (DC)$, then (X, ρ, μ) is called a space of homogeneous type (SHT). A quasimetric measure space (X, ρ, μ), where the doubling condition is not assumed, is also called a non-homogeneous space.

The measure μ on X satisfies the reverse doubling condition ($\mu \in (RDC)$) if there are constants η_1 and η_2 with $\eta_1 > 1$ and $\eta_2 > 1$ such that

$$\mu B(x, \eta_1 r) \geqslant \eta_2 \mu B(x, r). \tag{1}$$

It is known (see e.g., [19, p. 11]) that if $\mu \in (DC)$, then $\mu \in (RDC)$.

The next statements are from [12] (see also [5, Theorem 6.1.1, Corollary 6.1.1] and [11] in the case of Euclidean spaces).

Theorem A. *Let (X, ρ, μ) be a quasimetric measure space. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1$. Then I_α is bounded from $L^p(X)$ to $L^q(X)$ if and only if there exists a positive constant C such that*

$$\mu(B(a, r)) \leqslant C r^s, \quad s = \frac{pq(1-\alpha)}{pq + p - q}, \tag{2}$$

for all $a \in X$ and $r > 0$.

Corollary B. *Let (X, ρ, μ) be a quasimetric measure space, $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Then I_α is bounded from $L^p(X)$ to $L^q(X)$ if and only if $\mu \in (GC)$.*

The latter statement by different proof was also derived in [7] for metric spaces.

To prove some of our statements we need the following Hardy-type transform:

$$H_a f(x) := \int_{\rho(a,y) \leqslant \rho(a,x)} f(y)\, d\mu(y),$$

where a is a fixed point of X and $f \in L_{\mathrm{loc}}(X, \mu)$.

Theorem C. *Suppose that (X, ρ, μ) is a quasimetric measure space and $1 < p \leqslant q < \infty$. Assume that v is another measures on X. Let V (resp. W) be non-negative $v \times v$-measurable (resp. non-negative $\mu \times \mu$-measurable) function on $X \times X$. If there exists a positive constant C independent of $a \in X$ and $t > 0$ such that*

$$\left(\int_{\rho(a,y) \geqslant t} V(a, y)\, dv(y)\right)^{1/q} \left(\int_{\rho(a,y) \leqslant t} W(a, y)^{1-p'}\, d\mu(y)\right)^{1/p'} \leqslant C < \infty,$$

then there exists a positive constant c such that for all μ-measurable non-negative f and $a \in X$ the inequality

$$\left(\int_{B(a,r)} (H_a f(x))^q V(a, x)\, dv(x)\right)^{1/q} \leqslant c \left(\int_{B(a,r)} (f(x))^p W(a, x)\, d\mu(x)\right)^{1/p}$$

holds.

This statement was proved in [5, Section 1.1] for Lebesgue spaces.

Proof of Theorem C. Let $f \geqslant 0$. We define $S(s) := \int_{\rho(a,y) < s} f(y)\, d\mu(y)$, for $s \in [0, r]$. Suppose $S(r) < \infty$, then $2^m < S(r) \leqslant 2^{m+1}$, for some $m \in \mathbb{Z}$. Let

$$s_j := \sup\{t : S(t) \leqslant 2^j\}, \quad j \leqslant m \quad \text{and} \quad s_{m+1} := r.$$

Then it is easy to see that (see also [5, pp. 5–8] for details) $(s_j)_{j=-\infty}^{m+1}$ is a non-decreasing sequence, $S(s_j) \leqslant 2^j$, $S(t) \geqslant 2^j$ for $t > s_j$, and

$$2^j \leqslant \int_{s_j \leqslant \rho(a,y) \leqslant s_{j+1}} f(y)\, d\mu(y).$$

If $\beta := \lim_{j \to -\infty} s_j$, then

$$\rho(a, x) < r \quad \Leftrightarrow \quad \rho(a, x) \in [0, \beta] \cup \bigcup_{j=-\infty}^{m} (s_j, s_{j+1}].$$

If $S(r) = \infty$, then we may put $m = \infty$. Since

$$0 \leqslant \int_{\rho(a,y) < \beta} f(y)\, d\mu(y) \leqslant S(s_j) \leqslant 2^j,$$

for every j, therefore $\int_{\rho(a,y)<\beta} f(y)\,d\mu(y) = 0$. From these observations, we have

$$\begin{aligned}
&\int_{\rho(a,x)<r} (H_a f(x))^q V(a,x)\,dv(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}} (H_a f(x))^q V(a,x)\,dv(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}} V(a,x)\left(\int_{\rho(a,y)\leqslant s_{j+1}} (f(y))\,d\mu(y)\right)^q dv(x).
\end{aligned}$$

Notice that

$$\int_{\rho(a,y)\leqslant s_{j+1}} f\,d\mu \leqslant S(s_{j+2}) \leqslant 2^{j+2} \leqslant C \int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} f\,d\mu.$$

Using Hölder's inequality, we find that

$$\begin{aligned}
&\int_{\rho(a,x)<r} (H_a f(x))^q V(a,x)\,d\mu(x)\\
&\leqslant \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}} V(a,x)\left(\int_{\rho(a,y)\leqslant s_{j+1}} (f(y))\,d\mu(y)\right)^q dv(x)\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}} V(a,x)\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} (f(y))\,d\mu(y)\right)^q dv(x)\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,x)\leqslant s_{j+1}} V(a,x)\,dv(x)\\
&\quad\times\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\quad\times\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\\
&\leqslant C \sum_{j=-\infty}^{m} \int_{s_j\leqslant\rho(a,y)} V(a,y)\,dv(y)\left(\int_{\rho(a,y)\leqslant s_j} W(a,y)^{1-p'}\,d\mu(y)\right)^{q/p'}\\
&\left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\leqslant C \sum_{j=-\infty}^{m} \left(\int_{s_{j-1}\leqslant\rho(a,y)\leqslant s_j} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}\\
&\leqslant C \left(\int_{\rho(a,y)\leqslant r} (f(y))^p W(a,y)\,d\mu(y)\right)^{q/p}.
\end{aligned}$$

This completes the proof of the theorem. □

For our purposes we also need the following lemma (see [14] for the case of $\mathbf{R}^n$).

Lemma D. *Suppose that* (X, ρ, μ) *be an SHT. Let* $0 < \lambda < 1 \leqslant p < \infty$. *Then there exists a positive constant C such that for all balls* B_0,

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)} \leqslant C\mu(B_0)^{(1-\lambda)/p}.$$

Proof. Let $B_0 := B(x_0, r_0)$ and $B := B(a, r)$. We have

$$\|\chi_{B_0}\|_{L^{p,\lambda}(X,\mu)} = \sup_B \left(\frac{\mu(B_0 \cap B)}{\mu(B)^\lambda}\right)^{1/p}.$$

Suppose that $B_0 \cap B \neq \emptyset$. Let us assume that $r \leqslant r_0$. Then (see [19, Lemma 1] or [10, p. 9]) $B \subset B(x_0, br_0)$, where $b = a_1(1 + a_0)$. By the doubling condition it follows that

$$\begin{aligned}\frac{\mu(B \cap B_0)}{\mu(B)^\lambda} \leqslant \frac{\mu(B)}{\mu(B)^\lambda} &= \mu(B)^{1-\lambda} \leqslant \mu(B(x_0, br_0))^{1-\lambda}\\ &\leqslant C\mu(B_0)^{1-\lambda}.\end{aligned}$$

Let now $r_0 < r$. Then $\mu B_0 \leqslant c\mu B$, where the constant c depends only on a_1 and a_0. Then

$$\frac{\mu(B \cap B_0)}{\mu(B)^\lambda} \leqslant c\frac{\mu(B_0)}{\mu(B_0)^\lambda} = c\mu(B_0)^{1-\lambda}. \qquad \square$$

The next lemma may be well known but we prove it for the completeness.

Lemma E. *Let* (X, ρ, μ) *be a non-homogeneous space with the growth condition. Suppose that* $\sigma > -1$. *Then there exists a positive constant c such that for all* $a \in X$ *and* $r > 0$, *the inequality*

$$I(a, r, \sigma) := \int_{B(a,r)} \rho(a, x)^\sigma \, d\mu \leqslant cr^{\sigma+1}$$

holds.

Proof. Let $\sigma \geqslant 0$. Then the result is obvious because of the growth condition for μ. Further, assume that $-1 < \sigma < 0$. We have

$$\begin{aligned}I(a, r, \sigma) &= \int_0^\infty \mu\{x \in B(a, r) : \rho(a, x)^\sigma > \lambda\}\, d\lambda\\ &= \int_0^\infty \mu(B(a, r) \cap B(a, \lambda^{1/\sigma}))\, d\lambda = \int_0^{r^\sigma} + \int_{r^\sigma}^\infty := I^{(1)}(a, r, \sigma) + I^{(2)}(a, r, \sigma).\end{aligned}$$

By the growth condition for μ we have

$$I^{(1)}(a, r, \sigma) \leqslant r^\sigma \mu(B(a, r)) \leqslant cr^{\sigma+1},$$

while for $I^{(2)}(a, r, \sigma)$ we find that

$$I^{(2)}(a, r, \sigma) \leqslant c\int_{r^\sigma}^\infty \lambda^{1/\sigma}\, d\lambda = \frac{-c(\sigma + 1)}{\sigma} r^{\sigma+1} = c_1 r^{\sigma+1}$$

because $1/\sigma < -1$. $\square$

The following statement is the trace inequality for the operator K_α (see [1] for the case of Euclidean spaces and, e.g., [10] or [5, Theorem 6.2.1] for an SHT).

Theorem F. *Let (X, ρ, μ) be an SHT. Suppose that $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Assume that ν is another measure on X. Then K_α is bounded from $L^p(X, \mu)$ to $L^q(X, \nu)$ if and only if*

$$\nu B \leqslant c(\mu B)^{q(1/p-\alpha)},$$

for all balls B in X.

3. Main results

In this section we formulate the main results of the paper. We begin with the case of an SHT.

Theorem 3.1. *Let (X, ρ, μ) be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \lambda_1 < 1 - \alpha p$ and $\lambda_2/q = \lambda_1/p$. Then K_α is bounded from $L^{p,\lambda_1}(X, \mu)$ to $L^{q,\lambda_2}(X, \nu, \mu)$ if and only if there is a positive constant c such that*

$$\nu(B) \leqslant c\mu(B)^{q(1/p-\alpha)}, \tag{3}$$

for all balls B.

The next statement is a consequence of Theorem 3.1.

Theorem 3.2. *Let (X, ρ, μ) be an SHT and let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1/p$, $0 < \lambda_1 < 1 - \alpha p$ and $\lambda_2/q = \lambda_1/p$. Then for the boundedness of K_α from $L^{p,\lambda_1}(X, \mu)$ to $L^{q,\lambda_2}(X, \mu)$ it is necessary and sufficient that $q = p/(1 - \alpha p)$.*

For non-homogeneous spaces we have the following statements:

Theorem 3.3. *Let (X, ρ, μ) be a non-homogeneous space with the growth condition. Suppose that $1 < p \leqslant q < \infty$, $1/p - 1/q \leqslant \alpha < 1$ and $\alpha \neq 1/p$. Suppose also that $p\alpha - 1 < \beta < p - 1$, $0 < \lambda_1 < \beta - \alpha p + 1$ and $\lambda_1 q = \lambda_2 p$. Then I_α is bounded from $M_\beta^{p,\lambda_1}(X, \mu)$ to $M_\gamma^{q,\lambda_2}(X, \mu)$, where $\gamma = q(1/p + \beta/p - \alpha) - 1$.*

Theorem 3.4. *Suppose that (X, ρ, μ) is a quasimetric measure space and μ satisfies condition (2). Let $1 < p < q < \infty$. Assume that $0 < \alpha < 1$, $0 < \lambda_1 < p/q$ and $s\lambda_1/p = \lambda_2/q$. Then the operator I_α is bounded from $M^{p,\lambda_1 s}(X, \mu)$ to $M^{q,\lambda_2}(X, \mu)$.*

4. Proof of the main results

In this section we give the proofs of the main results.

Proof of Theorem 3.1. *Necessity*: Suppose K_α is bounded from $L^{p,\lambda_1}(\mu)$ to $L^{q,\lambda_2}(X, \nu, \mu)$. Fix $B_0 := B(x_0, r_0)$. For $x, y \in B_0$, we have that

$$B(x, \rho(x, y)) \subseteq B(x, a_1(a_0 + 1)r_0) \subseteq B(x_0, a_1(1 + a_1(a_0 + 1))r_0).$$

Hence using the doubling condition for μ, it is easy to see that

$$\mu(B_0)^{\alpha} \leqslant cK_{\alpha}\chi_{B_0}(x), \quad x \in B_0.$$

Consequently, using the condition $\lambda_2/q = \lambda_1/p$, the boundedness of K_α from $L^{p,\lambda}(X,\mu)$ to $L^{q,\lambda_2}(X,\nu,\mu)$ and Lemma D we find that

$$\mu(B_0)^{\alpha-\lambda_1/p}\nu(B_0)^{1/q} \leqslant c\|K_\alpha\chi_{B_0}\|_{L^{q,\lambda_2}(X,\nu,\mu)} \leqslant c\|\chi_{B_0}\|_{L^{p,\lambda_1}(X,\mu)} \leqslant c\mu(B_0)^{(1-\lambda_1)/p}.$$

Since c does not depend on B_0 we have condition (3).

Sufficiency: Let $B := B(a,r)$, $\tilde{B} := B(a, 2a_1 r)$ and $f \geqslant 0$. Write $f \in L^{p,\lambda_1}(\mu)$ as $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^C}$, where χ_B is a characteristic function of B. Then we have

$$S := \int_B (K_\alpha f(x))^q \, d\nu(x) \leqslant c\left(\int_B (K_\alpha f_1(x))^q \, d\nu(x) + \int_B (K_\alpha f_2(x))^q \, d\nu(x)\right) := c(S_1 + S_2).$$

Applying Theorem F and the fact $\mu \in (DC)$ we find that

$$S_1 \leqslant \int_X (K_\alpha f_1)^q(x)\, d\nu(x) \leqslant c\left(\int_{B(a,2a_1 r)} (f(x))^p \, d\mu(x)\right)^{q/p}.$$

Now observe that if $\rho(a,x) < r$ and $\rho(a,y) > 2a_1 r$, then $\rho(a,y) > 2a_1\rho(a,x)$. Consequently, using the facts $\mu \in (RDC)$ (see (1)), $0 < \lambda_1 < 1 - \alpha p$ and condition (3) we have

$$\begin{aligned}
S_2 &\leqslant c\int_{B(a,r)} \left(\int_{\rho(a,y)>r} \frac{f(y)}{\mu B(a,\rho(a,y))^{1-\alpha}} \, d\mu(y)\right)^q d\nu(x) \\
&= \nu(B)\left[\sum_{k=0}^{\infty} \int_{B(a,\eta_1^{k+1}r)\setminus B(a,\eta_1^k r)} \frac{f(y)}{\mu B(a,\rho(a,y))^{1-\alpha}} \, d\mu(y)\right]^q \\
&\leqslant c\nu(B)\left[\sum_{k=0}^{\infty}\left(\int_{B(a,\eta_1^{k+1}r)} (f(y))^p \, d\mu(y)\right)^{1/p}\right. \\
&\quad \left.\times\left(\int_{B(a,\eta_1^{k+1}r)\setminus B(a,\eta_1^k r)} \mu B(a,\rho(a,y))^{(\alpha-1)p'} \, d\mu(y)\right)^{1/p'}\right]^q \\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\nu(B)\left(\sum_{k=0}^{\infty} \mu B(a,\eta_1^{k+1}r)^{\lambda_1/p+\alpha-1+1/p'}\right)^q \\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\nu(B)\mu(B)^{(\lambda_1/p+\alpha-1/p)q}\left(\sum_{k=0}^{\infty} \eta_2^{k(\lambda_1/p+\alpha-1/p)}\right)^q \\
&\leqslant c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\mu(B)^{q\lambda_1/p} = c\|f\|^q_{L^{p,\lambda_1}(X,\mu)}\mu(B)^{\lambda_2},
\end{aligned}$$

where the positive constant c does not depend on B. Now the result follows immediately. □

Proof of Theorem 3.2. *Sufficiency*: Assuming $\alpha = 1/p - 1/q$ and $\mu = \nu$ in Theorem 3.1 we have that K_α is bounded from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\mu)$.

Necessity: Suppose that K_α is bounded from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\mu)$. Then by Theorem 3.1 we have

$$\mu(B)^{1/q-1/p+\alpha} \leqslant c.$$

The conditions $\mu(X) = \infty$ and $\mu\{x\} = 0$, for all $x \in X$, implies that $\alpha = 1/p - 1/q$. □

Proof of Theorem 3.3. Let $f \geqslant 0$. For $x,\ a \in X$, let us introduce the following notation:

$$E_1(x) := \left\{ y : \frac{\rho(a,y)}{\rho(a,x)} < \frac{1}{2a_1} \right\}, \quad E_2(x) := \left\{ y : \frac{1}{2a_1} \leqslant \frac{\rho(a,y)}{\rho(a,x)} \leqslant 2a_1 \right\},$$
$$E_3(x) := \left\{ y : 2a_1 < \frac{\rho(a,y)}{\rho(a,x)} \right\}.$$

For $i = 1, 2, 3$, $r > 0$ and $a \in X$, we denote

$$S_i := \int_{\rho(a,x)<r} \rho(a,x)^\gamma \left(\int_{E_i(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^q d\mu(x).$$

If $y \in E_1(x)$, then $\rho(a,x) < 2a_1 a_0 \rho(x,y)$. Hence, it is easy to see that

$$S_1 \leqslant C \int_B \rho(a,x)^{\gamma+q(\alpha-1)} \left(\int_{\rho(a,y)<\rho(a,x)} f(y)\, d\mu(y) \right)^q d\mu(x).$$

Taking into account the condition $\gamma < (1-\alpha)q - 1$ we have

$$\int_{\rho(a,x)>t} \rho(a,x)^{\gamma+q(\alpha-1)}\, d\mu(x) = \sum_{n=0}^{\infty} \int_{B(a,2^{k+1}t)\setminus B(a,2^k t)} (\rho(a,x))^{\gamma+(\alpha-1)q}\, d\mu(x)$$
$$\leqslant c \sum_{n=0}^{\infty} (2^k t)^{\gamma+q(\alpha-1)+1} = ct^{\gamma+q(\alpha-1)+1},$$

while the condition $\beta < p - 1$ implies

$$\int_{\rho(a,x)<t} \rho(a,x)^{\beta(1-p')+1}\, d\mu(x) \leqslant ct^{\beta(1-p')+1}.$$

Hence

$$\sup_{a\in X, t>0} \left(\int_{\rho(a,x)>t} \rho(a,x)^{\gamma+q(\alpha-1)}\, d\mu(x) \right)^{1/q} \left(\int_{B(a,t)} \rho(a,y)^{\beta(1-p')}\, d\mu(y) \right)^{1/p'}$$
$$< \infty.$$

Now using Theorem C we have

$$S_1 \leqslant c \left(\int_B \rho(a,x)^\beta (f(y))\, d\mu(y) \right)^{q/p} \leqslant c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M_\beta^{p,\lambda_1}(X,\mu)} r^{\lambda_2}.$$

Further, observe that if $\rho(a,y)>2a_1\rho(a,x)$, then $\rho(a,y)\leqslant a_1\rho(a,x)+a_1\rho(a,y)\leqslant \rho(a,y)/2+a_1\rho(x,y)$. Hence $\rho(a,y)/(2a_1)\leqslant\rho(x,y)$. Consequently, using the growth condition for μ, the fact $\lambda_1<\beta-\alpha p+1$ and Lemma E we find that

$$
\begin{aligned}
S_3 &\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\int_{\rho(a,y)>\rho(a,x)}\frac{f(y)}{\rho(a,y)^{1-\alpha}}\,d\mu(y)\right)^q d\mu(x)\\
&\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\sum_{k=0}^{\infty}\int_{B(a,2^{k+1}\rho(a,x))\setminus B(a,2^k\rho(a,x))}\frac{f(y)}{\rho(a,y)^{1-\alpha}}\,d\mu(y)\right)^q d\mu(x)\\
&\leqslant c\int_{B(a,r)}\rho(a,x)^{\gamma}\left[\sum_{k=0}^{\infty}\left(\int_{B(a,2^{k+1}\rho(a,x))}f^p(y)\rho(a,y)^{\beta}\,d\mu(y)\right)^{1/p}\right.\\
&\quad\left.\times\left(\int_{B(a,2^{k+1}\rho(a,x))\setminus B(a,2^k\rho(a,x))}\rho(a,y)^{\beta(1-p')+(\alpha-1)p'}\,d\mu(y)\right)^{1/p'}\right]^q d\mu(x)\\
&\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\gamma}\\
&\quad\times\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1-\beta/p}(\mu B(a,2^{k+1}\rho(a,x)))^{1/p'}\right)^q d\mu(x)\\
&\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\gamma}\left(\sum_{k=0}^{\infty}(2^k\rho(a,x))^{\lambda_1/p+\alpha-1/p-\beta/p}\right)^q d\mu(x)\\
&\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{(\lambda_1/p+\alpha-1/p-\beta/p)q+\gamma}\,d\mu(x)\\
&= c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}\int_{B(a,r)}\rho(a,x)^{\lambda_1 q/p-1}\,d\mu(x)\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}r^{\lambda_1 q/p}\\
&= c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}r^{\lambda_2}.
\end{aligned}
$$

So, we conclude that

$$S_3\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)}r^{\lambda_2}.$$

To estimate S_2 we consider two cases. First assume that $\alpha<1/p$. Let

$$E_{k,r}:=\{x:2^k r\leqslant\rho(a,x)<2^{k+1}r\},$$

$$F_{k,r}:=\{x:2^{k-1}r/a_1\leqslant\rho(a,x)<a_1 2^{k+2}r\}.$$

Assume that $p^* = p/(1-\alpha p)$. By Hölder's inequality, Corollary B and the assumption $\gamma = q(1/p + \beta/p - \alpha) - 1$ we have

$$\begin{aligned}
S_2 &= \sum_{k=-\infty}^{-1} \int_{E_{k,r}} \rho(a,x)^{\gamma} \left(\int_{E_2(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^q d\mu(x) \\
&\leqslant \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma} \left(\int_{E_2(x)} f(y)\rho(x,y)^{\alpha-1}\, d\mu(y) \right)^{p^*} d\mu(x) \right)^{q/p'} \\
&\quad \times \left(\int_{E_{k,r}} \rho(a,x)^{\gamma p^*/(p^*-q)}\, d\mu(x) \right)^{(p^*-q)/p^*} \\
&\leqslant c \sum_{k=-\infty}^{-1} 2^{k(\gamma+(p^*-q)/p^*)} \left(\int_X I_\alpha(f\chi_{F_{k,r}})(x)^{p^*}\, d\mu(x) \right)^{q/p^*} \\
&\leqslant c \sum_{k=-\infty}^{-1} 2^{k(\gamma+(p^*-q)/p^*)} \left(\int_{F_{k,r}} (f(x))^p\, d\mu(x) \right)^{q/p} \\
&\leqslant c \left(\int_{B(a,2a_1r)} \rho(a,x)^{\beta} (f(x))^p\, d\mu(x) \right)^{q/p} \\
&\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)} r^{\lambda_2}.
\end{aligned}$$

Let us now consider the case $1/p < \alpha < 1$.

First notice that (see [13])

$$\int_{E_2(x)} (\rho(x,y)^{(\alpha-1)p'}\, d\mu(y) \leqslant c\rho(a,x)^{1+(\alpha-1)p'},$$

where the positive constant c does not depend on a and x.

This estimate and Hölder's inequality yield

$$\begin{aligned}
S_2 &\leqslant c \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma+[(\alpha-1)p'+1)]q/p'} \left(\int_{E_2(x)} (f(y))^p\, d\mu(y) \right)^{q/p} d\mu(x) \right)^{q/p'} \\
&\leqslant c \sum_{k=-\infty}^{-1} \left(\int_{E_{k,r}} \rho(a,x)^{\gamma+[(\alpha-1)p'+1)]q/p'}\, d\mu(x) \right) \left(\int_{F_{k,r}} (f(y))^p\, d\mu(y) \right)^{q/p} \\
&\leqslant c \sum_{k=-\infty}^{-1} (2^k r)^{\gamma+[(\alpha-1)p'+1)]q/p'+1} \left(\int_{F_{k,r}} (f(y))^p\, d\mu(y) \right)^{q/p} \\
&= c \sum_{k=-\infty}^{-1} 2^{k\beta q/p} \left(\int_{F_{k,r}} (f(y))^p\, d\mu(y) \right)^{q/p} \leqslant c \left(\int_{B(a,2a_1r)} (f(y))^p \rho(a,y)^{\beta}\, d\mu(y) \right)^{q/p} \\
&\leqslant c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)} r^{\lambda_1 q/p} = c\|f\|^q_{M^{p,\lambda_1}_{\beta}(X,\mu)} r^{\lambda_2}.
\end{aligned}$$

Now the result follows immediately. □

Proof of Theorem 3.4. Let $f \geqslant 0$. Suppose that $a \in X$ and $r > 0$. Suppose also that $f_1 = f\chi_{B(a,2a_1r)}$ and $f_2 = f - f_1$. Then $I_\alpha f = I_\alpha f_1 + I_\alpha f_2$. Consequently,

$$\int_{B(a,r)} (I_\alpha f(x))^q \, d\mu(x) \leqslant 2^{q-1} \left(\int_{B(a,r)} (I_\alpha f_1(x))^q \, d\mu(x) + \int_{B(a,r)} (I_\alpha f_2(x))^q \, d\mu(x) \right)$$
$$:= 2^{q-1}(S^{(1)}_{a,r} + S^{(2)}_{a,r}).$$

Due to Theorem A and the condition $s\lambda_1/p = \lambda_2/q$ we have

$$S^{(1)}_{a,r} \leqslant c \left(\int_{B(a,2a_1r)} (f(x))^p \, d\mu(x) \right)^{q/p}$$
$$= c \left(\frac{1}{(2a_1r)^{\lambda_1 s}} \int_{B(a,2a_1r)} (f(x))^p \, dx \right)^{q/p} r^{\lambda_1 sq/p} \leqslant c \|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_2}.$$

Now observe that if $x \in B(a, r)$ and $y \in X \backslash B(a, 2a_1r)$, then $\rho(a, y)/2a_1 \leqslant \rho(x, y)$. Hence Hölder's inequality, condition (2) and the condition $0 < \lambda_1 < p/q$ yield

$$I_\alpha f_2(x) = \int_{X\backslash B(a,2a_1r)} f(y)/\rho(x, y)^{1-\alpha} \, d\mu(y)$$
$$= \sum_{k=0}^{\infty} \left(\int_{B(a,2^{k+2}a_1r)\backslash B(a,2^{k+1}a_1r)} (f(y))^p \, d\mu(y) \right)^{1/p}$$
$$\times \left(\int_{B(a,2^{k+2}a_1r)\backslash B(a,2^{k+1}a_1r)} \rho(a, y)^{(\alpha-1)p'} \, d\mu(y) \right)^{1/p'}$$
$$\leqslant c \sum_{k=0}^{\infty} \left(\frac{1}{(2^{k+1}a_1r)^{\lambda_1 s}} \int_{B(a,2^{k+1}a_1r)} (f(y))^p \, d\mu(y) \right)^{1/p}$$
$$\times (2^k a_1 r)^{\lambda_1 s/p+\alpha-1+s/p'}$$
$$\leqslant c \|f\|_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_1 s/p+\alpha-1+s/p'}.$$

Consequently, by the assumptions $s\lambda_1/p = \lambda_2/q$ and $s = pq(1-\alpha)/(pq + p - q)$ we conclude that

$$S^{(2)}_{a,r} \leqslant c \|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{(\lambda_1 s/p+\alpha-1+s/p')q+s} = c \|f\|^q_{M^{p,\lambda_1 s}(X,\mu)} r^{\lambda_2}.$$

Summarizing the estimates derived above we finally have the desired result. □

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