V. Kokilashvili, A. Meskhi, M. Sarwar, One and two weight estimates for one-sided operators in $L^{p(\cdot)}$ spaces, *Eurasian Math. J.*, 2010, Volume 1, Number 1, 73–110

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 1 (2010), 73 – 110

ONE AND TWO WEIGHT ESTIMATES FOR ONE–SIDED OPERATORS IN $L^{p(\cdot)}$ SPACES

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Communicated by R. Oinarov

Keywords and phrases: one-sided maximal functions, one-sided potentials, one-weight inequality, two-weight inequality, trace inequality.

Mathematics Subject Classification: 26A33, 42B25, 46E30.

Abstract. Various type weighted norm estimates for one-sided maximal functions and potentials are established in variable exponent Lebesgue spaces $L^{p(\cdot)}$. In particular, sufficient conditions (in some cases necessary and sufficient conditions) governing one and two weight inequalities for these operators are derived. Among other results generalizations of the Hardy–Littlewood, Fefferman–Stein and trace inequalities are given in $L^{p(\cdot)}$ spaces.

1 Introduction

This paper deals with the boundedness of one-sided maximal functions and potentials in weighted Lebesgue spaces with variable exponent. In particular, we derive one-weight inequality for one-sided maximal functions; sufficient conditions (in some cases necessary and sufficient conditions) governing two-weight inequalities for one-sided maximal and potential operators; criteria for the trace inequality for one-sided fractional maximal functions and potentials; Fefferman–Stein type inequality for one-sided fractional maximal functions; generalization of the Hardy-Littlewood theorem for the Riemann–Liouville and Weyl transforms. It is worth mentioning that some results of this paper implies the following fact: the one-weight inequality for one-sided maximal functions automatically holds when both the exponent of the space and the weight are monotonic functions.

The boundedness of one-sided integral operators in $L^{p(\cdot)}$ spaces was proved in [13]. In that paper the authors established the boundedness of the one-sided Hardy–Littlewood maximal functions, potentials and singular integrals in $L^{p(\cdot)}(I)$ spaces with the condition on p which is weaker than the log-Hölder continuity (weak Lipschitz) condition.

Solution of the one-weight problem for one-sided operators in classical Lebesgue spaces was given in [48], [1]. Trace inequalities for one-sided potentials in L^p spaces were characterized in [38], [40], [22]. It should be emphasized that a complete solution of the two-weight problem with transparent integral conditions on weights for one-sided maximal functions and potentials in the non-diagonal case are given in the monographs [16, Chapters 2 and 3], [9, Chapter 2]. For Sawyer-type two-weight criteria for one-sided fractional operators we refer to [35], [36], [34].

Weighted inequalities for classical integral operators in $L^{p(\cdot)}$ spaces were derived in [6], [8], [10]–[14], [19], [23]–[32], [45], [47], etc (see also [21], [44]).

The one-weight problem for the two-sided Hardy–Littlewood maximal operator in $L^{p(\cdot)}$ spaces was solved in [7]. Earlier, some generalizations of the Muckenhoupt condition in these spaces defined on bounded sets were discussed in [30] and [31].

Criteria for the boundedness of two-sided fractional maximal operators from L^p_w to $L^{q(\cdot)}_v$ were given in [24]. Two-weight Sawyer type criteria for two-sided maximal functions on the real line were announced in [23], [25].

In [2] necessary and sufficient conditions on a weight v governing the boundedness compactness of the generalized Riemann–Liouville transform $R_{\alpha(\cdot)}$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}_v(\mathbb{R}_+)$, $\alpha_- > 1/p_-$, were derived.

In Section 1 we give the definition and some essential well-known properties of the Lebesgue space with variable exponent and formulate Carleson–Hörmander type inequalities. In Section 2 we study the one-weight problem for one-sided Hardy–Littlewood maximal operators in $L^{p(\cdot)}$ spaces, while Section 3 is devoted to the same problem for one-sided fractional maximal functions. In Section 4 we derive sufficient (in some cases necessary and sufficient) conditions guaranteeing two-weight $p(\cdot)$ – $q(\cdot)$ norm estimates for one-sided fractional maximal operators. Fefferman–Stein type inequalities in variable exponent spaces are discussed in Section 5. In Section 6 we established criteria governing the trace inequality for the Riemann–Liouville and Weyl operators in $L^{p(\cdot)}$ spaces. In Section 7 we formulate generalization of the Hardy–Littlewood theorem for one-sided potentials in these spaces. Section 8 is dedicated to two-weight inequalities for one-sided operators.

Finally, we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c or C.

2 Preliminaries

Let Ω be an open set in $\mathbb{R}^n$ and let p be a measurable function on Ω. Suppose that

$$1 \leq p_- \leq p_+ < \infty, \tag{1}$$

where p_- and p_+ are the infimum and the supremum respectively of p on Ω. Suppose that ρ is a weight function on Ω, i.e. ρ is an almost everywhere positive locally integrable function on Ω. We say that a measurable function f on Ω belongs to $L^{p(\cdot)}_\rho(\Omega)$ (or $L^{p(x)}_\rho(\Omega)$) if

$$S_{p,\rho}(f) = \int_\Omega \big|f(x)\rho(x)\big|^{p(x)} dx < \infty.$$

It is known that (see, e.g., [33], [26], [28], [42]) $L^{p(\cdot)}_\rho(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}_\rho(\Omega)} = \inf\big\{\lambda>0 : S_{p(\cdot),\rho}\big(f/\lambda\big)\le 1\big\}.$$

If $\rho\equiv 1$, then we use the symbol $L^{p(\cdot)}(\Omega)$ (resp. S_p) instead of $L^{p(\cdot)}_\rho(\Omega)$ (resp. $S_{p(\cdot),\rho}$). It is clear that $\|f\|_{L^{p(\cdot)}_\rho(\Omega)} = \|f\rho\|_{L^{p(\cdot)}(\Omega)}$. It should be also emphasized that when p is constant, then $L^{p(\cdot)}_\rho(\Omega)$ coincides with the classical weighted Lebesgue space.

Further, we denote

$$p_-(E):=\inf_E p;\ \ p_+(E):=\sup_E p, \qquad E\subset\Omega,$$

$$p_-(\Omega)=p_-;\ \ p_+(\Omega)=p_+.$$

The following statement is well-known (see, e.g., [33], [42]):

Proposition A. *Let E be a measurable subset of Ω. Then the following inequalities hold:*

$$\|f\|^{r_+(E)}_{L^{r(\cdot)}(E)} \le S_{r(\cdot)}(f\chi_E) \le \|f\|^{r_-(E)}_{L^{r(\cdot)}(E)}, \ \ \|f\|_{L^{r(\cdot)}(E)}\le 1;$$

$$\|f\|^{r_-(E)}_{L^{r(\cdot)}(E)} \le S_{r(\cdot)}(f\chi_E) \le \|f\|^{r_+(E)}_{L^{r(\cdot)}(E)}, \ \ \|f\|_{L^{r(\cdot)}(E)}\ge 1;$$

$$\Big|\int_E f(x)g(x)dx\Big| \le \Big(\frac{1}{r_-(E)}+\frac{1}{(r_+(E))'}\Big)\ \|f\|_{L^{r(\cdot)}(E)}\ \|g\|_{L^{r'(\cdot)}(E)},$$

where $r'(x)=\frac{r(x)}{r(x)-1}$ and $1<r_-\le r_+<\infty$.

Let I be an open set in $\mathbb{R}$. In the sequel we shall use the notation:

$$I_+(x,h):=[x,x+h]\cap I, \quad I_-(x,h):=[x-h,x]\cap I;$$

$$I(x,h):=[x-h,x+h]\cap I.$$

We introduce the following one–sided maximal operators:

$$\big(M_{\alpha(\cdot)}f\big)(x)=\sup_{h>0}\frac{1}{(2h)^{1-\alpha(x)}}\int_{I(x,h)}|f(t)|dt,$$

$$\big(M^-_{\alpha(\cdot)}f\big)(x)=\sup_{h>0}\frac{1}{h^{1-\alpha(x)}}\int_{I_-(x,h)}|f(t)|dt,$$

$$\big(M^+_{\alpha(\cdot)}f\big)(x)=\sup_{h>0}\frac{1}{h^{1-\alpha(x)}}\int_{I_+(x,h)}|f(t)|dt,$$

where $0<\alpha_-\le\alpha_+<1$, I is an open set in $\mathbb{R}$ and $x\in I$.

If $\alpha\equiv 1$, then $M_{\alpha(\cdot)}$, $M^-_{\alpha(\cdot)}$ and $M^+_{\alpha(\cdot)}$ are the one–sided Hardy–Littlewood maximal operators which are denoted by M, M^- and M^+ respectively.

In [4] L. Diening proved the following statement:

Theorem A. *Let Ω be a bounded open set in $\mathbb{R}^n$. Then the maximal operator*

$$\big(M_\Omega f\big)(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)\bigcap\Omega} |f(y)|dy, \quad x \in \Omega,$$

is bounded in $L^{p(\cdot)}(\Omega)$ if $p \in \mathcal{P}(\Omega)$, that is,

a) $1 < p_- \le p(x) \le p_+ < \infty$;

b) *p satisfies the Dini–Lipschitz (log-Hölder continuity) condition ($p \in DL(\Omega)$): there exists a positive constant A such that for all $x, y \in \Omega$ with $0 < |x-y| \le \frac{1}{2}$ the inequality*

$$\big|p(x) - p(y)\big| \le \frac{A}{\ln \frac{1}{|x-y|}} \tag{2}$$

holds.

The next statement was proved in [3].

Theorem B. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $1 < p_- \le p_+ < \infty$. Then the maximal operator M_Ω is bounded in $L^{p(\cdot)}(\Omega)$ if*

(i) $p \in \mathcal{P}(\Omega)$;

(ii)

$$|p(x) - p(y)| \le \frac{C}{\ln(e + |x|)} \tag{3}$$

for all $x, y \in \Omega$, $|y| \ge |x|$.

We shall also need the following statements:

Proposition B ([33], [42]). *Let $1 \le p(x) \le q(x) \le q_+ < \infty$. Suppose that Ω is an open set in $\mathbb{R}^n$ with $|\Omega| < \infty$. Then the inequality*

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le (1 + |\Omega|)\|f\|_{L^{q(\cdot)}(\Omega)}$$

holds.

Proposition C ([4]). *Let Ω be an open set in $\mathbb{R}^n$ and let p and q be bounded exponents on Ω. Then*

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

if and only if $p(x) \le q(x)$ almost everywhere on Ω and there is a constant $0 < K < 1$ such that

$$\int_\Omega K^{p(x)q(x)/(|q(x)-p(x)|)} dx < \infty. \tag{4}$$

Remark A. In the previous statement it is used the convention $K^{1/0} := 0$.

Definition A ([13]). Let $\mathcal{P}_-(I)$ be the class of all measurable positive functions $p : I \to \mathbb{R}$ satisfying the following condition: there exist a positive constant C_1 such that for a.e $x \in I$ and a.e $y \in I$ with $0 < x - y \le \frac{1}{2}$ the inequality

$$p(x) \le p(y) + \frac{C_1}{\ln\left(\frac{1}{x-y}\right)} \tag{5}$$

holds. Further, we say that p belongs to $\mathcal{P}_+(I)$ if p is positive function on I and there exists a positive constant C_2 such that for a.e $x \in I$ and a.e $y \in I$ with $0 < y-x \leq \frac{1}{2}$ the inequality

$$p(x) \leq p(y) + \frac{C_2}{\ln\left(\frac{1}{y-x}\right)} \tag{6}$$

is fulfilled.

Definition B. We say that a measurable positive function on I belongs to the class $\mathcal{P}_\infty(I)$ ($p \in \mathcal{P}_\infty(I)$) if (3) holds for all $x, y \in I$ with $|y| \geq |x|$. .

We shall also need the following definition:

Definition C. Let p be a measurable function on unbounded interval I in $\mathbb{R}$. We say that $p \in \mathcal{G}(I)$ if there is a constant $0 < K < 1$ such that

$$\int_I K^{p(x)p_-/(p(x)-p_-)}dx < \infty.$$

Theorem C ([13]). *Let I be a bounded interval in $\mathbb{R}$. Suppose that $1 < p_- \leq p_+ < \infty$. Then*

(i) *if $p \in \mathcal{P}_-(I)$, then M^- is bounded in $L^{p(\cdot)}(I)$;*

(ii) *if $p \in \mathcal{P}_+(I)$, then M^+ is bounded in $L^{p(\cdot)}(I)$.*

In the case of unbounded set we have

Theorem D ([13]). *Let I be an arbitrary open set in $\mathbb{R}$. Suppose that $1 < p_- \leq p_+ < \infty$. If $p \in \mathcal{P}_+(I) \cap \mathcal{P}_\infty(I)$, then the operator M^+ is bounded in $L^{p(\cdot)}(I)$. Further, if $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$. Then the operator M^- is bounded in $L^{p(\cdot)}(I)$*

In particular, the previous statement yields

Theorem E ([13]). *Let $I = \mathbb{R}_+$ and let $1 < p_- \leq p_+ < \infty$. Suppose that $p \in \mathcal{P}_+(I)$ and there is a positive number a such that $p \in \mathcal{P}_\infty((a,\infty))$. Then M^+ is bounded in $L^{p(\cdot)}(I)$. Further, if $p \in \mathcal{P}_-(I)$ and there is a positive number a such that $p \in \mathcal{P}_\infty((a,\infty))$, then M^- is bounded in $L^{p(\cdot)}(I)$.*

The next statement gives one–weight criteria for one-sided maximal operators in classical Lebesgue spaces (see [48], [1]).

Theorem F ([1]). *Let $I \subseteq \mathbb{R}$ be an interval. Assume that $0 \leq \alpha < 1$ and $1 < p < 1/\alpha$, where p and α are constants ($1/\alpha = \infty$ if $\alpha = 0$). We set $1/q = 1/p - \alpha$.*

(i) *Let $T := M_\alpha^-$. Then the inequality*

$$\left[\int_I |Tf(x)|^q v(x)dx\right]^{1/q} \leq C\left[\int_I |f(x)|^p v^{p/q}(x)dx\right]^{1/p} \tag{7}$$

holds if and only if

$$\sup_{h>0}\left(\frac{1}{h}\int\limits_{I_+(x,x+h)} v(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int\limits_{I_-(x-h,x)} v^{-p'/q}(t)dt\right)^{\frac{1}{p'}} < \infty. \tag{8}$$

(ii) *Let* $T := M_\alpha^+$. *Then* (7) *holds if and only if*

$$\sup_{h>0}\bigg(\frac{1}{h}\int\limits_{I_-(x-h,x)} v(t)dt\bigg)^{\frac{1}{q}}\bigg(\frac{1}{h}\int\limits_{I_+(x,x+h)} v^{-p'/q}(t)dt\bigg)^{\frac{1}{p'}} < \infty. \tag{9}$$

Definition D. Let $I \subseteq \mathbb{R}_+$ be an interval. Suppose that $1 < p < q < \infty$, where p and q are constants. We say that the weight $v \in A^-_{p,q}(I)$ (resp. $v \in A^+_{p,q}(I)$) if (8) (resp. (9)) holds.

If $p = q$, then we denote the class $A^+_{p,q}(I)$ (resp. $A^-_{p,q}(I)$) by $A^+_p(I)$ (resp. $A^-_p(I)$).

Notice that $v \in A^+_{p,q}(I)$ (resp. $v \in A^-_{p,q}(I)$) is equivalent to the condition $v \in A^+_{1+q/p'}(I)$ (resp. $v \in A^-_{1+q/p'}(I)$).

Further, we denote by $D(\mathbb{R})$ (resp. $D(\mathbb{R}_+)$) a dyadic lattice in $\mathbb{R}$ (resp. in $\mathbb{R}_+$).

Definition E. We say that a measure μ belongs to the class $RD^{(d)}(\mathbb{R}^n)$ (dyadic reverse doubling condition) if there exists a constant $\delta > 1$, such that for all dyadic cubes Q and Q', $Q \subset Q'$, $|Q| = \frac{|Q'|}{2^n}$, the inequality

$$\mu(Q') \geq \delta\mu(Q)$$

holds.

Definition F. We say that measure μ on $\mathbb{R}^n$ satisfies the doubling condition ($\mu \in DC(\mathbb{R}^n)$) if there is a positive number b such that

$$\mu B(x, 2r) \leq b\mu B(x, r)$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

It is known (see [51], p. 11) that if $\mu \in DC(\mathbb{R}^n)$, then $\mu \in RD(\mathbb{R}^n)$, i.e., there are positive constants η_1 and η_2, $0 < \eta_1, \eta_2 < 1$, such that

$$\mu B(x, \eta_1 r) \leq \eta_2\mu B(x, r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

It is easy to check that if $\mu \in DC(\mathbb{R}^n)$, then $\mu \in RD^{(d)}(\mathbb{R})$.

We shall need some lemmas giving Carleson-Hörmandar type inequalities.

Lemma 2.1 ([52])**.** *Let* $1 < p \leq r < \infty$ *and let* $\rho^{-p'} \in RD^{(d)}(\mathbb{R}^n)$, *where* ρ *is a weight function on* $\mathbb{R}^n$. *Then there is a positive constant* C *such that for all non-negative* f *the inequality*

$$\sum_{Q\in D(\mathbb{R}^n)}\bigg(\int\limits_Q \rho^{-p'}(x)dx\bigg)^{-\frac{r}{p'}}\bigg(\int\limits_Q f(y)dy\bigg)^r \leq C\bigg(\int\limits_{\mathbb{R}^n}(f(x)\rho(x))^p dx\bigg)^{\frac{1}{p}}$$

holds.

Lemma 2.2 ([50], [53]). *Let $u(x) \geq 0$ on $\mathbb{R}^n$; $\{Q_i\}_{i\in A}$ is a countable collection of dyadic cubes in $\mathbb{R}^n$ and $\{a_i\}_{i\in A}$, $\{b_i\}_{i\in A}$ be positive numbers satisfying*

$$(i) \qquad \int_{Q_i} u(x)dx \leq Ca_i \qquad \text{for all} \quad i \in A;$$

$$(ii) \qquad \sum_{j:\ Q_j\subset Q_i} b_j \leq Ca_i \qquad \text{for all } i \in A.$$

Then

$$\left(\sum_{i\in I} b_i \left(\frac{1}{a_i}\int_{Q_i} g(x)u(x)dx\right)^p\right)^{\frac{1}{p}} \leq C_p\left(\int_{\mathbb{R}^n} g^p(x)u(x)dx\right)^{\frac{1}{p}}$$

for all $g \geq 0$ on $\mathbb{R}^n$ and $1 < p < \infty$.

3 Hardy–Littlewood one-sided maximal functions. One-weight problem

In this section we discuss the one-weight problem for the one-sided Hardy–Littlewood maximal operators.

We begin with the following statement:

Lemma 3.1 ([13]). *Let I be a bounded interval and let (1) hold on I. If $p \in \mathcal{P}_+(I)$, then there is a positive constant depending only on p such that for all f, $\|f\|_{L^{p(\cdot)}(I)} \leq 1$, the inequality*

$$\big(M^+f(x)\big)^{p(x)} \leq C\left(1 + M^+\big(|f|^{p(\cdot)}\big)(x)\right)$$

holds.

Now we formulate the main results of this section.

Theorem 3.1. *Let I be a bounded interval in $\mathbb{R}$ and let $1 < p_- \leq p_+ < \infty$.*

(i) *If $p \in \mathcal{P}_+(I)$ and a weight function w satisfies the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then for all $f \in L^{p(\cdot)}_w(I)$ the inequality*

$$\|(Nf)w\|_{L^{p(\cdot)}(I)} \leq C\|wf\|_{L^{p(\cdot)}(I)} \tag{12}$$

holds, where $N = M^+$.

(ii) *Let $p \in \mathcal{P}_-(I)$ and let $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$. Then inequality (12) holds for all $f \in L^{p(\cdot)}_w(I)$, where $N = M^-$.*

The result similar to Theorem 3.1 has been derived in [30], [31] for M_Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

In the case of unbounded intervals we have the next statement:

Theorem 3.2. *Let $I = \mathbb{R}_+$ and let $1 < p_- \leq p_+ < \infty$. Suppose that there is a positive number a such that $p(x) \equiv p_c \equiv const$ outside $(0, a)$.*

(i) *If $p \in \mathcal{P}_+(I)$ and $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then (12) holds for $N = M^+$.*

(ii) *If $p \in \mathcal{P}_-(I)$ and $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$, then (12) holds for $N = M^-$.*

Theorem 3.1 yields the following corollaries:

Corollary 3.1. *Let p be increasing function on an interval $I = (a, b)$ such that $1 < p(a) \le p(b) < \infty$. Suppose that w is increasing positive function on I. Then the one–weight inequality*

$$\|w^{1/p(\cdot)}(M^+ f)(\cdot)\|_{L^{p(\cdot)}(I)} \le c\|w^{1/p(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(I)}$$

holds.

Corollary 3.2. *Let p be decreasing function on an interval $I = (a, b)$ such that $1 < p(b) \le p(a) < \infty$. Suppose that w is decreasing positive function on I. Then the one–weight inequality*

$$\|w^{1/p(\cdot)}(M^- f)(\cdot)\|_{L^{p(\cdot)}(I)} \le c\|w^{1/p(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(I)}$$

holds.

Now we prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Since the proof of the second part is similar to the first one, we prove only (i). It is enough to show that

$$S_p\big(wM^+(f/w)\big) \le C$$

for f satisfying the condition $\|f\|_{L^{p(\cdot)}(I)} \le 1$.

First we prove that $S_{p^*}\left(\frac{f}{w}\right) < \infty$, where $p^*(x) = \frac{p(x)}{p_-}$.

By using Hölder's inequality we find that

$$S_{p^*}\left(\frac{f}{w}\right) = \int_I [f/w]^{p^*(x)}(x)dx \le \left(\int_I |f(x)|^{p(x)} dx\right)^{\frac{1}{p_-}} \cdot$$

$$\left(\int_I w(x)^{p(x)(1-(p_-)')} dx\right)^{\frac{1}{(p_-)'}} < \infty,$$

because $w^{p(\cdot)}(\cdot) \in A^+_{p_-}(I)$.

Thus Lemma 3.1 might be applied for p^*. Consequently,

$$S_p\big(w(M^+ f/w)\big) = \int_I \left[M^+\left(\frac{f}{w}\right)(x)\right]^{p(x)} w^{p(x)}(x)dx$$

$$= \int_I \left(\left[M^+ (f/w)(x)\right]^{p^*(x)}\right)^{p_-} w^{p(x)}(x)dx$$

$$\le C\int_I \left(1 + M^+\left(\left|\frac{f}{w}\right|^{p^*(\cdot)}\right)(x)\right)^{p_-} (w(x))^{p(x)}dx$$

$$\le C\int_I (w(x))^{p(x)}\, dx + C\int_I \left(M^+\left(\left|\frac{f}{w}\right|^{p^*(\cdot)}\right)(x)\right)^{p_-} w^{p(x)}(x)dx$$

$$\leq C + C\int_I \big|f/w\big|^{p(x)} w^{p(x)}(x)dx \leq C. \qquad \square$$

Proof of Theorem 3.2. First we prove (i). Without loss of generality we can assume that $M^+f(a) < \infty$. Since M^+ is sub-linear operator it is enough to prove that $S_{p,w}(M^+f) < \infty$, whenever $S_{p,w}(f) < \infty$. We have

$$\int_{\mathbb{R}_+} \big(M^+f\big)^{p(x)}(x)w(x)^{p(x)}dx \leq c\bigg[\int_0^a \big(M^+f\chi_{[0,a]}\big)^{p(x)}(x)w(x)^{p(x)}dx$$

$$+\int_0^a \big(M^+(f\chi_{[a,\infty)})\big)^{p(x)}(x)w(x)^{p(x)}dx + \int_a^\infty \big(M^+(f\chi_{[0,a]})\big)^{p(x)}(x)w(x)^{p(x)}dx$$

$$+\int_a^\infty \big(M^+f\chi_{[a,\infty)}\big)^{p(x)}(x)w(x)^{p(x)}dx\bigg] = c[I_1+I_2+I_3+I_4].$$

Since $M^+f(x) = M^+(f\chi_{[0,a]})(x)$ for $x \in [0,a]$, using the assumptions $w(\cdot)^{p(\cdot)} \in A^+_{p_-}([0,a])$, $p_+ \in \mathcal{P}_+((0,a))$ and Theorem 3.1 we find that $I_1 < \infty$.

Further, the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$ implies that $w(\cdot)^{p(\cdot)} \in A^+_{p_-}((a,\infty))$. Consequently, since $p \equiv p_c \equiv$ const on (a,∞), by Theorem F we have $I_4 < \infty$.

Now observe that $M^+(f\chi_{[0,a]})(x) = 0$ when $x \in (a,\infty)$. Therefore $I_3 = 0$.

It remains to estimate I_2. For this notice that if $x \in (0,a)$, then

$$M^+\big(f\cdot\chi_{[a,\infty)}\big)(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy$$

$$= \sup_{h>a-x}\frac{1}{h}\int_a^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy$$

$$\leq \sup_{h>a-x}\frac{1}{x+h-a}\int_a^{a+(x+h-a)}|f(y)|\chi_{[a,\infty)}(y)dy \leq M^+f(a) < \infty.$$

Hence,

$$I_2 \leq c\int_0^a w(x)^{p(x)}dx < \infty$$

because $w(\cdot)^{p(\cdot)}$ is locally integrable on $\mathbb{R}_+$.

To prove (ii) we use the notation of the proof of (i) substituting M^+ by M^+. In fact, the proof is similar to that of (i). The only difference is in the estimates of

$$I_2 := \int_0^a \big(M^-(f\chi_{[a,\infty)})\big)^{p(x)}(x)w(x)^{p(x)}dx$$

and

$$I_3 := \int_a^\infty \big(M^-(f\cdot\chi_{[0,a]})(x)\big)^{p(x)}(x)w(x)^{p(x)}dx.$$

Obviously, we have that $I_2 = 0$. Further, we represent I_3 as follows:

$$I_3 = \int_a^\infty \big(M^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x)w(x)^{p_c}dx$$

$$= \int_a^{2a} \big(M^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x) w(x)^{p_c}dx$$
$$+\int_{2a}^{\infty} \big(M^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x) w(x)^{p_c}dx := I_3^{(1)} + I_3^{(2)}.$$

Observe that for $x \in (a, 2a]$,

$$M^-(f\cdot\chi_{[0,a]})(x) \le \sup_{x-a<h<x} \frac{1}{a-x+h}\int_{a-(a-x+h)}^{a} |f(y)|dy \le M^- f(a) < \infty.$$

Hence,

$$I_3^{(1)} \le (M^- f)^{p_c}(a) \int_a^{2a} (w(x))^{p_c}\, dx < \infty.$$

If $x > 2a$, then

$$\big(M^- f\big)(x) \le \frac{1}{a-x}\int_0^a |f(y)|dy.$$

Therefore by using Hölder's inequality with respect to the exponent $p(\cdot)$ (see proposition A) we find that

$$I_3^{(2)} \le \left(\int_{2a}^{\infty} (w(x))^{p_c}(a-x)^{-p_c}dx\right)\left(\int_0^a |f(x)|dx\right)^{p_c}$$
$$\le c\left(\int_{2a}^{\infty} (w(x))^{p_c}(a-x)^{-p_c}dx\right) \|fw\|^{p_c}_{L^{p(\cdot)}_{([0,a])}} \|w^{-1}\|^{p_c}_{L^{p'(\cdot)}_{([0,a])}}$$
$$:= cJ_1\cdot J_2\cdot J_3.$$

It is clear that $J_2 < \infty$. Further, since $w(\cdot)^{p(\cdot)} \in A^-_{p_-}\big((a,\infty)\big)$, by Hölder's inequality we have that $w(\cdot)^{p(\cdot)} \in A^-_{p_c}\big((a,\infty)\big)$ because $p_c \ge p_-$. Hence, by applying Theorem F (for $\alpha = 0$) we have that the operator $\overline{M}^- f := M^-(f\chi_{(a,\infty)})$ is bounded in $L^{p_c}_w((a,\infty))$. Consequently, the Hardy operator

$$H_a f(x) = \frac{1}{x-a}\int_a^x |f(t)|dt, \quad x \in (a,\infty),$$

is bounded in $L^{p_c}_w((a,\infty))$. This implies (see, e.g., [20], [37]) that $J_1 < \infty$.

It remains to see that $J_3 < \infty$. Indeed, Proposition B yields

$$\|w^{-1}\|_{L^{p'(\cdot)}_{([0,a])}} \le (1+a)\|w^{-1}\|_{L^{(p_-)'\cdot}([0,a])}$$
$$\le c\|\chi_{\{w^{-1}\ge 1\}}(\cdot)w^{-1}(\cdot)\|_{L^{(p_-)'(\cdot)}([0,a])} + \|\chi_{\{w^{-1}<1\}}(\cdot)w^{-1}(\cdot)\|_{L^{(p_-)'}([0,a])}$$
$$\le c\big\|\chi_{\{w^{-1}\ge 1\}}(\cdot)w^{-\frac{p(\cdot)}{p_-}}(x)\big\|_{L^{(p_-)'}([0,a])} + c$$
$$\le c\bigg(\int_0^a w^{p(x)(1-(p_-)')}(x)dx\bigg)^{1/(p_-)'} + c.$$

Thus $I_3^{(2)} < \infty$. □

4 Fractional maximal operators. One-weight problem

In this section we derive the one-weight inequality for one-sided fractional maximal operators. Our main results are the following statements:

Theorem 4.1. *Let I be a bounded interval and let $1 < p_- \le p_+ < \infty$. Suppose that α is constant satisfying $0 < \alpha < 1/p_+$. Let $q(x) = \frac{p(x)}{1-\alpha p(x)}$.*

(i) *If $p \in \mathcal{P}_+(I)$ and a weight w satisfies the condition $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, then the inequality*

$$\|(N_\alpha f)w\|_{L^{q(\cdot)}(I)} \le C\|wf\|_{L^{p(\cdot)}(I)}, \qquad f \in L^{p(\cdot)}_w(I) \tag{10}$$

holds for $N_\alpha = M^+_\alpha$.

(ii) *Let $p \in \mathcal{P}_-(I)$ and let $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$. Then inequality (13) holds for $N_\alpha = M^-_\alpha$.*

Theorem 4.2. *Let $I = \mathbb{R}_+$, $1 < p_- \le p_+ < \infty$ and let $p(x) \equiv p_c \equiv const$ outside some interval $(0, a)$. Suppose that $q(x) = \frac{p(x)}{1-\alpha p(x)}$, where α is constant satisfying $0 < \alpha < 1/p_+$.*

(i) *If $p \in \mathcal{P}_+(I)$ and $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, then (10) holds for $N_\alpha = M^+_\alpha$.*

(ii) *If $p \in \mathcal{P}_-(I)$ and $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$, then (10) holds for $N_\alpha = M^-_\alpha$.*

Proof of Theorem 4.1. We prove (i). The proof of (ii) is the same. First we show that the inequality

$$M^+_\alpha(f/w)(x) \le \big(M^+\big(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)}\big)(x)\big)^{s(x)/q(x)}\bigg(\int_I f^{p(y)}(y)dy\bigg)^{\alpha},$$

holds, where $s(x) = 1 + q(x)/p'(x)$.

Indeed, denoting $g(\cdot) := (f(\cdot))^{p(\cdot)/s(\cdot)}(w(\cdot))^{-q(\cdot)/s(\cdot)}$ we see that $f(\cdot)/w(\cdot) = (g(\cdot))^{s(\cdot)/p(\cdot)}w^{q(\cdot)/p(\cdot)-1} = (g(\cdot))^{1-\alpha}g^{s(\cdot)/p(\cdot)+\alpha-1}w^{\alpha q(\cdot)}$. By using Hölder's inequality with respect to the exponent $(1-\alpha)^{-1}$ and the facts that $s(\cdot)/q(\cdot) = 1-\alpha$, $(s(y)/p(y)+\alpha-1)/\alpha = s(\cdot)$ we have

$$\begin{aligned}
&\frac{1}{h^{1-\alpha}}\int_{I_+(x,x+h)}\frac{f(y)}{w(y)}dy \\
&\le \bigg(\frac{1}{h}\int_{I_+(x,x+h)}g(y)dy\bigg)^{1-\alpha}\bigg(\int_{I_+(x,x+h)}g^{(s(y)/p(y)+\alpha-1)/\alpha}(y)w^{q(y)}(y)dy\bigg)^{\alpha} \\
&\le \big(M^+g(x)\big)^{s(x)/q(x)}\bigg(\int_{I_+(x,x+h)}g^{s(y)}(y)w^{q(y)}(y)\bigg)^{\alpha} \\
&\le \big(M^+g(x)\big)^{s(x)/q(x)}\bigg(\int_I f^{p(y)}(y)dy\bigg)^{\alpha}.
\end{aligned}$$

Now we prove that $S_q\big(wM^+_\alpha(f/w)\big) \le C$ when $S_p(f) \le 1$. By applying the above–derived inequality we find that

$$S_q\big(wM^+_\alpha(f/w)\big) \le c\int_I \big(M^+_\alpha(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})\big)^{s(x)}(x)w^{q(x)}(x)dx$$

$$= cS_s\big(M^+(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})w^{q(\cdot)/s(\cdot)}\big).$$

Observe now that the condition on the weight w is equivalent to the assumption $w^{q(\cdot)}(\cdot) \in A^+_{s_-}(I)$. On the other hand, $\|f^{p(\cdot)/s(\cdot)}\|_{L^{s(\cdot)}(I)} \le 1$. Therefore taking Theorem 3.1 into account we have the desired result. □

Proof of Theorem 4.2. (i) Let $f \ge 0$ and let $S_{p,w}(f) < \infty$. We have

$$S_{q,w}(M^+_\alpha f) = \int_I \big(M^+_\alpha f\big)^{q(x)}(x)w(x)^{q(x)}dx$$

$$\le c\Big[\int_0^a \big(M^+_\alpha f\chi_{[0,a]}(x)\big)^{q(x)}(x)w(x)^{q(x)}dx$$

$$+\int_0^a \big(M^+_\alpha(f\cdot\chi_{[a,\infty)})(x)\big)^{q(x)}(x)w(x)^{q(x)}dx$$

$$+\int_a^\infty \big(M^+_\alpha(f\cdot\chi_{[0,a]})(x)\big)^{q(x)}(x)w(x)^{q(x)}dx$$

$$+\int_a^\infty \big(M^+_\alpha(f\chi_{[a,\infty)})(x)\big)^{q(x)}(x)w(x)^{q(x)}dx\Big] := c[I_1+I_2+I_3+I_4].$$

It is easy to see that $I_1 < \infty$ because of Theorem 4.1 and the condition $w^{q(\cdot)}(\cdot) \in A^+_{p_-,q_-}([0,a])$. Further, it is obvious that $I_3 < \infty$ because $M^+_\alpha(f\chi_{[0,a]})(x) = 0$ for $x > a$. Further, observe that

$$I_2 \le c\int_0^a w(x)^{q(x)}dx < \infty,$$

where the positive constant depends on α, f, p, a.

It is easy to check that by Hölder's inequality with respect to the power

$$\big((p_c)'/q_c\big)/\big((p_-)'/q_-\big)$$

the condition $w(\cdot)^{q_c} \in A^+_{p_-,q_-}([a,\infty))$ implies $w(\cdot)^{q_c} \in A^+_{p_c,q_c}([a,\infty))$. Hence, by using Theorem F we find that $I_4 < \infty$.

(ii) We keep the notation of the proof of (i) but substitute M^+_α by M^-_α. The only difference between the proofs of (i) and (ii) is in the estimates of I_2 and I_3.

It is obvious that $I_2 = 0$, while for I_3 we have

$$I_3 = \int_a^{2a} \big(M^-_\alpha(f\cdot\chi_{[0,a]})(x)\big)^{q(x)}(x)w(x)^{q(x)}dx$$

$$+\int_{2a}^\infty \big(M^-_\alpha(f\cdot\chi_{[0,a]})(x)\big)^{q_c}(x)w(x)^{q_c}dx := I_3^{(1)} + I_3^{(2)}.$$

If $x > a$, then

$$M_\alpha^- f(x) \le \sup_{x-a<h<x} h^{\alpha-1} \int_{x-h}^{a} |f(y)|dy \le cM_\alpha^- f(a).$$

Consequently,

$$I_3^{(1)} \le c\big(M_\alpha^- f(a)\big)^{q_c} \int_a^{2a} \big(w(x)\big)^{q_c} dx < \infty.$$

Now observe that when $x > a$ we have the following pointwise estimates:

$$M_\alpha^-(f\chi_{[0,a]})(x) \le (x-a)^{\alpha-1} \int_0^a |f(y)|dy \le (x-a)^{\alpha-1} \|fw\|_{L^{p(\cdot)}([0,a])} \|w^{-1}\|_{L^{p'(\cdot)}([0,a])}$$

$$:= (x-a)^{\alpha-1} J_1 \cdot J_2.$$

Hence,

$$I_3^{(2)} \le \bigg(\int_{2a}^{\infty} (x-a)^{(\alpha-1)q_c} (w(x))^{q_c} dx \bigg) (J_1 \cdot J_2)^{q_c}.$$

It is obvious that $J_1 < \infty$. Further,

$$J_2 \le \|w^{-1}(\cdot)\chi_{w^{-1}>1}(\cdot)\|_{L^{p'(\cdot)}([0,a])} + \|w^{-1}(\cdot)\chi_{w^{-1}\le 1}(\cdot)\|_{L^{p'(\cdot)}([0,a])}$$

$$:= J_2^{(1)} + J_2^{(2)}.$$

It is clear that $J_2^{(2)} < \infty$. To estimate $J_2^{(1)}$ observe that by Proposition B we have

$$J_2^{(1)} \le (1+a)\|w^{-1}\chi_{w^{-1}>1}\|_{L^{p_-}([0,a])} \le (1+a)\|w^{-q(\cdot)/q_-}\chi_{w^{-1}>1}\|_{L^{p_-}([0,a])}$$

$$\le (1+a)\|w^{-q(\cdot)/q_-}\|_{L^{p_-}([0,a])} < \infty.$$

Since M_α^- is bounded from $L_w^{p_c}([a,\infty))$ to $L_w^{q_c}([a,\infty))$ we have the Hardy inequality

$$\bigg(\int_a^\infty (x-a)^{(\alpha-1)q_c} w^{q_c}(x) \bigg(\int_a^x |f(t)|dt \bigg)^{q_c} dx \bigg)^{1/q_c} \le c \bigg(\int_a^\infty |f(x)|^{p_c} w^{p_c}(x) dx \bigg)^{1/p_c}.$$

From this inequality it follows that (see, e.g., [20], [37])

$$\int_{2a}^\infty (x-a)^{(\alpha-1)q_c} (w(x))^{q_c} dx < \infty. \quad \Box$$

5 Generalized fractional maximal operators. Two-weight problem

Let $I = [a,b]$ be a bounded interval and let $I^+ := [b, 2b-a)$; $I^- := [2a-b, a)$. Let $Q = I_1 \times I_2 \times \cdots \times I_n$ be a cube in $\mathbb{R}^n$. We denote:

$$Q^+ := I_1^+ \times I_2^+ \times \cdots \times I_n^+, \quad Q^- := I_1^- \times I_2^- \times \cdots \times I_n^-.$$

Let α be a measurable function on $\mathbb{R}^n$, $0 < \alpha_- \le \alpha(x) \le \alpha_+ < n$. Let us define dyadic fractional maximal functions on $\mathbb{R}^n$:

$$\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x) = \sup_{\substack{Q\ni x\\ Q\in D(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}} \int_{Q^+} |f(y)|dy;$$

$$\big(M^{-,(d)}_{\alpha(\cdot)} f\big)(x) = \sup_{\substack{Q\ni x\\ Q\in D(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}} \int_{Q^-} |f(y)|dy.$$

If $\alpha(x) \equiv 0$, then we have Hardy-Littlewood dyadic maximal functions $M^{+,(d)}$, $M^{-,(d)}$.

In the paper [39] the two-weight weak-type inequality was proved in the classical Lebesgue spaces for one-sided dyadic Hardy-Littlewood maximal functions defined on $\mathbb{R}^n$.

Theorem 5.1. *Let p be constant and let $1 < p < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < n$ where q and α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'} \in RD^{(d)}(\mathbb{R}^n)$. Then $M^{+,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^{q(\cdot)}_v(\mathbb{R}^n)$ if and only if*

$$A := \sup_{Q, Q\in D(\mathbb{R}^n)} \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \big\|\chi_{Q^+} w^{-1}\big\|_{L^{p'}(\mathbb{R}^n)} < \infty. \tag{11}$$

Proof. *Necessity.* Assuming $f = \chi_{Q^+} w^{-p'}$ ($Q \in D(\mathbb{R}^n)$) in the inequality

$$\big\|M^{+,(d)}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \le C\|f\|_{L^p_w(\mathbb{R}^n)} \tag{12}$$

we have that

$$\bigg\|\chi_Q(\cdot)\bigg(\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int_{Q^+} f\bigg)\bigg\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} = \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)}\bigg(\int_{Q^+} w^{-p'}(y)dy\bigg)$$

$$\le \big\|M^{+,(d)}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \le C\bigg(\int_{Q^+} w^{-p'}(y)dy\bigg)^{\frac{1}{p}}.$$

Thus, to show that (11) holds it remains to prove that for all dyadic cubes Q, $S_Q = \int_Q w^{-p'}(y)dy < \infty$. Indeed, suppose the contrary that $S_Q = \infty$ for some cube Q. Then $S_Q = \|w^{-1}\|_{L^{p'}(Q)} = \infty$. This implies that there is a non-negative function g such that $g \in L^p(Q)$ and $\int_Q g(y)w^{-1}(y)dy = \infty$. Further, let $Q = \bar{Q}^+$, where $\bar{Q} \in D(\mathbb{R}^n)$. Then taking $f = \chi_Q g w^{-1}$ we have

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \bigg(\int_{\bar{Q}^+} g^p(x)dx\bigg)^{\frac{1}{p}} < \infty;$$

$$\big\|M^{+,(d)}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \ge \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)}\Big(\int\limits_{\bar{Q}^+} f(y)dy\Big)$$

$$= \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \int\limits_{\bar{Q}^+} g(y)w(y)^{-1}dy = \infty.$$

This contradicts (12).

Sufficiency. For every $x \in \mathbb{R}^n$ we take $Q_x \in D(\mathbb{R}^n)$ ($Q_x \ni x$) so that

$$|Q_x|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q_x^+} |f(y)|dy > \frac{1}{2}\,\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x). \tag{13}$$

Assume that f be non-negative bounded with compact support. Then it is easy to see that we can take maximal cube Q_x containing x for which (13) holds. Let $Q \in D(\mathbb{R}^n)$ and let us introduce the set

$$F_Q := \Big\{x \in Q : Q \text{ is maximal for which } |Q|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q^+} f(y)dy > \frac{1}{2}\, M^{+,(d)}_{\alpha(\cdot)} f(x)\Big\}.$$

Dyadic cubes have the following property: if $Q_1, Q_2 \in D(\mathbb{R}^n)$, and $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$, where $\overset{o}{Q}$ denotes the inner part of a cube Q.

Now observe that $F_{Q_1} \bigcap F_{Q_2} \neq \emptyset$ if $Q_1 \neq Q_2$. Indeed, if $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 = \emptyset$, then it is clear. If $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Let us take $x \in F_{Q_1} \bigcap F_{Q_2}$. Then $x \in Q_1$, $x \in Q_2$ and

$$\frac{1}{|Q_1|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q_1^+} f(y)dy > \frac{1}{2}\,\Big(M^{+,(d)}_{\alpha(\cdot)} f\Big)(x);\ \frac{1}{|Q_2|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q_2^+} f(y)dy > \frac{1}{2}\,\Big(M^{+,(d)}_{\alpha(\cdot)} f\Big)(x).$$

If $Q_1 \subset Q_2$, then Q_2 would be the maximal cube for which (13) holds. Consequently $x \notin F_{Q_1}$ and $x \in F_{Q_2}$. Analogously we have that if $Q_2 \subset Q_1$, then $x \in F_{Q_1}$ and $x \notin F_{Q_2}$.

Further, it is clear that $F_Q \subset Q$ and $\bigcup\limits_{Q \in D_m(\mathbb{R}^n)} F_Q = \mathbb{R}^n$, where $D_m(\mathbb{R}^n) = \big\{Q : Q \in D(\mathbb{R}^n), F_Q \neq \emptyset\big\}$.

Suppose that $\|f\|_{L^p_w(\mathbb{R}^n)} \le 1$ and that r is a number satisfying the condition $p < r < q_-$. We have

$$\big\|M^{+,(d)}_{\alpha(\cdot)} f\big\|^r_{L^{q(\cdot)}_v(\mathbb{R}^n)} = \big\|v^r\big(M^{+,(d)}_{\alpha(\cdot)} f\big)^r\big\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} = \sup\int\limits_{\mathbb{R}^n} v^r(x)\Big(M^{+,(d)}_{\alpha(\cdot)} f\Big)^r(x)h(x)dx,$$

where the supremum is taken over all functions h, $\|h\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \le 1$. Now for such an h, using Lemma 2.1, we have that

$$\int\limits_{\mathbb{R}^n} v^r(x)\Big(M^{+,(d)}_{\alpha(\cdot)} f\Big)^r(x)h(x)dx = \sum_{Q \in D_m(\mathbb{R}^n)} \int\limits_{F_Q} v^r(x)\Big(M^{+,(d)}_{\alpha(x)} f\Big)^r(x)h(x)dx$$

$$\leq C \sum_{Q\in D_m(\mathbb{R}^n)} \Big(\int\limits_{F_Q} v^r(x)|Q|^{(\frac{\alpha(x)}{n}-1)r} h(x)dx\Big)\Big(\int\limits_{Q^+} f(y)dy\Big)^r$$

$$\leq C \sum_{Q\in D_m(\mathbb{R}^n)} \big\|v^r(\cdot)|Q|^{(\frac{\alpha(\cdot)}{n}-1)r}\chi_Q(\cdot)\big\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \big\|h\big\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \Big(\int\limits_{Q^+} f(y)dy\Big)^r$$

$$= C \sum_{Q\in D_m(\mathbb{R}^n)} \big\|v(\cdot)\ |Q|^{\frac{\alpha(\cdot)}{n}-1}\chi_Q(\cdot)\big\|^r_{L^{q(\cdot)}(\mathbb{R}^n)} \big\|h\big\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \Big(\int\limits_{Q^+} f(y)dy\Big)^r$$

$$\leq C\ A^r \sum_{Q\in D_m(\mathbb{R}^n)} \Big(\int\limits_{Q^+} w^{-p'}(y)dy\Big)^{-\frac{r}{p}} \Big(\int\limits_{Q^+} f(y)dy\Big)^r \leq CA^r\|f\|^r_{L^p_w(\mathbb{R}^n)}.$$

In the last inequality we used also the fact that $Q^+ \in D(\mathbb{R}^n)$ if and only if $Q \in D(\mathbb{R}^n)$.

Let us pass now to an arbitrary f, where $f \in L^p_w(\mathbb{R}^n)$. For such an f we take the sequence $f_m = f\chi_{Q(0,k_m)}\chi_{\{f<j_m\}}$, where

$$Q(0,k_m) := \{(x_1,\cdots,x_n) : |x_i| < k_m,\ i = 1,\cdots,n\}.$$

and $k_m, j_m \to \infty$ as $m \to \infty$. Then it is easy to see that $f_m \to f$ in $L^p_w(\mathbb{R}^n)$ and also pointwise. Moreover, $f_m(x) \leq f(x)$. On the other hand, $\big\{M^{+,(d)}_{\alpha(\cdot)} f_m\big\}$ is a Cauchy sequence in $L^{q(\cdot)}_v(\mathbb{R}^n)$, because

$$\big\|M_{\alpha(\cdot)}f_m - M_{\alpha(\cdot)}f_j\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq \big\|M_{\alpha(\cdot)}\big(f_m - f_j\big)\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq\ C\big\|f_m - f_j\big\|_{L^p_w(\mathbb{R}^n)}.$$

Since $L^{q(\cdot)}_v(\mathbb{R}^n)$ is a Banach space, there exists $g \in L^{q(\cdot)}_v(\mathbb{R}^n)$ such that

$$\big\|\big(M_\alpha f_m\big) - g\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \to 0.$$

Taking Proposition A into account we can conclude that there is a subsequence $M_{\alpha(\cdot)}f_{m_k}$ which converges to g in $L^{q(\cdot)}_v(\mathbb{R}^n)$ and also almost everywhere. But f_{m_k} converges to f in $L^p_w(\mathbb{R}^n)$ and almost everywhere. Consequently,

$$\|g\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq C\|f\|_{L^p_w(\mathbb{R}^n)}, \tag{14}$$

where the positive constant C does not depend on f. Now observe that since f_{m_k} is non-decreasing, for fixed $x \in Q,\ Q \in D(\mathbb{R}^n)$, we have that

$$|Q|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q^+} f(y)dy = \lim_{k\to\infty}|Q|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q^+} f_{m_k}(y)dy$$

$$\leq \lim_{k\to\infty}\sup_{Q\ni x}|Q|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q^+} f_{m_k}(y)dy = \lim_{k\to\infty}\Big(M^{+,(d)}_{\alpha(\cdot)} f_{m_k}\Big)(x)$$

and the last limit exists because it converges to g almost everywhere. Hence,

$$\left(M^{+,(d)}_{\alpha(\cdot)} f\right)(x) \le \lim_{k\to\infty} \left(M^{+,(d)}_{\alpha(\cdot)} f_{m_k}\right)(x) = g(x).$$

for almost every x. Finally, (14) yields

$$\big\|M^{+,(d)}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \le C\|f\|_{L^p_w(\mathbb{R}^n)}.$$

□

The proof of the next statement is similar to that of Theorem 4.1; therefore it is omitted.

Theorem 5.2. *Let $1 < p < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < n$, where p is constant and q, α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'} \in RD^{(d)}(\mathbb{R}^n)$. Then $M^{-,(d)}_{\alpha(\cdot)}$ is bounded from from $L^p_w(\mathbb{R}^n)$ to $L^{q(\cdot)}_v(\mathbb{R}^n)$ if and only if*

$$\sup_{Q, Q\in D(\mathbb{R}^n)} \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \big\|w^{-1}(\cdot)\chi_{Q^-}(\cdot)\big\|_{L^{p'}(\mathbb{R}^n)} < \infty.$$

Let us now consider the case when $p \equiv q \equiv \text{const}$.

Theorem 5.3. *Let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \le \alpha_+ < n$. Then $M^{+,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} v^p(x)\Big(M^{+,(d)}_{\alpha(\cdot)}\big(w^{-p'}\chi_Q\big)(x)\Big)^p dx \le C \int_Q w^{-p'}(x)dx < \infty,$$

for all dyadic cubes $Q \subset \mathbb{R}^n$.

Proof. For *sufficiency* it is enough to show that the inequality

$$\Big\|v\, M^{+,(d)}_{\alpha(\cdot),u}\, f\Big\|_{L^p(\mathbb{R}^n)} \le C\, \Big\|u^{\frac{1}{p}}\, f\Big\|_{L^p(\mathbb{R}^n)} \tag{15}$$

holds if for all $Q \in D(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} v^p(x)\Big(M^{+,(d)}_{\alpha(\cdot),u}\, \chi_Q\Big)^p(x)\, dx \le C \int_Q |f(x)|^p u(x)\, dx,$$

where

$$\left(M^{+,(d)}_{\alpha(\cdot),u}\, f\right)(x) = M^{+,(d)}_{\alpha(\cdot)}\, (fu)(x).$$

To prove (15) we argue in the same manner as in the proof of Theorem 4.1. Let us construct the set F_Q for $Q \in D(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} v^p(x)\Big(M^{+,(d)}_{\alpha(\cdot),u}\Big)^p(x)\,dx \\
\leq\ & 2^p \sum_{Q\in D_m} \int\limits_{F_Q} v^p(x)\Big(\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^+} f(y)u(y)dy\Big)^p\,dx \\
=\ & C \sum_{Q\in D_m} \Big(\int\limits_{F_Q} v^p(x)\,|Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\,dx\Big)\Big(\int\limits_{Q^+} f(y)u(y)dy\Big)^p \\
=\ & C \sum_{Q\in D_m} \Big(\int\limits_{F_Q} v^p(x)\,|Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\,dx\Big)\big(u(Q^+)\big)^p\Big(\frac{1}{u(Q^+)}\int\limits_{Q^+} f(y)u(y)dy\Big)^p.
\end{aligned}$$

Taking Lemma 1.2 into account it is enough to show that

$$S := \sum_{\substack{j:\ Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in D(\mathbb{R}^n)}} \Big(\int\limits_{F_{Q_j^-}} v^p(x)\big|Q_j^-\big|^{\left(\frac{\alpha(x)}{n}-1\right)p}dx\Big)u^p\big(Q_j\big) \leq C\int\limits_Q u(x)dx.$$

Indeed, we have

$$\begin{aligned}
S &= \sum_{\substack{j:\ Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in D(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\Big(\big|Q_j^-\big|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q_j} u(y)dy\Big)^p dx \\
&\leq \sum_{\substack{j:\ Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in D(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \\
&= \int\limits_{\bigcup_{Q_j\subset Q} F_{Q_j^-}} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \\
&\leq \int\limits_{\mathbb{R}^n} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \leq C\int\limits_Q u(y)dy.
\end{aligned}$$

Necessity. Taking the test function $f_Q = \chi_Q w^{-p'}$ in the two-weight inequality

$$\Big\|v\,\Big(M^{+,(d)}_{\alpha(\cdot)}\,f\Big)\Big\|_{L^p(\mathbb{R}^n)} \leq C\,\Big\|f\,w\Big\|_{L^p(\mathbb{R}^n)}$$

and observing that $\int\limits_Q w^{-p'}(y)dy < \infty$ for every $Q \in D(\mathbb{R}^n)$ we have the desired result. $\square$

The proof of the next statement is similar to that of the previous theorem. The proof is omitted.

Theorem 5.4. *Suppose that $1 < p < \infty$, where p is constant. Then $M^{-,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if there is a positive constant C such that for all $Q \in D(\mathbb{R}^n)$,*

$$\int\limits_{\mathbb{R}^n} v^p(x)\Big(M^{-,(d)}_{\alpha(\cdot)}\Big(w^{-p'}\chi_Q\Big)\Big)^p(x)dx \le C\int\limits_Q w^{-p'}(x)dx < \infty.$$

Let us now discuss the two–weight problem for the one-sided maximal functions $M^+_{\alpha(\cdot)}$, $M^-_{\alpha(\cdot)}$ defined on $\mathbb{R}$.

Recall that by $M^{+,(d)}_{\alpha(\cdot)}$ and $M^{-,(d)}_{\alpha(\cdot)}$ we denote one–sided dyadic maximal functions. Now we assume that they are defined on $\mathbb{R}$.

Together with these operators we need the following maximal operators:

$$\begin{aligned}
\Big(\bar{M}^+_{\alpha(\cdot)}f\Big)(x) &= \sup_{h>0}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(y)|dy;\\
\Big(\bar{M}^-_{\alpha(\cdot)}f\Big)(x) &= \sup_{h>0}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x-h}^{x-\frac{h}{2}}|f(y)|dy;\\
\Big(\widetilde{M}^+_{\alpha(\cdot)}f\Big)(x) &= \sup_{j\in\mathbb{Z}}\frac{1}{2^{(j-1)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^j}|f(y)|dy.
\end{aligned}$$

To prove the next statements we need some lemmas.

Lemma 5.1. *Let $f \in L_{loc}(\mathbb{R})$. Then the following pointwise estimates hold:*

$$\big(M^+_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^+_{\alpha(\cdot)}f\big)(x);$$

$$\big(M^-_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^-_{\alpha(\cdot)}f\big)(x) \tag{16}$$

for every $x \in \mathbb{R}$.

Proof. Observe that

$$\frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+h}|f(t)|dt = \frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+\frac{h}{2}}|f(t)|dt + \frac{1}{h^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$

$$= 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_x^{x+\frac{h}{2}}|f(t)|dt + 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$

$$\leq 2^{\alpha(x)-1}\big(M^{+}_{\alpha(\cdot)}f\big)(x)+2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Hence,

$$\big(M^{+}_{\alpha(\cdot)}f\big)(x)\leq 2^{\alpha(x)-1}\big(M^{+}_{\alpha(\cdot)}f\big)(x)+2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Consequently,

$$\big(1-2^{\alpha(x)-1}\big)\big(M^{+}_{\alpha(\cdot)}f\big)(x)\leq 2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x),$$

which implies

$$\big(M^{+}_{\alpha(\cdot)}f\big)(x)\leq \frac{2^{\alpha(x)-1}}{1-2^{\alpha(x)-1}}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x)\leq \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Analogously the inequality (16) follows. □

Lemma 5.2. *The following inequality*

$$\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x)\leq C\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x) \tag{17}$$

holds with a positive constant C independent of f and x.

Proof. Let us take $h>0$. Then $h\in[2^{j-1},2^{j})$ for some $j\in\mathbb{Z}$. Consequently,

$$\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+h}^{x+\frac{h}{2}}|f(t)|dt\leq \frac{1}{(2^{j-2})^{1-\alpha(x)}}\int\limits_{x+2^{j-2}}^{x+2^{j}}|f(t)|dt$$

$$=\frac{1}{2^{(j-2)1-\alpha(x)}}\int\limits_{x+2^{j-2}}^{x+2^{j-1}}|f(t)|dt+\frac{1}{2^{(j-2)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^{j}}|f(t)|dt$$

$$=\frac{1}{2^{(j-2)(1-\alpha(x))}}\int\limits_{x+2^{j-2}}^{x+2^{j-1}}|f(t)|dt+\frac{2^{\alpha(x)-1}}{2^{(j-1)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^{j}}|f(t)|dt$$

$$\leq\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x)+2^{\alpha_+-1}\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x)=\big(1+2^{\alpha_+-1}\big)\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Hence, (17) holds for $C=1+2^{\alpha_+-1}$. □

Lemma 5.3. *There exists a positive constant C depending only on α such that for all f, $f\in L_{loc}(\mathbb{R})$, and $x\in\mathbb{R}$,*

$$\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x)\leq C\big(M^{+,(d)}_{\alpha(\cdot)}f\big)(x). \tag{18}$$

Proof. Let $h = 2^j$ for some integer j. Suppose that I and I' are dyadic intervals such that $I \bigcup I'$ is again dyadic, $|I| = |I'| = 2^{j-1}$ and $[x + \frac{h}{2}, x + h) \subset (I \bigcup I')$. Then $x \in (I \bigcup I')^-$, where $(I \bigcup I')^-$ is dyadic and

$$\int_{x+\frac{h}{2}}^{x+h} |f(t)|dt \le \int_{I \bigcup I'} |f(t)|dt \le 2^{j(1-\alpha(x))} \big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x),$$

whence

$$\big(\widetilde{M}^{+}_{\alpha(\cdot)} f\big)(x) \le 2^{1-\alpha_-} \big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x).$$

If $I \bigcup I'$ is not dyadic, then we take $I_1 \in D(\mathbb{R})$ with length 2^j containing I'. Consequently, $x \in (I_1)^-$, where I_1^- is dyadic. Observe that $x \in I^-$, where I^- is also dyadic. Consequently,

$$\int_{x+\frac{h}{2}}^{x+h} |f(t)|dt \le \int_{I \bigcup I_1} |f(t)|dt = \int_{I} |f(t)|dt + \int_{I_1} |f(t)|dt \le C\ h^{1-\alpha(x)} \big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x),$$

with positive constant C independent of j. Finally, we have (18). □

Lemma 5.4. *There exists a positive constant C depending only on α such that*

$$\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x) \le C\big(M^{+}_{\alpha(\cdot)} f\big)(x) \tag{19}$$

for all f, $f \in L_{loc}(\mathbb{R})$, $x \in \mathbb{R}$.

Proof. Let $x \in I$, $I \in D(\mathbb{R})$. Denote $I = [a, b)$. Then $I^+ = [b, 2b - a)$. Let $h = 2b - a - x$. We have

$$\frac{1}{|I|^{1-\alpha(x)}} \int_{I^+} |f(t)|dt \le \frac{2^{1-\alpha(x)}}{|I \bigcup I^+|^{1-\alpha(x)}} \int_{x}^{x+h} |f(t)|dt$$

$$\le 2^{1-\alpha_-} \frac{1}{h^{1-\alpha(x)}} \int_{x}^{x+h} |f(t)|dt \le 2^{1-\alpha_-} M^{+}_{\alpha(\cdot)} f(x).$$

Since I is arbitrary dyadic cube containing x, then (19) holds for $C = 2^{1-\alpha_-}$. □

Summarizing Lemmas 5.1–5.4, we have the next statement:

Proposition 5.1. There exists positive constants C_1 and C_2 such that for all f, $f \in L_{loc}(\mathbb{R})$ and $x \in \mathbb{R}$ the two-sided inequality

$$C_1\big(M^{+}_{\alpha(\cdot)} f\big)(x) \le \big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x) \le C_2\big(M^{+}_{\alpha(\cdot)} f\big)(x)$$

holds.

Now Theorem 5.1 (for $n = 1$) and Proposition 5.1 yield the following theorem:

Theorem 5.5. *Let p, q and α be measurable functions on $I = \mathbb{R}$, $1 < p_- < q_- \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$. Suppose also that $p \in \mathcal{G}(I)$. Further, assume that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}_w(I)$ to $L^{q(\cdot)}_v(I)$ if*

$$B := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(\mathbb{R})} \left\|\chi_{(a,a+h)} w^{-1}\right\|_{L^{(p_-)'}(\mathbb{R})} < \infty.$$

Proof. By using Theorem 5.1 we have that the condition $B < \infty$ implies

$$\|M^{+,(d)}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R})} \leq C\|fw\|_{L^{p_-}(\mathbb{R})}$$

Now Propositions C and 5.1 complete the proof. □

Analogously the next statement can be proved:

Theorem 5.6. *Let p, q and α be measurable functions on $I := \mathbb{R}$, $1 < p_- < q_- \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$. Suppose also that $p \in \mathcal{G}(I)$ and that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if*

$$B_1 := \sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a,a+h)}(\cdot) h^{\alpha(\cdot)-1} v(\cdot)\right\|_{L^{q(\cdot)}(I)} \left\|\chi_{(a-h,a)} w^{-1}\right\|_{L^{(p_-)'}(I)} < \infty.$$

The results of this section deduce the following corollaries:

Corollary 5.1. *Let $I := \mathbb{R}$ and $1 < p < q_- \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$, where p is constant. Assume that $w^{-p'} \in RD^{(d)}(\mathbb{R})$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(I)} \left\|\chi_{(a,a+h)} w^{-1}\right\|_{L^{p'}(I)} < \infty.$$

Corollary 5.2. *Let $I := \mathbb{R}$ and let $1 < p < q_- \leq q_+ < \infty$, where p is constant. Suppose that α is measurable function on $\mathbb{R}$ satisfying $0 < \alpha_- \leq \alpha_+ < 1$. Suppose also that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^-_{\alpha(\cdot)}$ is bounded from from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a,a+h)}(\cdot) h^{\alpha(\cdot)-1} v(\cdot)\right\|_{L^{q(\cdot)}(I)} \left\|\chi_{(a-h,a)} w^{-1}\right\|_{L^{p'}(I)} < \infty.$$

Corollary 5.3. *Let $I = \mathbb{R}$, $1 < p_- < q_- \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha_+ < 1$. Suppose that $p_- = p(\infty)$ and $p \in \mathcal{P}_\infty(I)$. Assume that $w^{-(p_-)'} \in RD^{(d)}(\mathbb{R})$. Then:*

(i) *$M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if $B < \infty$;*

(ii) *$M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if $B_1 < \infty$.*

Proof of Corollary 5.1. *Sufficiency* is a direct consequence of Theorem 5.5.

Necessity follows immediately by applying the two-weight inequality for the test function $f(x) = \chi_{(a,a+h)}(x) w^{-p'}(x)$ (see also necessity of the proof of Theorem 5.1 for the details). □

Proof of Corollary 5.2. Similar to that of Corollary 5.1. □

Proof of Corollary 5.3. (i) The result follows from Theorem 4.5 because the condition $p \in \mathcal{P}_\infty(I)$ implies that

$$\int_I K^{p(x)p(\infty)/|p(x)-p(\infty)|}dx < \infty.$$

Hence, by using the assumption $p(\infty) = p_-$ we have that $p \in \mathcal{G}(I)$.

The second part of the theorem is obtained in a similar manner; therefore it is omitted. □

The next statement gives the boundedness of $M^+_{\alpha(\cdot)}$ in the diagonal case $p \equiv q \equiv$ *const*.

Theorem 5.7. *Let $I := \mathbb{R}$ and let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \leq \alpha_+ < \infty$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^p_v(I)$ if and only if there is a positive constant C such that for all bounded intervals $J \subset \mathbb{R}$,*

$$\int_{\mathbb{R}} v^p(x)\Big(M^+_{\alpha(\cdot)}\big(w^{-p'}\chi_J\big)(x)\Big)^p dx \leq C\int_J w^{-p'}(x)dx < \infty.$$

Proof. *Sufficiency* follows from Proposition 5.1 and Theorem 5.3 for $n = 1$. For *necessity* we take $f = \chi_J\, w^{p'}$ in the two weight inequality

$$\big\|v\, M^+_{\alpha(\cdot)}\, f\big\|_{L^p_v(I)} \leq C\, \big\|w\, f\big\|_{L^p_v(I)}$$

and we are done. □

Analogously the following theorem follows:

Theorem 5.8. *Let $I := \mathbb{R}$ and let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \leq \alpha_+ < \infty$.. Then $M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^p_v(I)$ if and only if*

$$\int_{\mathbb{R}} v^p(x)\Big(M^-_{\alpha(\cdot)}\big(w^{-p'}\chi_J\big)(x)\Big)^p dx \leq C\int_J w^{-p'}(x)dx < \infty$$

for all bounded intervals $J \subset \mathbb{R}$.

Finally we mention that the results similar to those of this section were derived in [24] for generalized two-sided fractional maximal functions and Riesz potentials.

6 Fefferman–Stein type inequality

In this section we derive Fefferman–Stein type inequality for the operators $M^-_{\alpha(\cdot)}$, $M^+_{\alpha(\cdot)}$. Notice that this inequality for the classical Riesz potentials for the diagonal case was established by E. Sawyer (see, e.g., [49]).

The main statement of this section reeds as follows:

Theorem 6.1. *Let α, p and q be measurable functions on $I=\mathbb{R}$. Suppose that $1<p_-<q_-\le q_+<\infty$ and $0<\alpha_-\le\alpha_+<1/p_-$. Suppose that $p\in\mathcal{G}(I)$. Then the following inequalities hold:*

$$\|v(\cdot)(M^{+}_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}\le c\|f(\cdot)(\widetilde{N}^{-}_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}; \tag{20}$$

$$\|v(\cdot)(M^{-}_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}\le c\|f(\cdot)(\widetilde{N}^{+}_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}, \tag{21}$$

where

$$\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(x)=\sup_{h>0}h^{-1/p_-}\|v(\cdot)h^{\alpha(\cdot)}\chi_{(x-h,x)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})},$$

$$\big(\widetilde{N}^{+}_{\alpha(\cdot)}v\big)(x)=\sup_{h>0}h^{-1/p_-}\|v(\cdot)h^{\alpha(\cdot)}\chi_{(x,x+h)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}.$$

Proof. We prove (20). The proof of (21) is the same. First we show that the inequality

$$\|v(\cdot)(M^{+,(d)}_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}\le c\|f(\cdot)(\widetilde{N}^{-}_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}$$

holds.

Repeating the arguments of the proof of Theorem 5.1 for one-dimensional dyadic intervals J we construct the sets F_J. Take h, $\|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R})}\le 1$, where $p_-<r<q_-$. By using Lemma 2.1 and Proposition C we have

$$\int_{\mathbb{R}}v^r(x)\big(M^{+,(d)}_{\alpha()}f(x)\big)^r h(x)dx=\sum_{J\in D_m(\mathbb{R})}\int_{F_J}v(x)^r\big(M^{+,(d)}_{\alpha(\cdot)}f\big)^r(x)h(x)dx$$

$$\le c\sum_{J\in D_m(\mathbb{R})}\bigg(\int_{F_J}v^r(x)|J|^{(\alpha(x)-1)r}h(x)dx\bigg)\bigg(\int_{J+}f(t)dt\bigg)^r$$

$$\le c\sum_{J\in D_m(\mathbb{R})}\Big\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r}h(\cdot)\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)/r}(\mathbb{R})}\Big\|h\Big\|_{L^{(q(\cdot)/r)'}(\mathbb{R})}\bigg(\int_{J+}f(t)dt\bigg)^r$$

$$\le c\sum_{J\in D_m(\mathbb{R})}\Big\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)/r}(\mathbb{R})}\bigg(\int_{J+}f(t)dt\bigg)^r$$

$$=c\sum_{J\in D_m(\mathbb{R})}\bigg(\int_{J^+}f(x)\Big\|v(\cdot)|J|^{\alpha(\cdot)-1}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R})}dx\bigg)^r$$

$$=c\sum_{J\in D_m(\mathbb{R})}|J|^{-r/(p_-)'}\bigg(\int_{J^+}f(x)\Big\|v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R})}dx\bigg)^r$$

$$\le c\sum_{J\in D_m(\mathbb{R})}|J|^{-r/(p_-)'}\bigg(\int_{J^+}f(x)\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(x)dx\bigg)^r$$

$$\le c\|f(\cdot)\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(\cdot)\|^r_{L^{p_-}(\mathbb{R})}\le c\|f(\cdot)\widetilde{N}^{-}_{\alpha(\cdot)}v(\cdot)\|^r_{L^{p(\cdot)}(\mathbb{R})}.$$

Here we used the inequality

$$\Big\|v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R})}\le C_{\alpha,p}\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(x),\quad x\in J_+,$$

which follows in the same manner as Lemma 5.4 was proved. Now Proposition 5.1 completes the proof. □

7 The trace inequality for one-sided potentials

Let

$$R_{\alpha(\cdot)}f(x) = \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{1-\alpha(x)}}dt; \qquad x \in \mathbb{R},$$

$$W_{\alpha(\cdot)}f(x) = \int_{x}^{\infty} \frac{f(t)}{(t-x)^{1-\alpha(x)}}dt; \qquad x \in \mathbb{R},$$

where α is a measurable function on $\mathbb{R}$ with $0 < \alpha_- \le \alpha_+ < 1$.

Here we establish criteria which guarantees the boundedness of $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$.

Theorem G ([24]). *Suppose that $1 < p < q_- \le q_+ < \infty$, where p is constant. Let $0 < \alpha_- \le \alpha_+ < 1$. Then the generalized Riesz potential*

$$T_{\alpha(\cdot)}f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha(x)}}dy, \qquad x \in \mathbb{R},$$

is bounded from $L^p(\mathbb{R})$ to $L^{q(\cdot)}_v(\mathbb{R})$ if and only if

$$\sup_{J\subset\mathbb{R}} \big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}_v(\mathbb{R})}|J|^{-\frac{1}{p}} < \infty, \tag{22}$$

where the supremum is taken over all bounded intervals $J \subset \mathbb{R}$.

Now we prove the following statement:

Theorem 7.1. *Let $I := \mathbb{R}$ and let measurable functions p, q, and α satisfy the conditions $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. Further, suppose that $p \in \mathcal{G}(I)$.*

If

$$\sup_{J\subset\mathbb{R}} \big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}_v(\mathbb{R})}|J|^{-\frac{1}{p_-}} < \infty,$$

where the supremum is taken over all bounded intervals $J \subset \mathbb{R}$, then $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ are bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$.

Proof. The result is a direct consequence of the inequalities

$$\big(R_{\alpha(\cdot)}f\big)(x) \le \big(T_{\alpha(\cdot)}f\big)(x), \ \ \big(W_{\alpha(\cdot)}f\big)(x) \le \big(T_{\alpha(\cdot)}f\big)(x) \quad (f \ge 0),$$

Theorem G and Proposition C. □.

Theorem 7.2. *Let $I := \mathbb{R}$ and let p, q and α satisfy the conditions of Theorem G. Then the following conditions are equivalent:*

(i) $R_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}_v(I)$;

(ii) $W_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}_v(I)$;

(iii) *condition* (22) *holds.*

Proof. The implications (iii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (i) follow from Theorems 7.1 and G.

Let us now show that (i) $\Rightarrow$ (iii). Let $f(x) = \chi_{(a,a+h)}(x)$, where $a \in \mathbb{R}$ and $h > 0$. Then $\|f\|_{L^p(\mathbb{R})} = h^{\frac{1}{p}}$. On the other hand,

$$\big\|R_{\alpha(\cdot)}f\big\|_{L_v^{q(\cdot)}(\mathbb{R})} \geq \Big\|\chi_{(a,a+h)}(\cdot)\Big(\int\limits_{a-h}^{a} \frac{dt}{(x-t)^{1-\alpha(x)}}\Big)\Big\|_{L_v^{q(\cdot)}(\mathbb{R})}$$

$$\geq C\big\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)}\big\|_{L_v^{q(\cdot)}(\mathbb{R})}.$$

Hence, (i) implies that

$$\big\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)}\big\|_{L_v^{q(\cdot)}(\mathbb{R})}h^{-\frac{1}{p}} \leq C$$

for all $a \in \mathbb{R}$ and $h > 0$. This implies (iii). Analogously the implication (ii)$\Rightarrow$(iii) can be derived. $\Box$

8 Hardy–Littllewood type inequalities

The results of the previous section enable us to formulate necessary and sufficient conditions governing the Hardy–Littlewood (see [17]) type inequalities for one–sided potentials. For these inequalities in classical Lebesgie spaces we refer also to [46]. In particular, we give necessary and sufficient conditions on q, p and α for which $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ are bounded from L^p to $L^{q(\cdot)}$, where p is constant.

Theorem 8.1. *Let $I = \mathbb{R}$ and let p, q and α satisfy the conditions of Theorem G. Then the following conditions are equivalent:*

(i) $R_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$;

(ii) $W_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$;

(iii) $\sup\limits_{J\subset\mathbb{R}} \big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}(J)}|J|^{-\frac{1}{p}} < \infty$,

where the supremum is taken over all bounded intervals J in $\mathbb{R}$.

9 Two-weight inequalities for monotonic weights

Let

$$(T_{v,w}f)(x) = v(x)\int_0^x f(y)w(y)dy, \quad x \in \mathbb{R}_+,$$

$$(T'_{v,w}f)(x) = v(x)\int_x^\infty f(y)w(y)dy, \quad x \in \mathbb{R}_+.$$

In the sequel we will use the following notation:

$$v_\alpha(x) := \frac{v(x)}{x^{1-\alpha}}, \quad \widetilde{w}(x) := \frac{1}{w(x)}, \quad \overline{w}(x) := \frac{1}{w(x)x}, \quad \overline{w}_\alpha(x) := \frac{1}{x^{1-\alpha}w(x)}.$$

Let us fix a positive number a and let

$$p_0(x) := p_-([0,x]), \quad \widetilde{p}_0(x) := \begin{cases} p_{\scriptscriptstyle 0}(x), & \text{if } x \le a; \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$

$$p_1(x) := p_-([x,a]); \quad \widetilde{p}_1(x) := \begin{cases} p_{\scriptscriptstyle 1}(x), & \text{if } x \le a; \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$

$$I_k := [2^{k-1}, 2^{k+2}]; \quad k \in \mathbb{Z}, I_k = [2^k, 2^{k+1}]; \quad k \in \mathbb{Z},$$

where $(0,x)$ and $[0,x]$ are open and close intervals respectively.

Recall that a function p satisfies the Dini–Lipschitz condition on $\mathbb{R}_+$, i.e, $p \in DL(\mathbb{R}_+)$ if (2) holds for $x, y \in \mathbb{R}_+$ satisfying the condition $0 < |x-y| \le \frac{1}{2}$.

The following two statement are known (see [15]):

Theorem 9.1. *Let $1 < \widetilde{p}_{\scriptscriptstyle 0}(x) \le p(x) \le p_+ < \infty$. Suppose that there exists a positive number a such that $p(x) = p_c = const$ when $x > a$. If*

$$\sup_{t>0} \int_t^\infty \Big(v(x)\Big)^{p(x)} \left(\int_0^t w(y)^{(\widetilde{p}_0)'(x)} dy\right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx < \infty$$

then $T_{v,w}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

Theorem 9.2. *Let $1 < \widetilde{p}_{\scriptscriptstyle 1}(x) \le p(x) \le p_+ < \infty$. Suppose that there exists a positive number a such that $p(x) = p_c = const$, when $x > a$. If*

$$\sup_{t>0} \int_0^t (v(x))^{p(x)} \left(\int_t^\infty w(y)^{(\widetilde{p}_1)'(x)} dy\right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} dx < \infty,$$

then $T'_{v,w}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

The next two lemmas will be useful for us.

Lemma 9.1 ([2]). *Let $1 \le p_- \le p(x) \le q(x) \le q_+ < \infty$, $p \in DL(\mathbb{R}_+)$ and let $p(x) = p_c = const$, $q(x) = q_c = const$ when $x > a$ for some positive number a. Then there exist a positive constant c such that*

$$\sum_i \| f\chi_{I_i} \|_{L^{p(\cdot)}(\mathbb{R}_+)} \| g\chi_{I_i} \|_{L^{q'(\cdot)}(\mathbb{R}_+)} \le c \| f \|_{L^{p(\cdot)}(\mathbb{R}_+)} \| g \|_{L^{q'(\cdot)}(\mathbb{R}_+)}$$

for all $f$and g with $f \in L^{p(\cdot)}(\mathbb{R}_+)$ and $g \in L^{q'(\cdot)}(\mathbb{R}_+)$.

Lemma 9.2 ([4]). *Let $p \in DL(\mathbb{R}_+)$. Then there exist a positive constant c such that for all open intervals I in $\mathbb{R}_+$ satifying the condition $| I |> 0$ we have*

$$| I |^{p_-(I) - p_+(I)} \le c.$$

Now we prove some lemmas.

Lemma 9.3. *Let $1 < p_- \le p_0(x) \le p(x) \le p_+ < \infty$ and let $p(x) \equiv p_c \equiv const$ if $x > a$ for some positive constant a. Suppose that v and w are positive increasing functions on $\mathbb{R}_+$ satisfying the condition*

$$B := \sup_{t>0} \int_t^\infty \Big(\frac{v(x)}{x}\Big)^{p(x)} \Big(\int_0^t w(y)^{-(\widetilde{p}_0)'(x)} dy\Big)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx < \infty. \tag{23}$$

Then $v(4x) \le cw(x)$ for all $x > 0$, where the positive constant c is independent of x.

Proof. First assume that $0 < t < a$. The fact that $\overline{c} = \overline{\lim\limits_{t\to 0}} \frac{v(4t)}{w(t)} < \infty$ follows from the inequalities:

$$\int_t^\infty \Big(\frac{v(x)}{x}\Big)^{p(x)} \Big(\int_0^t w(y)^{-(\widetilde{p}_0)'(x)} dy\Big)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx$$

$$\ge \int_{4t}^{8t} \Big(\frac{v(4t)}{w(t)}\Big)^{p(x)} \cdot t^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} \cdot x^{-p(x)} dx$$

$$\ge \Big(\frac{v(4t)}{w(t)}\Big)^{p_-} \int_{4t}^{8t} t^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} \cdot x^{-p(x)} dx \ge c \Big(\frac{v(4t)}{w(t)}\Big)^{p_-},$$

where the positive constant c is independent of a small positive number t.

Further, suppose that δ is a positive number such that $v(4t) \le (\overline{c}+1)w(t)$ when $t < \delta$. If $\delta < a$, then for all $\delta < t < a$, we have that

$$v(4t) \le v(4a) \le \widetilde{c}w(\delta) \le \widetilde{c}w(t),$$

where $\overline{c}$ depends on v, w and δ. Now it is enough to take $c = \max\{(\overline{c}+1), \overline{c}\}$.

Let now $a \le t < \infty$. Then $p(x) \equiv p_c \equiv$ const for $x > t$ and, consequently,

$$B \ \ge \ \sup_{t>0} \Big(\int_t^\infty \Big(\frac{v(x)}{x}\Big)^{p_c} dx\Big) \Big(\int_0^t w(x)^{-p_c'} dx\Big)^{p_c - 1} \ge c \Big(\frac{v(4t)}{w(t)}\Big)^{p_c}.$$

The lemma is proved. □

The proof of the next lemma is similar to that of the previous one; therefore we omit it.

Lemma 9.4. *Let $1 < p_- \le p_1(x) \le p(x) \le p_+ < \infty$, and let $p(x) \equiv p_c \equiv const$ if $x > a$ for some positive constant a. Suppose that v and w are positive decreasing functions on $\mathbb{R}_+$. If*

$$\widetilde{B} := \sup_{t>0} \int_0^t (v(x))^{p(x)} \Big(\int_t^\infty (\overline{w}(y))^{(\widetilde{p}_1)'(x)} dy\Big)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} dx < \infty, \tag{24}$$

then $v(x) \le cw(4x)$, where the positive constant c does not depend on $x > 0$.

Theorem 9.3. *Let $1 < p_- \le p_+ < \infty$ and let $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $\in (a,\infty)$ for some positive number a. Let v and w be weights on $\mathbb{R}_+$ such that*

(a) *$T'_{v_0,\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$;*

(b) *there exists a positive constant b such that one of the following two conditions hold:*

$$\text{(i)}\quad \operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \le bw(x) \ \ \text{for almost all} \ \ x \in \mathbb{R}_+;$$

$$\text{(ii)}\quad v(x) \le b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y) \ \ \text{for almost all} \ \ x \in \mathbb{R}_+.$$

Then M^- is bounded from $L^{p(\cdot)}_w(\mathbb{R}_+)$ to $L^{p(\cdot)}_v(\mathbb{R}_+)$.

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \le 1$. We have

$$\int_0^\infty \big(M^- f(x)\big)v(x)g(x)dx \le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{1,k}(x)\big)v(x)g(x)dx$$

$$+\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{2,k}(x)\big)v(x)g(x)dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{3,k}(x)\big)v(x)g(x)dx := S_1+S_2+S_3,$$

where $f_{1,k} = f\cdot\chi_{[0,2^{k-1}]}$, $f_{2,k} = f\cdot\chi_{[2^{k+1},\infty]}$, $f_{3,k} = f\cdot\chi_{[2^{k-1},2^{k+2}]}$.

If $y\in[0,2^{k-1})$ and $x\in[2^k,2^{k+1}]$, then $y<x/2$. Hence $x/2 \le x-y$. Consequently, if $h<x/2$, then for $x\in[2^{k-1},2^{k+2}]$, we have

$$\frac{1}{h}\int_{x-h}^{x} \mid f_{1,k}(y)\mid dy \;=\; \frac{1}{h}\int_{x-h}^{x} \mid f\cdot\chi_{[0,2^{k-1}]}\mid dy = 0.$$

Further, if $h>\frac{x}{2}$, then

$$\frac{1}{h}\int_{x-h}^{x} \mid f_{1,k}(y)\mid dy \;=\; \frac{1}{h}\int_{x-h}^{x} \mid f\cdot\chi_{[0,2^{k-1}]}\mid dy \le c\,\frac{1}{x}\int_0^x \mid f(y)\mid dy.$$

This yields that

$$M^- f_{1,k}(x) \le c\,\frac{1}{x}\int_0^x \mid f(y)\mid dy \quad \text{for} \ \ x\in[2^k,2^{k+1}].$$

Hence, due to the boundedness of $T_{\bar{v},\widetilde{w}}$ in $L^{p(x)}(\mathbb{R}_+)$ we have that

$$\begin{aligned} S_1 &\le c\int_0^\infty \left(T_{v_0,1}|f|\right)(x)\ v(x)g(x)dx \\ &\le c\,\|(T_{v_0,1}|f|)\ v\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \le c\,\|fw\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

Observe now that $S_2 =$ because $f_{2,k} = f \cdot \chi_{[2^{k+2},\infty]}$. Let us estimate S_3. By using condition (i) of (b), boundedness of the operator M^- in $L^{p(\cdot)}(\mathbb{R}_+)$ and lemma 9.1 we have that

$$\begin{aligned} S_3 &\leq c\sum_k (\text{ess}\sup_{E_k} \text{v}) \|M^- f_{3,k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leq c\sum_k (\text{ess}\sup_{E_k} \text{v}) \|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leq c\, \|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

If condition (ii) of (b) holds, then

$$v(z) \leq b \ \text{ess}\inf_{y\in[\frac{z}{4},4z]} w(y) \leq b \ \text{ess}\inf_{y\in(2^{k-1},2^{k+2})} w(y) \leq bw(x)$$

for $z \in E_k$ and $x \in I_k$. Hence,

$$\text{ess}\sup_{E_k} w(y) \leq bw(x)$$

if $x \in I_k$. Consequently, taking into account this inequality and the estimate of S_3 in the previous case we have the desire result for M^-. □

Theorem 9.4. *Let $1 < p_- \leq p_+ < \infty$ and let $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive number. Let v and w be weight functions on $\mathbb{R}_+$ such that*

(a) *$T'_{v,\overline{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$;*

(b) *there exists a positive constant b such that one of the following two conditions hold:*

(i) $\quad \text{\it ess sup}_{y\in[\frac{x}{4},4x]} v(y) \leq bw(x)$ *for almost all* $x \in \mathbb{R}_+$;

(ii) $\quad v(x) \leq b \,\text{\it ess inf}_{y\in[\frac{x}{4},4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then M^+ is bounded from $L^{p(\cdot)}_w(\mathbb{R}_+)$ to $L^{p(\cdot)}_v(\mathbb{R}_+)$.

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leq 1$. We have

$$\int_0^\infty \big(M^+f(x)\big)v(x)g(x)dx \leq \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{1,k}(x)\big)v(x)g(x)dx$$

$$+\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{2,k}(x)\big)v(x)g(x)dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{3,k}(x)\big)v(x)g(x)dx := S_1 + S_2 + S_3,$$

where $f_{i,k}$, $i = 1, 2, 3$ are defined in the proof of the previous theorem. It is easy to see that $S_1 = 0$. To estimate S_2 observe that

$$M^+f \cdot \chi_{[2^{k+1},\infty)}(x) \leq c \sup_{j\geq k+2} 2^{-j} \int\limits_{I_k} |f(y)|dy, \quad x \in E_k, \tag{25}$$

Indeed, notice that if $y \in (2^{k+2}, \infty)$ and $x \in E_k$, then $y - x \geq 2^{k+1}$. Hence,

$$\frac{1}{h}\int_x^{x+h} \mid f_{2,k}(y) \mid dy \leq \frac{1}{h}\int\limits_{\{y: y-x<h, y-x>2^{k+1}\}} |f(y)|dy = 0$$

for $h \leq 2^{k+1}$ and $x \in I_k$.

Let now $h > 2^{k+1}$. Then $h \in [2^j, 2^{j+1})$ for some $j \geq k+1$. If $y - x < h$, then it is clear that $y = y - x + x \leq h + x \leq 2^{j+1} + 2^{k+1} \leq 2^{j+1} + 2^j \leq 2^{j+2}$. Consequently, for such an h we have that

$$\frac{1}{h}\int\limits_x^{x+h} |f_{2,k}(y)|dy = \frac{1}{h}\int\limits_x^{x+h} |f \cdot \chi_{[2^{k+2},\infty)}(y)|dy \leq \frac{1}{h}\int\limits_{\{y: y-x<h, y>2^{k+2}\}} |f(y)|dy$$

$$\leq \frac{1}{x}\int\limits_{\{y:\ y\in[2^{k+2},2^{j+2}]\}} |f(y)|dy \leq \sum_{i=k+1}^{j+1} 2^{-j}\int\limits_{\{y:\ y\in[2^j,2^{j+2}]\}} |f(y)|dy$$

which proves inequality (25).

Taking into account estimate (25) and the boundedness of $T'_{v,\overline{w}}$ in $L^{p(\cdot)}(\mathbb{R}_+)$ we find that

$$S_2 \leq c\sum_k \int\limits_{E_k} v(x)g(x)\Big(\sup_{j\geq k+1} 2^{-j}\int\limits_{E_j} |f(y)|dy\Big)dx$$

$$\leq c\sum_k \Big(\int\limits_{I_k} v(x)g(x)dx\Big)\Big(\sum_{j=k+1}^{\infty} 2^{-j}\int\limits_{E_j} |f(y)|dy\Big)$$

$$= c\sum_j 2^{-j}\Big(\int\limits_{E_j} |f(y)|dy\Big)\sum_{k=-\infty}^{j-1}\Big(\int\limits_{E_k} v(x)g(x)dx\Big)$$

$$= c\sum_j 2^{-j}\Big(\int\limits_{E_j} |f(y)|dy\Big)\Big(\int\limits_0^{2^j} v(x)g(x)dx\Big) \leq c\sum_j \int\limits_{E_j} |f(y)|\, y^{-1}\Big(\int\limits_0^{y} v(x)g(x)dx\Big)dy$$

$$= c\int\limits_{\mathbb{R}_+} |f(y)|\, y^{-1}\Big(\int\limits_0^{y} v(x)g(x)dx\Big)dy = c\int\limits_{\mathbb{R}_+} v(x)g(x)\Big(\int\limits_x^{\infty} |f(y)|\, y^{-1}dy\Big)dx$$

$$\leq c\ \|g\|_{L^{p'(\cdot)}\mathbb{R}_+} \cdot \|T'_{v(\cdot),1/\cdot}f\|_{L^{p(\cdot)}\mathbb{R}_+} \leq c\|fw\|_{L^{p(\cdot)}\mathbb{R}_+}.$$

To estimate S_3 assume that condition (i) of (b) is satisfied. By Lemma 9.1 and the boundedness of the operator M^+ in $L^{p(\cdot)}(\mathbb{R}_+)$ we conclude that

$$\begin{aligned} S_3 &\leq c\sum_k (\text{ess}\sup_{E_k} \text{v})\|M^+ f_{3,k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leq c\sum_k (\text{ess}\sup_{E_k} \text{v})\|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leq c\ \|f(\cdot)w(\cdot)\chi_{I_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \end{aligned}$$

$$\leq c\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}\|g(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leq c\,\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

□

Theorem 9.5. *Let $1 < p_- \leq p_0(x) \leq p(x) \leq p_+ < \infty$ and let $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive constant. Assume that v and w are positive increasing weights on $(0,\infty)$. If condition (23) is satisfied, then M^- is bounded from $L^{p(\cdot)}_w(\mathbb{R}^+)$ to $L^{p(\cdot)}_v(\mathbb{R}^+)$.*

Proof. Follows from Lemma 9.3 and Theorem 9.3. □

Theorem 9.6. *Let $1 < p_- \leq p_1(x) \leq p(x) \leq p_+ < \infty$, and let $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive constant. Let v and w be positive decreasing weights on $(0,\infty)$. If condition (24) is satisfied, then M^+ is bounded from $L^{p(\cdot)}_w(\mathbb{R}^+)$ to $L^{p(\cdot)}_v(\mathbb{R}^+)$.*

Proof. Follows immediately from Lemma 9.4 and Theorem 9.4. □

Let us discuss two–weight estimates for one-sided potentials defined on $\mathbb{R}_+$:

$$\mathcal{R}_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \qquad \mathcal{W}_\alpha f(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}dt,$$

where $x > 0$ and $0 < \alpha < 1$.

The following statements were proved in [13]:

Theorem H. *Let $I = \mathbf{R}_+$ and let $p \in \mathcal{P}_+(I)$. Suppose that there exists a positive constant a such that $p \in \mathcal{P}_\infty((a,\infty))$. Suppose that α is a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Theorem I. *Let $I = \mathbf{R}_+$ and let $p \in \mathcal{P}_+(I)$. Let α be a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and let $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive number a. Then $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Remark A. Theorems H and I are true if we replace the condition $p \in \mathcal{P}_\infty((a,\infty))$ by the condition: p is constant outside an interval $(0,a)$ for some positive number a.

Our next statements regarding one–sided potentials read as follows:

Theorem 9.7. *Let $1 < p_- \leq p_+ < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive number. Let v and w be a.e. positive measurable functions on $\mathbb{R}_+$ satisfying the conditions:*

(a) *$T_{v_\alpha,\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$,*

(b) *there exists a positive constant b such that one of the following two conditions hold:*

$$\text{(i)} \qquad \operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \leq b w(x) \ \text{ for almost all } \ x \in \mathbb{R}_+;$$

$$\text{(ii)} \qquad v(x) \leq b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y) \quad \text{for almost all } \ x \in \mathbb{R}_+.$$

Then $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}_w(\mathbb{R}_+)$ to $L^{q(\cdot)}_v(\mathbb{R}_+)$.

Theorem 9.8. *Let $1 < p_- \le p_+ < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in DL(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive number. Let v and w be a.e. positive measurable functions on $\mathbb{R}_+$ satisfying the conditions:*

(a) *$T'_{v,\overline{w}_\alpha}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$,*

(b) *there exists a positive constant b such that one of the following two conditions hold:*

$$\text{(i)} \qquad \operatorname*{ess\ sup}_{y\in[\frac{x}{4},4x]} v(y) \le bw(x) \ \text{ for almost all } \ x \in \mathbb{R}_+;$$

$$\text{(ii)} \qquad v(x) \le b \operatorname*{ess\ inf}_{y\in[\frac{x}{4},4x]} w(y) \quad \text{for almost all } \ x \in \mathbb{R}_+.$$

Then $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}_w(\mathbb{R}_+)$ to $L^{q(\cdot)}_v(\mathbb{R}_+)$.

Proof of Theorem 9.7. Let $f \ge 0$ and let $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \le 1$. It is obvious that

$$\int_0^\infty (\mathcal{R}_\alpha f(x))\, v(x) g(x) dx \le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{1,k}(x))\, v(x) g(x) dx$$

$$+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{2,k}(x))\, v(x) g(x) dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{3,k}(x))\, v(x) g(x) dx :=$$

$$= S_1 + S_2 + S_3,$$

where $f_{i,k}$, $i = 1, 2, 3$ are defined in the proof of Theorem 8.3

If $y \in [0, 2^{k-1})$ and $x \in [2^k, 2^{k+1}]$, then $y < \frac{x}{2}$. Hence

$$\mathcal{R}_\alpha f_{1,k}(x) \le \frac{c}{x^{1-\alpha}} \int_0^x f(t) dt, \quad x \in [2^{k-1}, 2^{k+2}].$$

By using Hölder's inequality, Theorem 9.1, Remark A we find that condition (i) guarantees the estimate

$$S_1 \le c\|fw\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Further, observe that if $x \in [2^k, 2^{k+1})$, then $\mathcal{R}_\alpha f_{2,k}(x) = 0$. Hence $S_2 = 0$.

To estimate S_3 we argue as in the case of the proof of Theorem 9.3. □

The proof of these theorems are based on the following lemmas which can be derived easily by using monotonicity of the weights v, w and the fact that $q(x) = \frac{p(x)}{1-\alpha p(x)}$:

The proof of the next two lemmas are similar to that of Lemma 9.3; therefore we omit it.

Lemma 9.5. *Let the conditions of Theorem 9.9 be satisfied. Then there is a positive constant c such that for all $t > 0$ the inequality*

$$v(4t) \le cw(t)$$

is satisfied.

Lemma 9.6. *Let the conditions of Theorem* 9.10 *be satisfied. Then there is a positive constant b such that for all $t > 0$ the inequality*

$$v(t) \leq bw(4t)$$

holds.

These lemmas and Theorems 9.7 and 9.8 immediately imply the following statements:

Theorem 9.9. *Let $1 < p_- \leq p_+ < \infty$ and let α be a constant satisfying the condition $\alpha < 1/p_+$. Suppose that $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $p \in DL(\mathbb{R}_+)$. Assume that $p(x) \equiv p_c \equiv$ const outside some interval $[0, a]$, where a is a positive constant. Let v and w be positive increasing functions on $\mathbb{R}_+$ satisfying the condition*

$$\int\limits_t^\infty (v_\alpha(x))^{q(x)} \left(\int\limits_0^t w^{-(\widetilde{p}_0)'(x)}(y)dy \right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} dx < \infty.$$

Then $\mathcal{R}_\alpha$ is bounded from $L_w^{p(\cdot)}(\mathbb{R})$ to $L_v^{q(\cdot)}(\mathbb{R})$.

Theorem 9.10. *Let $1 < p_- \leq p_+ < \infty$ and let α be a constant satisfying the condition $\alpha < 1/p_+$. Suppose that $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $p \in DL(\mathbb{R}_+)$. Suppose also that $p(x) \equiv p_c \equiv$ const outside some interval $[0, a]$, where a is a positive constant and that v and w are positive decreasing functions on $\mathbb{R}_+$ satisfying the condition*

$$\sup_{t>0} \int_0^t (v(x))^{p(x)} \left(\int_t^\infty (\overline{w}_\alpha(y))^{(\widetilde{p_1})'(x)} \, dy \right)^{\frac{p(x)}{(\widetilde{p_1})'(x)}} dx < \infty.$$

Then $\mathcal{W}_\alpha$ is bounded from $L_w^{p(\cdot)}(\mathbb{R})$ to $L_v^{q(\cdot)}(\mathbb{R})$.

Acknowledgement. The first and second authors were partially supported by the Georgian National Science Foundation Grant No. GNSF/ST07/3-169.

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Received: 18.09.2009