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TWO–WEIGHT INEQUALITIES FOR FRACTIONAL MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN $L^{p(\cdot)}$ SPACES

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Various type of two–weight criteria for fractional maximal operators (involving the Hardy–Littlewood maximal functions) and singular integrals in variable exponent Lebesgue spaces defined on the real line are established. Bibliography: 30 titles.

Introduction

We study the two-weight problem for fractional maximal operators (involving the Hardy–Littlewood maximal functions) and singular integrals in variable exponent Lebesgue spaces $L^{p(\cdot)}$. In particular, we derive two–weight criteria of various type for maximal operators and the Hilbert transform on the line. For a bounded interval we assume that the exponent p satisfies the local log-Hölder continuity condition and for the real line we require that p is constant outside some interval. In the framework of variable exponent analysis such a condition first appeared in the paper [1], where the author established the boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Unfortunately, we do not know whether the established criteria remain valid when p satisfies log–Hölder decay condition at infinity (cf., for example, [2] for the log–Hölder decay condition). It is known that the local log-Hölder continuity condition for the exponent p together with the log-Hölder decay condition guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (cf. [2]–[5]).

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The boundedness of the maximal, potential and singular operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces was derived in [1]–[7]. Weighted inequalities for classical operators in $L^{p(\cdot)}_w$ spaces, where w is a power type weight, were established in the papers [8]–[14] etc, while the same problems with general weights for Hardy, maximal and fractional integral operators were studied in [10] and [15]–[21]. Moreover, in [21], a complete solution of the one–weight problem for the Hardy–Littlewood maximal functions defined on Euclidean spaces are given in terms of Muckenhoupt type conditions. Finally, we note that, in [17], modular type sufficient conditions governing the two–weight inequality for maximal and singular operators were established.

Some results of this paper without proofs were announced in [22].

Throughout the paper, J denotes an interval (bounded or unbounded) in $\mathbb{R}$.

Let p be a nonnegative function on $\mathbb{R}$. Suppose that E is a measurable subset of $\mathbb{R}$. We use the following notation:

$$p_-(E) := \inf_E p, \quad p_+(E) := \sup_E p, \qquad p_- := p_-(\mathbb{R}), \quad p_+ := p_+(\mathbb{R}).$$

Assume that $1 \leqslant p_-(J) \leqslant p_+(J) < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(J)$ (sometimes it is denoted by $L^{p(x)}(J)$) is the class of all μ-measurable functions f on X for which

$$S_p(f) := \int\limits_J |f(x)|^{p(x)} dx < \infty.$$

The norm in $L^{p(\cdot)}(J)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(J)} = \inf\{\lambda > 0 : S_p(f/\lambda) \leqslant 1\}.$$

It is known (cf., for example, [23, 24, 8]) that $L^{p(\cdot)}$ is a Banach space. For other properties of $L^{p(\cdot)}$ spaces we refer, for example, to [23]–[25].

Finally, we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c or C. The symbol $f(x) \approx g(x)$ means that there are positive constants c_1 and c_2 independent of x such that the inequality $f(x) \leqslant c_1 g(x) \leqslant c_2 f(x)$ holds. Throughout the paper, $p'(x)$ denotes the function $p(x)/(p(x)-1)$.

1 Sawyer Type Condition for Maximal Operators in $L^{p(x)}$ Spaces

1.1 The case of bounded interval

Let J be a bounded interval in $\mathbb{R}$, and let

$$(M_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int\limits_I |f(y)| dy, \quad x \in J,$$

where $x \in J$ and α is a constant satisfying the condition $0 \leqslant \alpha < 1$.

For a weight function u we denote

$$u(E) := \int\limits_E u(x) dx.$$

Definition. Let J be a bounded interval in $\mathbb{R}$. We say that a nonnegative function u satisfies the *doubling condition* on J ($u \in DC(J)$) if there is a positive constant b such that for all $x \in J$ and all r, $0 < r < |J|$,

$$u\big(I(x-2r, x+2r) \cap J\big) \leqslant bu\big(I(x-r, x+r) \cap J\big).$$

Definition. We say that $p \in LH(J)$ (p satisfies the local log-Hölder condition) if there is a positive constant c such that

$$|p(x) - p(y)| \leqslant \frac{c}{-|x-y|}$$

for all $x, y \in J$ satisfying the condition $|x - y| \leqslant 1/2$.

Theorem 1.1. *Let $1 < p_- \leqslant p(x) \leqslant p_+ < \infty$, and let the measure $d\nu(x) = w(x)^{-p'(x)}dx$ belong to $DC(J)$. Suppose that $0 \leqslant \alpha < 1$ and $p \in LH(J)$. Then the inequality*

$$\|v(\cdot)M_\alpha^{(J)} f\|_{L^{p(\cdot)}(J)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds if and only if there exists a positive constant c such that for all intervals $I \subset J$

$$\int_I (v(x))^{p(x)} (M_\alpha^{(J)}(w(\cdot)^{-p'(\cdot)}\chi_{I(\cdot)}))^{p(x)}(x)dx \leqslant c \int_I w^{-p'(x)}(x)dx < \infty.$$

To prove Theorem 1.1, we need some auxiliary statements.

Proposition A ([26, Lemma 3.20])**.** *Let s be a constant satisfying the condition $1 < s < \infty$, and let $u \geqslant 0$ on $\mathbb{R}$. Suppose that $\{Q_i\}_{i\in A}$ is a countable collection of dyadic intervals in $\mathbb{R}$ and $\{a_i\}_{i\in A}$,$\{b_i\}_{i\in A}$ are sequences of positive numbers satisfying the following conditions*:

(i) $\displaystyle\int_{Q_i} u \leqslant a_i$ *for all* $i \in A$,

(ii) $\displaystyle\sum_{\{j\in A: Q_j \subset Q_i\}} b_j \leqslant ca_i$ *for all* $i \in A$.

Then there is a positive constant c_s depending on s such that

$$\bigg(\sum_{i\in A} b_i \bigg(\frac{1}{a_i}\int_{Q_i} gu\bigg)^s\bigg)^{1/s} \leqslant c_s \bigg(\int_{\mathbb{R}} g^s u\bigg)^{1/s}$$

for all nonnegative functions g.

Corollary A. *Let $1 < s < \infty$, and let u be a nonnegative measurable function on $\mathbb{R}$. Suppose that $\{Q_i\}_{i\in A}$ is a sequence of dyadic cubes in $\mathbb{R}^n$ and that $\{b_i\}_{i\in A}$ is a sequence of positive numbers satisfying the condition*

$$\sum_{\{j\in A: Q_j \subset Q_i\}} b_j \leqslant cu(Q_i).$$

Then there is a positive constant c such that for all nonnegative functions g

$$\sum_{i\in A} b_i\Big(\frac{1}{u(Q_i)}\int\limits_{Q_i} gu\Big)^s \leqslant c\Big(\int\limits_{\mathbb{R}} g^s u\Big)^{1/s}.$$

Lemma A. *Let J be a bounded interval, and let $1 \leqslant r_-(J) \leqslant r_+(J) < \infty$. Suppose that $r \in LH(J)$ and the measure μ satisfies the condition $\mu \in DC(J)$. Then there is a positive constant c such that for all f, $\|f\|_{L^{r(\cdot)}(J,\mu)} \leqslant 1$, intervals $I \subseteq J$, and $x \in I$*

$$\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|d\mu(y)\Big)^{r(x)} \leqslant c\Big[\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|^{r(y)}d\mu(y)\Big)+1\Big].$$

Proof. We follow the idea of Diening [1] (cf. also [27] for the similar statement in the case of metric measure spaces with doubling measure). We give the proof for completeness.

First recall that (cf., for example, [27]), since J with the Euclidean distance and the measure μ is a bounded doubling metric measure space with the finite measure μ, the condition $r \in LH(J)$ implies the inequality

$$\big(\mu(I)\big)^{r_-(I)-r_+(I)} \leqslant C \tag{1.1}$$

for all subintervals I of J.

Assume that $\nu B \leqslant 1/2$. By the Hölder inequality,

$$\begin{aligned}\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|d\mu(y)\Big)^{r(x)} &\leqslant \Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|^{r_-(I)}d\mu(y)\Big)^{r(x)/r_-(I)}\\ &\leqslant c\mu(I)^{-r(x)/r_-(I)}\Big[\frac{1}{2}\int\limits_I |f(y)|^{r(y)}d\mu(x)+\frac{1}{2}\mu(I)\Big]^{r(x)/r_-(I)}.\end{aligned}$$

Note that the expression in the bracket is less than or equal to 1. Consequently, by (1.1), we find

$$\begin{aligned}\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|d\mu(y)\Big)^{r(x)} &\leqslant c\mu(I)^{1-r(x)/r_-(I)}\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|^{r(y)}d\mu(y)+1\Big)\\ &\leqslant c\mu(I)^{(r_-(I)-r_+(I))/r_-(I)}\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|^{r(y)}d\mu(y)+1\Big)\\ &\leqslant c\Big(\frac{1}{\mu(I)}\int\limits_I |f(y)|^{r(y)}d\mu(y)+1\Big).\end{aligned}$$

The case $\mu(I) > 1/2$ is trivial. □

Suppose that S is an interval in $\mathbb{R}$ and introduce the dyadic maximal operator

$$\big(M_\alpha^{(d),S}\big)f(x) = \sup_{\substack{x\in I\\ I\in D(S)}} |I|^{\alpha-1}\int\limits_I |f(y)|dy,$$

where $0 \leqslant \alpha < 1$ and $D(S)$ is a dyadic lattice in S.

To prove Theorem 1.1, we need some assertions.

Lemma 1.1. *Let S be a bounded interval on $\mathbb{R}$, and let J be a subinterval of S. Suppose that $\sigma(x) := w^{-p'(x)}$ belongs to the class $DC(J)$ and $p \in LH(J)$, where $1 < p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$. Let $0 \leqslant \alpha < 1$. If there is a positive constant c such that for all interval $I \subset J$*

$$\int\limits_I (v(x))^{p(x)}\Big(M_\alpha^{(d),S}\big(\chi_I(\cdot)\sigma(\cdot)\big)\Big)^{p(x)}(x)dx \leqslant c\int\limits_I \sigma(x)dx < \infty,$$

then the following estimate holds:

$$\|v(\cdot)M_\alpha^{(d),S}\big(f(\cdot)\chi_J(\cdot)\big)\|_{L^{p(\cdot)}(J)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}.$$

Proof. Suppose that $\|f\|_{L_w^{p(\cdot)}(J)} \leqslant 1$. Assume that $f_1 := \chi_J f$. Let us introduce the set

$$J_k = \{x \in S : 2^k < (M_\alpha^{(d),S}f_1)(x) \leqslant 2^{k+1}\}, \quad k \in \mathbb{Z}.$$

Suppose that for k, $J_k \neq \varnothing$, $\{I_j^k\}$ is a maximal dyadic interval, $I_j^k \subset D(S)$, such that

$$\frac{1}{|I_j^k|^{1-\alpha}}\int\limits_{I_j^k} |f_1(y)|dy > 2^k. \tag{1.2}$$

It is obvious that such a maximal interval always exists. Now observe that

(i) $\{I_j^k\}$ are disjoint for fixed k,

(ii) $\overline{J}_k := \{x \in S : \big(M_\alpha^{(d),S}f_1\big)(x) > 2^k\} = \cup_j I_j^k$.

Indeed, (i) holds because $I_i^k \cap I_j^k \neq \varnothing$ implies $I_i^k \subset I_j^k$ or $I_j^k \subset I_i^k$. Consequently, if $I_i^k \subset I_j^k$, then I_j^k is a maximal interval for which (1.2) is fulfilled.

To prove (ii), we note that if $x \in \overline{J}_k$, then $M_\alpha^{(d),S}f_1(x) \geqslant 2^k$. Hence there is a maximal dyadic interval I_j^k containing x such that (1.2) holds for I_j^k. Let $x \in \bigcup\limits_j I_j^k$. Then $x \in I_{j_0}^k$ for some j_0. Hence $\big(M_\alpha^{(d),S}f_1\big)(x) > 2^k$ because (1.2) holds for $I_{j_0}^k$.

Denote

$$E_j^k := I_j^k \backslash \{x \in S : M_\alpha^{(d),S}f_1(x) > 2^{k+1}\}.$$

Then $E_j^k = I_j^k \cap J_k$. Indeed, if $x \in E_j^k$, then $x \in I_j^k$ and $M_\alpha^{(d),S}f_1(x) \leqslant 2^{k+1}$. Hence, by (1.2),

$$2^k < |I_i^k|^{\alpha-1}\int\limits_{I_j^k} |f_1(y)|dy \leqslant M_\alpha^{(d),S}f_1(x) \leqslant 2^{k+1}.$$

This means that $x \in I_j^k \cap J_k$. Let $x \in I_j^k \cap J_k$. Then $M_\alpha^{(d),S} f_1(x) \leqslant 2^{k+1}$. Consequently, $x \in E_j^k$.

Note that $\{E_j^k\}$ are disjoint for every j, k because, as we have seen,

$$E_j^k = \{x \in I_j^k : 2^k < M_\alpha^{(d),S} f_1(x) \leqslant 2^{k+1}\}.$$

Moreover, $E_j^k \subset I_j^k$. Assume that $\|w(\cdot) f_1(\cdot)\|_{L^{p(\cdot)}(S)} \leqslant 1$. Denote

$$v_1 := v\chi_J, \quad \sigma_1 := \sigma\chi_J.$$

By the above arguments and Lemma A with $r(\cdot) = p(\cdot)/p_-$ and the measure $d\mu(x) = \sigma(x)dx$, we have

$$\begin{aligned}
&\int_J (v(x))^{p(x)} \big(M_\alpha^{(d),S} f_1\big)^{p(x)}(x)dx \\
&\qquad = \int_S (v_1(x))^{p(x)} \big(M_\alpha^{(d),S} f_1\big)^{p(x)}(x)dx \leqslant \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} 2^{(k+1)p(x)} dx \\
&\qquad \leqslant c \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{1}{|I_j^k|^{1-\alpha}} \int_{I_j^k} |f_1(y)| dy \bigg)^{p(x)} dx \\
&\qquad = c \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} \bigg(\frac{1}{\sigma(I_j^k \cap J)} \int_{I_j^k} \Big| \frac{f_1}{\sigma} \Big| \sigma \bigg)^{p(x)} dx \\
&\qquad = c \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} \bigg(\frac{1}{\sigma(I_j^k \cap J)} \int_{I_j^k} \Big| \frac{f_1}{\sigma} \Big| \sigma \bigg)^{p(x)} dx \\
&\qquad \leqslant c \sum_{j,k} \bigg(\int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} dx \bigg) \bigg(\frac{1}{\sigma(I_j^k \cap J)} \int_{I_j^k} \Big| \frac{f_1(y)}{\sigma(y)} \Big|^{\frac{p(y)}{p_-}} \sigma(y) dy \bigg)^{p_-} \\
&\qquad\quad + c \sum_{j,k} \bigg(\int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} dx \bigg) \\
&\qquad \equiv c \bigg(\sum_{j,k} A_j^k + \sum_{j,k} B_j^k \bigg).
\end{aligned}$$

Note that the sign of sum is taken over all those j and k for which $\sigma(I_j^k \cap J) > 0$).

To use Corollary A, we note that

$$\begin{aligned}
&\sum_{\substack{I_j^k \subset I_i \\ I_j^k, I_i \in D(S)}} \int_{E_j^k} (v_1(x))^{p(x)} \bigg(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \bigg)^{p(x)} dx \leqslant \sum_{I_j^k \subset I_i} \int_{E_j^k} (v_1(x))^{p(x)} \big(M_\alpha^{(d),S}(\chi_{I_i \cap J}\sigma)\big)^{p(x)}(x)dx \\
&\leqslant \int_{I_i} (v_1(x))^{p(x)} \big(M_\alpha^{(d),S}(\chi_{I_i \cap J}\sigma)\big)^{p(x)}(x)dx \leqslant c \int_{I_i \cap J} \sigma(x)dx = c \int_{I_i} \sigma_1(x)dx.
\end{aligned}$$

Corollary A implies

$$\sum_{j,k} A_j^k = \sum_{j,k}\bigg(\int_{E_j^k}(v_1(x))^{p(x)}\bigg(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\bigg)^{p(x)}dx\bigg)\bigg(\frac{1}{\sigma_1(I_j^k)}\int_{I_j^k}\bigg|\frac{f_1(y)}{\sigma(y)}\bigg|^{\frac{p(x)}{p_-}}\sigma_1(y)dy\bigg)^{p_-}$$

$$\leqslant c\int_S |f_1(x)|^{p(x)}\sigma(x)^{-p(x)}\sigma_1(x)dx = c\int_S |f_1(x)|^{p(x)}w^{p(x)}dx \leqslant c.$$

For the second term we have

$$\sum_{j,k} B_j^k = \sum_{j,k}\int_{E_j^k}(v_1(x))^{p(x)}\bigg(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\bigg)^{p(x)}dx \leqslant \sum_{j,k}\int_{E_j^k}(v_1(x))^{p(x)}\big(M_\alpha^{(d),S}(\chi_J\sigma)\big)^{p(x)}(x)dx$$

$$= \int_J (v(x))^{p(x)}\big(M_\alpha^{(d),S}(\chi_J\sigma)\big)^{p(x)}(x)dx \leqslant c\int_J \sigma(x)dx < \infty.$$

Finally, we conclude that

$$\|v(\cdot)\big(M_\alpha^{(d),S}f_1\big)(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant c$$

for $\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant 1$. □

Proof of Theorem 1.1. *Sufficiency.* Let us take an interval S containing J. Without loss of generality we can assume that S is a maximal dyadic interval and $|J| \leqslant |S|/8$. Further, suppose that J and S have the same center. Without loss of generality we assume that $|S| = 2^{m_0}$ for some integer m_0. Then every interval $I \subset J$ has the length $|I|$ less than or equal to 2^{m_0-3}. Assume that $|I| \in [2^j, 2^{j+1})$ for some j, $j \leqslant m_0 - 4$. Let us introduce the set

$$F = \big\{t \in (-2^{m_0-4}, 2^{m_0-4}) : \text{ there is } I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}\big\}.$$

The simple geometric observation (cf. also [28], p. 431) shows that $|F| \geqslant 2^{m_0-4}$.

Further, let

$$(K_t f_1)(x) := \sup_{\substack{S\supset I_1\ni x\\ I_1\in D(S)-t}} \frac{1}{|I_1|^{1-\alpha}}\int_{I_1}|f_1|, \quad t \in F,$$

where $f_1 = \chi_J f$. Then for x ($x \in J$) there exists $I \ni x$, $I \subset J$, such that

$$|I|^{\alpha-1}\int_I |f_1| > \frac{1}{2}(M_\alpha^{(J)}f_1)(x).$$

For the interval I we have $|I| \in [2^j, 2^{j+1})$, $j \leqslant m_0 - 4$. Therefore, for $t \in F$ there is an interval I_1, $I_1 \in D(S) - t, I \subset I_1 \subset S, |I_1| = 2^{j+1}$, such that

$$|I|^{\alpha-1}\int_I |f_1| \leqslant \frac{c}{|I_1|^{1-\alpha}}\int_{I_1}|f_1|.$$

Hence

$$(M_\alpha^{(J)}f)(x) \leqslant c(K_t f_1)(x) \quad \text{for every} \quad t \in F,\ x \in J,$$

with the positive constant c depending only on α. Consequently,

$$(M_\alpha^{(J)} f)(x) \leqslant \frac{1}{|F|} \int\limits_F (K_t f_1)(x) dt \leqslant \frac{c}{|I(0, 2^{m_0-4})|} \int\limits_{I(0,2^{m_0-4})} (K_t f_1)(x) dt.$$

Suppose that $\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant 1$. By Lemma 1.1, we have

$$\begin{aligned}
S_t &:= \int\limits_J (v(x))^{p(x)} \big((K_t f_1)(x)\big)^{p(x)} dx \\
&= \int\limits_J (v(x))^{p(x)} \Bigg(\sup_{\substack{S \supset I_1 \ni x \\ I_1 \in D(S)-t}} \frac{1}{|I_1|^{1-\alpha}} \int\limits_{I_1} |f_1| \Bigg)^{p(x)} dx \\
&= \int\limits_{J+t} (v_t(x))^{p(x-t)} \Bigg(\sup_{\substack{S \supset I_1 \ni x \\ I_1 \in D(S)}} |I_1|^{\alpha-1} \int\limits_{I_1} \chi_J(s-t) f_1(s-t) ds \Bigg)^{p(x-t)} dx \\
&= \int\limits_{J+t} (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(S)}} |I_1|^{\alpha-1} \int\limits_{I_1} \chi_{J+t}(s) f_1(s-t) ds \Bigg)^{p_t(x)} dx \\
&= \int\limits_{J+t} (v_t(x))^{p_t(x)} \big(M_\alpha^{(d),S} \big(\chi_{J+t}(\cdot) f_1(\cdot - t)\big)\big)^{p_t(x)} dx \leqslant c
\end{aligned}$$

provided that

$$\int\limits_{J+t} (w_t(x))^{p_t(x)} (f_1(x-t))^{p_t(x)} dx = \int\limits_J w(x) |f(x)|^{p(x)} dx \leqslant 1,$$

where $v_t(x) = v(x-t)$, $w_t(x) = w(x-t)$, and $p_t(x) = p(x-t)$. To justify this conclusion, we need to check that for every $I \subset J + t$

$$\int\limits_I (v_t(x))^{p_t(x)} \big(M_\alpha^{(d),S} (\sigma_t \chi_I)(x)\big)^{p_t(x)} dx \leqslant c \int\limits_I \sigma_t(x) dx < \infty,$$

where the positive constant c is independent of I and t. Indeed, note that

$$\begin{aligned}
&\int\limits_I (v_t(x))^{p_t(x)} \big(M_\alpha^{(d),S} (\sigma_t \chi_I)(x)\big)^{p_t(x)} dx \\
&= \int\limits_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(S)}} |I_1|^{\alpha-1} \int\limits_{I_1} \chi_I(s) \sigma(s-t) ds \Bigg)^{p_t(x)} dx \\
&= \int\limits_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 - t \ni x - t \\ I_1 \in D(S)}} |I_1 - t|^{\alpha-1} \int\limits_{I_1 - t} \chi_I(s+t) \sigma(s) ds \Bigg)^{p_t(x)} dx \\
&= \int\limits_{I-t} (v(x))^{p(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(S)-t}} |I_1|^{\alpha-1} \int\limits_{I_1} \chi_{I-t}(s) \sigma(s) ds \Bigg)^{p(x)} dx
\end{aligned}$$

$$\leqslant \int\limits_{I-t} (v(x))^{p(x)} \big(M_\alpha^{(J)}(\chi_{I-t}\sigma)\big)^{p(x)}(x)dx \leqslant \int\limits_{I-t} \sigma(x)dx = \int\limits_{I} \sigma_t(x)dx < \infty.$$

Further, let $g \in L^{p'(\cdot)}(J)$ with $\|g\|_{L^{p'(\cdot)}(J)} \leqslant 1$. Then

$$\int\limits_J (M_\alpha^{(J)} f)(x)v(x)g(x)dx \leqslant \int\limits_J \bigg(\frac{1}{|I(0,2^{m_0-4})|} \int\limits_{I(0,2^{m_0-4})} (K_t f_1)(x)dt\bigg) v(x)g(x)dx$$

$$\leqslant \frac{1}{|I(0,2^{m_0-4})|} \int\limits_{I(0,2^{m_0-4})} \bigg(\int\limits_J (K_t f_1)(x)g(x)v(x)dx\bigg)dt$$

$$\leqslant \frac{1}{|I(0,2^{m_0-4})|} \int\limits_{I(0,2^{m_0-4})} \|(K_t f_1)v\|_{L^{p(\cdot)}(J)} \|g\|_{L^{p'(\cdot)}(J)} dt \leqslant c$$

provided that $\|f\|_{L_w^{p(\cdot)}(J)} \leqslant 1$.

Finally, we conclude that

$$\|(M_\alpha^{(J)} f)v\|_{L^{p(\cdot)}(J)} \leqslant c$$

if $\|fw\|_{L^{p'(\cdot)}(J)} \leqslant 1$. Sufficiency is proved.

Necessity. Let $f_I(t) = \chi_I(t)w^{-p'(t)}(t)$. Suppose that $\beta = \|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} \leqslant 1$. We have

$$\big\|v(\cdot)(M_\alpha^{(J)} f)^{p(\cdot)}(\cdot)\big\|_{L^{p(\cdot)}(J)} \geqslant \big\|\chi_I(\cdot)v(\cdot)\big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} =: A.$$

By the boundedness of $M_\alpha^{(J)}$ and (1.1) for $r = 1/p$ (recall that the measure $d\nu(x) = w(x)^{-p'(x)}dx$ satisfies the doubling condition and $1/p \in LH(J)$), we find

$$A = \big\|\chi_I(\cdot)v(\cdot)M_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \leqslant c\big\|w(\cdot)w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big\|_{L^{p(\cdot)}(J)}$$

$$\leqslant c\bigg(\int\limits_I w^{-p'(x)p(x)}(x)w^{p(x)}(x)dx\bigg)^{1/p_+(I)} \leqslant \overline{c}\bigg(\int\limits_I w^{-p'(x)}(x)dx\bigg)^{\frac{1}{p_-(I)}} \leqslant \overline{c}.$$

On the other hand,

$$A = \overline{c}\bigg\|\frac{1}{\overline{c}}\chi_I(\cdot)v(\cdot)M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(\cdot)\bigg\|_{L^{p(\cdot)}(J)}$$

$$\geqslant \overline{c}\bigg(\int\limits_I (\overline{c})^{-p(x)}(v(x))^{p(x)}\big[M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)\big](x)dx\bigg)^{\frac{1}{p_-(I)}}$$

$$\geqslant c\bigg[\int\limits_I (v(x))^{p(x)}\big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\big)^{p(x)}dx\bigg]^{\frac{1}{p_-(I)}}.$$

Summarizing these inequalities, we conclude that

$$\int\limits_I (v(x))^{p(x)}\big(M_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\big)^{p(x)}dx \leqslant c\int\limits_I w^{-p'(x)}(x)dx < \infty.$$

Suppose now that $\beta \geqslant 1$. Let us take

$$f(t) = \frac{w^{-p'(t)}(t)\chi_I(t)}{\beta}.$$

Then

$$\|f_I(\cdot)w(\cdot)\|_{L^{p(\cdot)}(J)} = \frac{\|w^{1-p'(\cdot)}(\cdot)\chi_I(\cdot)\|_{L^{p(\cdot)}(J)}}{\beta} \leqslant 1.$$

Arguing as above, we obtain the desired result. It remains to show that

$$A := \int\limits_J w^{-p'(x)}(x)dx < \infty.$$

Suppose that $A = \infty$. Then $\|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} = \infty$. Hence there exists a function g, $\|g\|_{L^{p(\cdot)}(J)}$, $g \geqslant 0$, such that

$$\int\limits_J g(x)w^{-1}(x)dx = \infty.$$

Let $f(x) = g(x)w^{-1}(x)$. Then

$$\big\|v(\cdot)\big(M_\alpha^{(J)}f\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \geqslant \Bigg(\int\limits_J w^{-1}(x)g(x)\Bigg)\big\|v(\cdot)|J|^{\alpha-1}\big\|_{L^{p(\cdot)}(J)} = \infty,$$

while $\|fw\|_{L^{p(\cdot)}(J)} = \|g\|_{L^{p(\cdot)}(J)} < \infty$. □

Corollary 1. *Let J be a bounded interval, let $1 < p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$, and let $0 \leqslant \alpha < 1$. Assume that $p \in LH(J)$. Then the inequality*

$$\big\|v(\cdot)\big(M_\alpha^{(J)}f\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \leqslant c\|f\|_{L^{p(\cdot)}(J)} \quad \textit{(trace inequality)}$$

holds if and only if

$$\sup_{I, I\subset J} \frac{1}{|I|}\int\limits_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx < \infty.$$

Proof. *Sufficiency.* By Theorem 1.1, it is enough to show that

$$\big(M_\alpha^{(J)}\chi_I\big)(x) \leqslant |I|^\alpha \quad \text{for } x \in I.$$

This is true because of the following estimates:

$$\sup_{\substack{S, S\subset J\\ S\ni x}} |S|^{\alpha-1}\int\limits_S \chi_I \leqslant \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^{\alpha-1}\int\limits_{S\cap I} dx = \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^\alpha = |I|^\alpha.$$

Necessity follows by choosing an appropriate test functions in the trace inequality. □

1.2 The case of an unbounded interval

Now we derive criteria for the two–weight inequality for the following maximal operators:

$$\big(M_\alpha^{(\mathbb{R}_+)} f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int\limits_{(x-h,x+h)\cap\mathbb{R}_+} |f(y)|dy$$

and

$$\big(M_\alpha^{(\mathbb{R})} f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int\limits_{x-h}^{x+h} |f(y)|dy,$$

where $0 \leqslant \alpha < 1$.

In the sequel, we assume that $v^{p(\cdot)}(\cdot)$ and $w^{-p'(\cdot)}(\cdot)$ are a.e. positive locally integrable functions.

Theorem 1.2. *Let $0 \leqslant \alpha < 1$, $1 < p_-(\mathbb{R}_+) \leqslant p \leqslant p_+(\mathbb{R}_+) < \infty$, and let $p \in LH(\mathbb{R}_+)$. Suppose that there is a bounded interval $[0,a]$ such that $w^{-p'(\cdot)}(\cdot) \in DC([0,a])$ and $p \equiv p_c \equiv$const outside $[0,a]$. Then the inequality*

$$\|vM_\alpha^{(\mathbb{R}_+)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)}$$

holds if and only if there is a positive constant b such that for all bounded intervals $I \subset \mathbb{R}_+$

$$\|vM_\alpha^{(\mathbb{R}_+)}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(I)} \leqslant c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty. \tag{1.3}$$

Proof. *Sufficiency.* Suppose that $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$. We show that

$$\|vM_\alpha^{(\mathbb{R}_+)}\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty.$$

We represent $M_\alpha^{(\mathbb{R}_+)} f(x)$ as follows:

$$\begin{aligned} M_\alpha^{(\mathbb{R}_+)} f(x) &= \chi_{[0,a]}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) + \chi_{[0,a]}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x) \\ &\quad + \chi_{(a,\infty)}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) + \chi_{(a,\infty)}(x) M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x) \\ &=: M_\alpha^{(1)} f(x) + M_\alpha^{(2)} f(x) + M_\alpha^{(3)} f(x) + M_\alpha^{(4)} f(x). \end{aligned}$$

Since $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$, we have $\|wf\|_{L^{p(\cdot)}([0,a])} < \infty$. Applying now Theorem 1.1, we find $\|vM_\alpha^{(1)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$.

Further, observe that

$$M_\alpha^{(2)} f(x) \leqslant \sup_{h>a-x} \frac{1}{h^{1-\alpha}} \int\limits_{a}^{x+h} |f(y)|dy \leqslant \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a) < \infty.$$

Hence

$$\|vM_\alpha^{(2)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \big(M_\alpha^{(\mathbb{R}_+)} f\big)(a) \cdot \|v\|_{L^{p(\cdot)}([0,a])} < \infty.$$

We use the following representation for $M_\alpha^{(3)}f(x)$:

$$\begin{aligned}\big(M_\alpha^{(3)}f\big)(x) &= \chi_{(a,2a]}(x)M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x)+\chi_{(2a,\infty)}(x)M_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x).\\ &=: \big(\overline{M}_\alpha^{(3)}f\big)(x)+\big(\widetilde{M}_\alpha^{(3)}f\big)(x).\end{aligned}$$

It is easy to check that for $x\in(a,2a]$

$$\big(\overline{M}_\alpha^{(3)}f\big)(x)\leqslant \sup_{h>x-a}\frac{1}{(a-x+h)^{1-\alpha}}\int\limits_{x-h}^{a}|f(y)|dy\leqslant\big(M_\alpha^{(\mathbb{R}_+)}f\big)(a).$$

Consequently,

$$\|v\overline{M}_\alpha^{(3)}f\|_{L^{p(\cdot)}(\mathbb{R}_+)}\leqslant\|f\|_{L^{p_c}\left((a,2a]\right)}\big(M_\alpha^{(\mathbb{R}_+)}f\big)(a)<\infty$$

because $v^{p(\cdot)}(\cdot)$ is locally integrable on $\mathbb{R}_+$. Further, for $x>2a$ we have

$$\big(\widetilde{M}_\alpha^{(3)}f\big)(x)\leqslant\frac{1}{(x-a)^{1-\alpha}}\int\limits_0^a|f(y)|dy.$$

Using the Hölder inequality in $L^{p(\cdot)}$, we find

$$\begin{aligned}\Big\|v\widetilde{M}_\alpha^{(3)}f\Big\|_{L^{p(\cdot)}(\mathbb{R}_+)} &\leqslant\Big\|\frac{v(x)}{(x-a)^{1-\alpha}}\Big\|_{L^{p_c}\left((2a,\infty)\right)}\bigg(\int\limits_0^a|f(y)|dy\bigg)\\ &\leqslant\Big\|\frac{v(x)}{(x-a)^{1-\alpha}}\Big\|_{L^{p_c}\left((2a,\infty)\right)}\big\|fw\big\|_{L^{p(\cdot)}\left((0,a]\right)}\big\|w^{-1}\big\|_{L^{p'(\cdot)}\left((0,a]\right)}\\ &=I_1\cdot I_2\cdot I_3.\end{aligned}$$

Since $I_2<\infty$ and $I_3<\infty$, we need to show that $I_1<\infty$. This follows from the fact that the condition (1.3) yields

$$\big\|v\overline{M}_\alpha\big(w^{-(p_c)'}\chi_I\big)\big\|_{L^{p_c}\left((2a,\infty)\right)}\leqslant\big\|w^{1-(p_c)'}(\cdot)\chi_I(\cdot)\big\|_{L^{p_c}\left((2a,\infty)\right)},\quad I\subset(2a,\infty),\tag{1.4}$$

where $\overline{M}_\alpha$ is the maximal operator defined on $(2a,\infty)$ as follows:

$$\big(\overline{M}_\alpha f\big)(x)=\sup_{h>0}\frac{1}{h^{1-\alpha}}\int\limits_{(2a,\infty)\cap(x-h,x+h)}|f(y)|dy.$$

Using the result of Sawyer [29] (cf. also [28, Chapter 4]) for Lebesgue spaces with constant parameter, we see that (1.4) implies the inequality

$$\big\|v\overline{M}_\alpha f\big\|_{L^{p_c}\left((2a,\infty)\right)}\leqslant c\big\|fw\big\|_{L^{p_c}\left((2a,\infty)\right)}.$$

Since

$$\overline{M}_\alpha f(x)\geqslant\frac{1}{(x-a)^{1-\alpha}}\int\limits_{2a}^{x}|f(y)|dy\quad x>2a,$$

for the Hardy operator

$$\big(H_a f\big)(x) = \int_{2a}^{x} f(t)dt, \quad x > 2a,$$

we have the two-weight inequality

$$\big\|v(x)(x-a)^{\alpha-1} H_a f\big\|_{L^{p_c}\big((2a,\infty)\big)} \leqslant \big\|wf\big\|_{L^{p_c}\big((2a,\infty)\big)}. \tag{1.5}$$

Recall that (cf., for example, [30, SEction 1.3]) a necessary condition for (1.5) is that

$$\sup_{t>2a} \bigg(\int_t^\infty \bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c} dx\bigg)^{\frac{1}{p_c}} \bigg(\int_{2a}^{t} w^{1-(p_c)'}(x)dx\bigg)^{\frac{1}{(p_c)'}} < \infty.$$

Hence

$$\int_{2a}^{\infty} \bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c} dx = \int_{2a}^{3a} (\cdots) + \int_{3a}^{\infty} (\cdots) \leqslant a^{\alpha-1} \int_{2a}^{3a} \big(v(y)\big)^{p_c} + \int_{3a}^{\infty} \bigg[\frac{v(x)}{(x-a)^{1-\alpha}}\bigg]^{p_c} dx < \infty.$$

It remains to estimate

$$I := \|vM_\alpha^{(4)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

But $I < \infty$ because of the two-weight result of Sawyer [29] (cf. also [28, Chapter 4]) for the maximal operator defined on (a,∞) in Lebesgue spaces with constant exponent. Sufficiency is proved.

Necessity easily follows by taking the test functions $f(\cdot) = \chi_I(\cdot) w^{-p'(\cdot)}(\cdot)$ in the two–weight inequality. □

The next assertion follows in the same way as the previous one, and we omit the proof.

Theorem 1.3. *Let* $0 \leqslant \alpha < 1$, $1 < p_- \leqslant p \leqslant p_+ < \infty$, *and let* $p \in LH(\mathbb{R})$. *Suppose that there is a positive number* a *such that* $w^{-p'(\cdot)}(\cdot) \in DC([-a,a])$ *and* $p \equiv p_c \equiv \mathrm{const}$ *outside* $[-a,a]$. *Then the inequality*

$$\|vM_\alpha^{(\mathbb{R})} f\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant \|wf\|_{L^{p(\cdot)}(\mathbb{R})}$$

holds if and only if there is a positive constant b *such that for all bounded intervals* $I \subset \mathbb{R}$

$$\|vM_\alpha^{(\mathbb{R})}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

2 Integral Operators on $\mathbb{R}_+$

In this section, we derive two–weight criteria of other type for the operators

$$(\mathscr{H} f)(x) = (\text{p.v.}) \int_0^\infty \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R}_+,$$

$$(\mathscr{M}f)(x) = \sup_{I \ni x} \frac{1}{|I|} \int\limits_I |f(t)|dt, \quad x \in \mathbb{R}_+,$$

provided that the weights are monotone, where the supremum is taken over all finite intervals $I \subset \mathbb{R}_+$ containing x.

In this section, use the notation

$$g_- := g_-(\mathbb{R}_+), \quad g_+ := g_+(\mathbb{R}_+)$$

for a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

First we present the following statement regarding the weighted Hardy transform

$$(H_{v,w}f)(x) = v(x) \int\limits_0^x f(t)w(t)dt$$

and its dual

$$(H'_{v,w}f)(x) = v(x) \int\limits_x^\infty f(t)w(t)dt$$

defined on $\mathbb{R}_+$.

Theorem A. *Let* $1 < p_- \leqslant p(x) \leqslant q(x) \leqslant q_- < \infty$ *and let* $p, q \in LH(\mathbb{R}_+)$. *Suppose that* $p = p_c \equiv \text{const}$, $q = q_c \equiv \text{const}$ *outside some interval* $(0, a)$. *Then*

(i) *the operator* $H_{v,w}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+)$ *to* $L^{q(\cdot)}(\mathbb{R}_+)$ *if and only if*

$$D := \sup_{t>0} D(t) := \sup_{t>0} \|v\|_{L^{q(\cdot)}\big((t,\infty)\big)} \|w\|_{L^{p'(\cdot)}\big((0,t)\big)} < \infty,$$

(ii) the operator $H'_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ $L^{q(\cdot)}(\mathbb{R}_+)$ if and only if

$$D' := \sup_{t>0} D'(t) := \sup_{t>0} \|v\|_{L^{q(\cdot)}\big((0,t)\big)} \|w\|_{L^{p'(\cdot)}\big((t,\infty)\big)} < \infty.$$

Proof. We prove part (i). Part (ii) follows from the duality arguments. Let $\|f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant 1$. We represent $H_{v,w}f$ as follows:

$$H_{v,w}f(x) = \chi_{[0,a]}v(x) \int\limits_0^x f(t)w(t)dt + \chi_{(a,\infty)}v(x) \int\limits_0^x f(t)w(t)dt := H^{(1)}_{v,w}f(x) + H^{(2)}_{v,w}f(x).$$

Note that the condition $D < \infty$ implies

$$D^{(a)} := \sup_{0<t<a} \|v\|_{L^{q(\cdot)}\big((t,a)\big)} \|w\|_{L^{p'(\cdot)}\big((0,t)\big)} < \infty.$$

Consequently (cf. [19]),

$$\|H^{(1)}_{v,w}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant c\|f\|_{L^{p(\cdot)}([0,a])} \leqslant c.$$

It remains to estimate $\|H^{(2)}_{v,w}f\|_{L^{q(\cdot)}(\mathbb{R}_+)}$. Let $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \leqslant 1$. Then

$$\int\limits_0^\infty (H^{(2)}_{v,w}f)(x)g(x)dx = \int\limits_a^\infty (H^{(2)}_{v,w}f)(x)g(x)dx$$

$$\leqslant \int_a^\infty v(x)\Big(\int_a^x f(t)w(t)dt\Big)g(x)dx + \Big(\int_a^\infty v(x)g(x)dx\Big)\Big(\int_0^a f(t)w(t)dt\Big) := S_1 + S_2.$$

Now, we can apply the boundedness of the Hardy transform

$$T^{(a)}_{v,w} f(x) = v(x) \int_a^x f(t)w(t)dt$$

from $L^{p_c}([a,\infty))$ to $L^{q_c}([a,\infty))$ (cf., for example, [30, Section 1.3]) because

$$\sup_{t>a} \|v\|_{L^{q_c}\left((t,\infty)\right)} \|w\|_{L^{(p_c)'}\left((a,t)\right)} \leqslant D < \infty.$$

By this fact and the Hölder inequality, we have

$$S_1 \leqslant \|T^{(a)}_{v,w} f\|_{L^{q_c}([a,\infty))} \|g\|_{L^{q_c}([a,\infty))} \leqslant c\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C.$$

Applying the Hölder inequality for $L^{p(\cdot)}$ spaces, we find that

$$S_2 \leqslant \Big(\int_a^\infty v(x)g(x)dx\Big) \|f\|_{L^{p(\cdot)}([0,a])} \|w\|_{L^{p'(\cdot)}([0,a])} \leqslant C.$$

Necessity follows in the standard way by choosing an appropriate test functions. □

Theorem B (cf. [17])**.** $1 < p_- \leqslant p_+ < \infty$. *Suppose that* $p \in LH(\mathbb{R}_+)$ *and* $p = p_c = \text{const}$ *outside some interval. Then*

$$\|vTf\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant c\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \tag{2.1}$$

where T *is* $\mathscr{M}$ *or* $\mathscr{H}$*, if*

(i) $H_{\overline{v},\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$, where $\overline{v}(x) := \frac{v(x)}{x}$, $\widetilde{w}(x) := \frac{1}{w(x)}$,

(ii) $H'_{v,\widetilde{w}_1}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$, where $\widetilde{w}_1(x) := \frac{1}{w(x)x}$,

(iii) $v_+([x/4, 4x]) \leqslant cw(x)$ a.e. or $v(x) \leqslant cw_-([x/4, 4x])$ a.e. (2.2)

Theorems A and B imply the following assertion.

Theorem 2.1. *Let* $1 < p_- \leqslant p_+ < \infty$*, and let* $p \in LH(\mathbb{R}_+)$*. Suppose that* $p = p_c \equiv \text{const}$ *outside some interval* $[0, a]$*. Suppose also that* v *and* w *are weights on* $\mathbb{R}_+$*. Then the inequality* (2.1)*, where* T *is* $\mathscr{M}$ *or* $\mathscr{H}$*, holds if*

(i) $E_1 := \sup_{t>0} E_1(t) := \sup_{t>0} \|v(x)x^{-1}\|_{L^{p(x)}\left((t,\infty)\right)} \|w^{-1}\|_{L^{p'(\cdot)}\left((0,t)\right)} < \infty,$ (2.3)

(ii) $E_2 := \sup_{t>0} E_2(t) := \sup_{t>0} \|v\|_{L^{p(\cdot)}\left((0,t)\right)} \|w^{-1}(x)x^{-1}\|_{L^{p'(x)}\left((0,t)\right)} < \infty,$ (2.4)

(iii) the condition (2.2) is satisfied.

Theorem 2.2. *Let $1 < p_- \leqslant p_+ < \infty$, and let $p \in LH(\mathbb{R}_+)$. Suppose that $p = p_c \equiv \mathrm{const}$ outside some interval $[0, a]$. Suppose also that v and w are positive increasing functions on $\mathbb{R}_+$. Then the inequality* (2.1)*, where T is $\mathscr{M}$ or $\mathscr{H}$, holds if and only if* (2.3) *is satisfied.*

Proof. *Sufficiency.* Taking Theorem 4 into account, it suffices to show that the condition (2.3) implies the conditions (2.4) and (2.2). For (2.2) we show that there is a positive constant c such that for all $t > 0$

$$v(4t) \leqslant cw(t), \quad t > 0. \tag{2.5}$$

Indeed, the inequality (1.1) with respect to the Lebesgue measure $d\mu(x) = dx$ and the exponent $r = p'$ which belongs to $LH([0, a])$, for small t yields

$$\begin{aligned} E_1(t) &\geqslant \|\chi_{[t,4t]}(\cdot)|\cdot|^{-1}\|_{L^{p(\cdot)}_{v(\cdot)}(\mathbb{R}_+)} \, \| \, \chi_{[0,t/4]}(\cdot)w^{-1}(\cdot) \, \|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\geqslant c\frac{v(t)}{t} t^{\frac{1}{p_-([t,4t])}} w^{-1}(t/4) t^{\frac{1}{(p')_-([0,t/4])}} \\ &\geqslant c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_-([0,4t])}} t^{\frac{1}{(p')_-([0,t/4])}} = c\frac{v(t)}{w(t/4)}. \end{aligned}$$

Further, for large t

$$\begin{aligned} E_1(t) &\geqslant \|v(x)x^{-1}\chi_{(t,2t)}(x)\|_{L^{p_c}(\mathbb{R}_+)} \|\chi_{[t/8,t/4]}(\cdot)w^{-1}(\cdot)\|_{L^{(p_c)'}(\mathbb{R}_+)} \\ &\geqslant c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_c}} t^{\frac{1}{(p_c)'}} = c\frac{v(t)}{w(t/4)} \end{aligned}$$

Thus, the condition (2.2) is satisfied.

Taking into account the inequality (2.5) and the fact that v and w are increasing, we easily conclude that the condition (2.4) is satisfied.

Necessity. First we observe that the inequality (2.1) implies $\|w^{-1}\|_{L^{p'(\cdot)}(0,t)} < \infty$ for all $t > 0$.

Let $T = \mathscr{M}$. Using the obvious inequality

$$\mathscr{M}f(x) \geqslant \frac{c}{x} \int_0^x f(t)dt, \quad x > 0,$$

and taking into account Theorem A, we obtain the necessity for $\mathscr{M}$. Let $T = \mathscr{H}$. We take $f \geqslant 0$ so that $\|f\|_{L^{p(\cdot)}_w(\mathbb{R}_+)} \leqslant 1$. Then

$$\|v\mathscr{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant C. \tag{2.6}$$

It is obvious that (2.6) yields t

$$C \geqslant \|v\mathscr{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \geqslant \|\chi_{(t,\infty)}(\cdot)v\mathscr{H}f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

If f has support on $(0, t)$, $t > 0$, then this inequality implies

$$C \geqslant \left\|\chi_{(t,\infty)}(\cdot)v(\cdot)\left(\int_0^t \frac{f(y)}{\cdot - y}dy\right)\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \geqslant c\|\chi_{(t,\infty)}(x)v(x)x^{-1}\|_{L^{p(\cdot)}(\mathbb{R}_+)}\left(\int_0^t f(y)dy\right).$$

Taking the supremum with respect to f and using the inequality

$$\|g\|_{L^{p(\cdot)}} \leqslant \sup_{\|h\|_{L^{p'(\cdot)}} \leqslant 1} \left| \int gh \right|$$

(cf., for example, [24]), we obtain the necessity. □

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