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WEIGHTED EXTRAPOLATION IN MIXED NORM FUNCTION SPACES

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Abstract. Rubio de Francía's weighted extrapolation results for pairs of functions in mixed-norm Banach function spaces defined on the product of quasi-metric measure spaces are obtained. As a consequence, we formulate appropriate results for the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. Here we treat only the weighted extrapolation in grand Lebesgue spaces with mixed norms.

1. Introduction and Preliminaries

In this note we formulate weighted extrapolation theorems for pairs of functions (f, g) in mixed norm spaces defined on the product of quasi-metric measure spaces with doubling measures (spaces of homogeneous type). Let (X, d, μ) and (Y, ρ, ν) be the spaces of homogeneous type. We showed that if the one-weight inequality holds in the classical weighted Lebesgue space for all weights from the "strong" Muckenhoupt class $A^{(S)}(X \times Y)$ defined with respect to products of balls $B_1 \times B_2$, $B_1 \subset X$, $B_2 \subset Y$, then appropriate inequality holds for the same pair of functions in mixed-norm Banach function spaces $(E_1(X), E_2(Y))$ provided that the Hardy–Littlewood maximal operators M_X and M_Y are bounded in the spaces $\big(E_1^{1/q_0}\big)'(X)$ and $\big(E_2^{1/q_0}\big)'(Y)$, respectively, for some $q_0 > 1$ (for a similar result in the Euclidean setting see [12]). We treat both cases: diagonal and off-diagonal ones.

Rubio de Francía's extrapolation theory is one of the important tools to study the boundedness of integral operators in the weighted function spaces.

By taking $(f, g) = (f, Tf)$, as a special case, one can obtain one-weighted inequalities for that multiple operator T of Harmonic Analysis for which the strong Muckenhoupt condition guarantees the one-weighted boundedness. To such operators belong, for example, strong maximal operators, Calderón–Zygmund singular integrals with product kernels and multiple fractional integral operators. Based on the extrapolation result for the mixed-norm Lebesgue spaces we can derive, for example, the one-weight mixed-norm inequality due to D. Kurtz [18] regarding the strong maximal operator in mixed-norm Lebesgue spaces under the $A_q(A_p)$ condition on weights, and formulate an appropriate weighted extrapolation result.

One of the novelties of this note is that together with the extrapolation results we determine weighted bounds in terms of the weighted Muckenhuopt characteristics. The derived extrapolation results are applied to the weighted extrapolation in the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. It should be emphasized that the majority of the results are new even for the case of Euclidean spaces with the Lebesgue measure.

Let (X, d, μ) be a quasi-metric measure space with a quasi-metric d and measure μ. In what follows, we will assume that the balls $B(x, r) := \{y \in X;\, d(x, y) < r\}$ with center x and radius r are measurable with positive μ for all $x \in X$ and $r > 0$.

If μ satisfies the doubling condition $\mu(B(x, 2r)) \le C_d \mu(B(x, r))$, with a positive constant C_d, independent of x and r, then we say that (X, d, μ) is a space of homogeneous type (SHT, shortly).

We assume that (X, d, μ) and (Y, ρ, ν) are the spaces of homogeneous type without atoms.

For the definition, examples and some properties of an SHT see, e.g., [3]. We also assume that the class of continuous functions is dense in L^1 defined on an SHT.

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For a given quasi-metric measure space (X, d, μ) and q, satisfying $1 < q < \infty$, we denote as usual by $L^q(\mu) = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm.

Let (X, d, μ) be an SHT. The Hardy–Littlewood maximal function defined on X and given by the formula

$$M_X f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \tag{1}$$

is the Hardy–Littlewood maximal operator defined on an SHT (X, d, μ).

For the sharp bounds of the norm of the maximal operator M_X in terms of characteristics of weights we refer to [13] and references cited therein.

Let $1 < r < \infty$. We say that a weight function w defined on $X \times Y$ belongs to the Muckenhoupt class $A_r^{(S)}$ if

$$[w]_{A_r^{(S)}} := \sup_{B_1 \times B_2} \left(\frac{1}{\mu(B_1)\nu(B_2)} \int_{B_1 \times B_2} w \, d\mu \times \nu \right) \left(\frac{1}{\mu(B_1)\nu(B_2)} \int_{B_1 \times B_2} w^{1-r'} \, d\mu \times \nu \right)^{r-1} < \infty,$$

where the supremum is taken over all products of the balls $B_1 \times B_2 \subset X \times Y$.

Let $1 < p, q < \infty$. Suppose that ρ is a μ-a.e. positive function on $X \times Y$ such that ρ^q is locally integrable. We say that $\rho \in \mathcal{A}_{p,q}^{(S)}$ if

$$[\rho]_{\mathcal{A}_{p,q}^{(S)}} := \sup_{B_1 \times B_2} \left(\frac{1}{\mu(B_1)\nu(B_2)} \int_{B_1 \times B_1} \rho^q \, d\mu \times \nu \right) \left(\frac{1}{\mu(B_1)\nu(B_2)} \int_B \rho^{-p'} \, d\mu \times \nu \right)^{q/p'} < \infty,$$

where the supremum is taken over all products of balls $B_1 \times B_2 \in X \times Y$.

If $p = q$, then we denote $\mathcal{A}_{p,q}^{(S)}$ by $\mathcal{A}_p^{(S)}$.

Let E be a Banach function space (BFS) on X (for the Definition and some essential properties of BFSs, see [1]). For a BFS E, we denote by E' its Köthe dual (or associated) space.

Now we define the mixed-norm space for BFSs E_1 and E_2 defined on quasi-metric measure spaces (X, d, μ) and (Y, ρ, ν) respectively. The mixed-norm space, denoted by $(E_1(X), E_2(Y))$ (or simply, (E_1, E_2)), is defined with respect to the norm defined for the $\mu \times \nu$- measurable function $f : X \times Y \to \mathbb{R}$:

$$\|f\|_{(E_1, E_2)} = \Big\| \|f\|_{E_1} \Big\|_{E_2}.$$

It can be checked that (E_1, E_2) is a BFS.

For a Banach space E and $0 < p < \infty$, the p-convexification of E is defined as follows:

$$E^p = \{f : |f|^p \in E\}.$$

E^p may be equipped with the quasi-norm $\|f\|_{E^p} = \||f|^p\|_E^{1/p}$. It can be observed that if $1 \le p < \infty$, then E^p is a Banach space, as well. For $1 \le p < \infty$ and BFSs E_1 and E_2, we have

$$(E_1, E_2)^p = (E_1^p, E_2^p).$$

Before formulating the main results we recall that Rubio de Francía's extrapolation in the setting of a strong Muckenhoupt condition was treated in [6] (see also [12, 15]).

We say that a BFS E belongs to $\mathbb{M}(X)$ if the maximal operator M_X defined with respect to the balls $B \subset X$ (see (1)) is bounded in E. The class $\mathbb{M}(Y)$ is defined similarly.

To formulate the main results, we need the following notation:

$$[M_X, M_Y] := \|M_X\|_{(E_1^{1/q_0})'} \|M_Y\|_{(E_2^{1/q_0})'}; \quad \overline{[M_X, M_Y]} := \|M_X\|_{(\overline{E}_1^{1/\tilde{q}_0})'} \|M_Y\|_{(\overline{E}_2^{1/\tilde{q}_0})'}.$$

2. Main Results

Now we formulate the main extrapolation results for the mixed-norm BFSs.

Theorem 2.1 (Diagonal Case). *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f and g defined on $X\times Y$. Suppose that for some $1<p_0<\infty$ and for every $w\in A_{p_0}^{(S)}$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{X\times Y} g^{p_0}(x,y)w(x,y)\,d\mu\times\nu\right)^{\frac{1}{p_0}}\le CN\big([w]_{A_{p_0}^{(S)}}\big)\left(\int\limits_{X\times Y} f^{p_0}(x,y)w(x,y)\,d\mu\times\nu\right)^{\frac{1}{p_0}},\tag{2}$$

with some non-decreasing function $s\to N(s)$, holds. Suppose that there exists $1<q_0<\infty$ such that E_1^{1/q_0} and E_2^{1/q_0} are again BFSs. If $(E_1^{1/q_0})'\in\mathbb{M}(X)$ and $(E_2^{1/q_0})'\in\mathbb{M}(Y)$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(E_1,E_2)}\le 16^{1/q_0}C\widetilde{C}J\big([M_X,M_Y],p_0,q_0\big)\|f\|_{(E_1,E_2)},$$

where the positive constant C is defined in (2),

$$J\big([M_X,M_Y],p_0,q_0\big)$$
$$:=\begin{cases} N\Big(2^{p_0-q_0}(\bar{c}q_0')^{2(p_0-q_0)}\big([M_X,M_Y]\big)^{(2((q_0)'-1)+1)(p_0-q_0)}\Big), & q_0<p_0\\ N\Big(2^{(q_0-p_0)/(q_0-1)}(\bar{c}(q_0)')^{2(q_0-p_0)/(q_0-1)}\big([M_X,M_Y]\big)^{(2q_0-p_0-1)(q_0-1)}\Big), & q_0>p_0\end{cases}$$

and $\widetilde{C}$ is defined by $\widetilde{C}=\max\Big\{2,2^{(q_0-p_0)/(q_0p_0-p_0)}\Big\}$.

Theorem 2.2 (Off-diagonal Case). *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions $f,g\in L^0(\mu\times\nu)$ defined on $X\times Y$. Suppose that for some $1<p_0\le q_0<\infty$ and for every $w\in A_{1+q_0/p_0'}^{(S)}$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{X\times Y} g^{q_0}(x,y)w(x,y)\,d\mu\times\nu\right)^{\frac{1}{q_0}}\le CN\big([w]_{A^{(S)}_{1+\frac{q_0}{p_0'}}}\big)\left(\int\limits_{X\times Y} f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,d\mu\times\nu\right)^{\frac{1}{p_0}},\tag{3}$$

with some positive constant C and non-decreasing function $s\to N(s)$, holds. Suppose that there exist $1<\widetilde{p}_0<\infty$, $1<\widetilde{q}_0<\infty$ such that

$$\frac{1}{\widetilde{p}_0}-\frac{1}{\widetilde{q}_0}=\frac{1}{p_0}-\frac{1}{q_0},$$

and $\overline{E}_1(X)^{1/\widetilde{q}_0}$, $E_1(X)^{1/\widetilde{p_0}}$, $\overline{E}_2(Y)^{1/\widetilde{q}_0}$, $E_2(Y)^{1/\widetilde{p_0}}$ are BFSs, and also the following condition

$$\big(\overline{E}_1(Y)^{1/\widetilde{q}_0}\big)'=\Big[\big(E_1(Y)^{1/\widetilde{p}_0}\big)'\Big]^{\widetilde{p}_0/\widetilde{q}_0};\;\;\big(\overline{E}_2(Y)^{1/\widetilde{q}_0}\big)'=\Big[\big(E_2(Y)^{1/\widetilde{p}_0}\big)'\Big]^{\widetilde{p}_0/\widetilde{q}_0}$$

is satisfied.

If $\left(\overline{E}_1^{1/\widetilde{q}_0}\right)'\in\mathbb{M}(X)$ and $\left(\overline{E}_2^{1/\widetilde{q}_0}\right)'\in\mathbb{M}(Y)$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(\overline{E}_1,\overline{E}_2)}\le 16^{\widetilde{q}_0}\;C\overline{C}\overline{J}\big(\overline{[M_X,M_Y]},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)\|f\|_{(E_1,E_2)},$$

where the constant C is the same as in (3),

$$\overline{J}\big(\overline{[M_X,M_Y]},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)$$
$$:=\begin{cases} N\left[\left(2\overline{c}^2\left(1+\dfrac{\widetilde{q}_0'}{\widetilde{q}_0}\right)^2\right)^{\gamma(\widetilde{q}_0-q_0)}\big(\overline{[M_X,M_Y]}\big)^{1+2\frac{\gamma\widetilde{q}_0(q_0-q)}{\widetilde{q}_0'}}\right], & \widetilde{q}_0<q_0,\\ N\left[\left(2\overline{c}^2\left(1+\frac{\widetilde{q}_0}{\widetilde{p}_0'}\right)^2\right)^{\frac{\gamma(\widetilde{q}_0-q_0)}{\gamma\widetilde{q}_0-1}}\big(\overline{[M_X,M_Y]}\big)^{(2\gamma\widetilde{q}_0-\gamma q_0-1)/(\gamma\widetilde{q}_0-1)}\right], & \widetilde{q}_0>q_0,\end{cases}$$

and $\overline{C}:=\max\left\{2^{\gamma\widetilde{q}_0\left(\frac{1}{\widetilde{q}_0}-\frac{1}{q_0}\right)},2^{\gamma(\widetilde{p}_0)'\left(\frac{1}{p_0}-\frac{1}{\widetilde{p}_0}\right)}\right\}$.

As corollaries, we have appropriate extrapolation results for the mixed-norm Lebesgue, Lorentz, Orlicz and grand Lebesgue spaces. Here, we give the statements about only grand Lebesgue spaces with mixed norms.

In 1992, T. Iwaniec and C. Sbordone [14], in their studies related to the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of spaces $L^{p),\theta}(\Omega)$ can be found in the work of L. Greco, T. Iwaniec and C. Sbordone [11].

Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during the last years due to their various applications, we mention here, e.g., [2, 7–10], the monograph [17] and references therein.

To formulate and prove the main result of this section we need to introduce the following notation: let σ_i, $i=1,2$, be sufficiently small positive numbers and let $\psi_1(\cdot)$ and $\psi_2(\cdot)$ be n-tuple positive increasing functions on the intervals $(0,\sigma_i)$, $i=1,2$, such that $\lim\limits_{\lambda\to 0}\psi_i(\lambda)=0$, $i=1,2$. In this case, we say that $\psi_i\in\Psi_{\sigma_i}$, $i=1,2$.

We say that for weight functions u and v on X and Y, respectively, a function $f: X\times Y$ belongs to $\big(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X), L_v^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ $1<p_1,p_2<\infty$, if

$$\|f\|_{\big(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X),L_v^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)}$$

$$=\sup_{0<\varepsilon_1<\sigma_1}\sup_{0<\varepsilon_2<\sigma_2}\Bigg(\psi_2(\varepsilon_2)\int\limits_Y\bigg(\psi_1(\varepsilon_1)\int\limits_X|f(x,y))|^{p_1-\varepsilon_1}u(x)d\mu(x)\bigg)^{\frac{p_2-\varepsilon_2}{p_1-\varepsilon_1}}v(y)d\nu(y)\Bigg)^{\frac{1}{p_2-\varepsilon_2}}<\infty.$$

If $\psi_i(\cdot)\equiv 1$, $i=1,2$, then the space $\big(L^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ is the mixed norm Lebesgue space. Further, if $\psi_i(\cdot)=\theta_i$, $i=1,2$, then we denote $\big(L^{p_1),\psi_1(\cdot),\sigma_1}(X), L^{p_2),\psi_2(\cdot),\sigma_2}(Y)\big)$ by $\big(L^{p_1),\theta_1,\sigma_1}(X),$ $L^{p_2),\theta_2,\sigma_2}(Y)\big)$.

Theorem 2.3. *Let $\mathcal{F}$ be a family of pairs (f,g) of non-negative functions $f,g\in L^0(\mu\times\nu)$ defined on $X\times Y$. Suppose that for some $1\le p_0<\infty$ and for every $w\in A_{p_0}^{(S)}(X\times Y)$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\bigg(\int\limits_{X\times Y}g^{p_0}(x,y)w(x,y)\,d\mu\times\nu\bigg)^{\frac{1}{p_0}}\le CN\big([w]_{A_{p_0}^{(S)}}\big)\bigg(\int\limits_{X\times Y}f^{p_0}(x,y)w(x,y)\,d\mu\times\nu\bigg)^{\frac{1}{p_0}},\tag{4}$$

with some non-decreasing function $s\to N(s)$, holds. Then there is a positive constant C such that for every $1<p_1,p_2<\infty$, $\psi_i\in\Psi_{\sigma_i}$, $i=1,2$, and $u\in A_{p_1}(X)$, $v\in A_{p_2}(Y)$,

$$\|g\|_{(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X),L^{p_1),\psi_2(\cdot),\sigma_2}(Y))}\le C\|f\|_{(L_u^{p_1),\psi_1(\cdot),\sigma_1}(X),L^{p_1),\psi_2(\cdot),\sigma_2}(Y))},$$

where $\sigma_1,\sigma_2>0$ are the numbers such that $u\in A_{p_1-\sigma_1}(X)$, $u\in A_{p_2-\sigma_2}(X)$.

Theorem 2.4 (Off-diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f,g) of non-negative functions $f,g\in L^0(\mu\times\nu)$ defined on $X\times Y$. Suppose that for some $1<p_0\le q_0<\infty$ and for every $w\in A_{1+q_0/p_0'}^{(S)}(X\times Y)$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\bigg(\int\limits_{X\times Y}g^{q_0}(x,y)w(x,y)\,d\mu\times\nu\bigg)^{\frac{1}{q_0}}\le CN\big([w]_{A_{1+\frac{q_0}{p_0'}}^{(S)}}\big)\bigg(\int\limits_{X\times Y}f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,d\mu\times\nu\bigg)^{\frac{1}{p_0}},$$

with some non-decreasing function $s\to N(s)$, holds.

Then for any $1<p_1,q_1,p_2,q_2<\infty$, satisfying the condition

$$\frac{1}{p_1}-\frac{1}{q_1}=\frac{1}{p_2}-\frac{1}{q_2}=\frac{1}{p_0}-\frac{1}{q_0},$$

any $\theta_1,\theta_2>0$, $u\in\mathcal{A}_{p_1,q_1}(X)$, $v\in\mathcal{A}_{p_2,q_2}(Y)$, and for all $(f,g)\in\mathcal{F}$

we have

$$\Big\| v(y) \| u(x) g(x,y) \|_{L_u^{q_1), \frac{\theta_1 q_1}{p_1}, \sigma_1}(X)} \Big\|_{L_v^{q_2), \frac{\theta_2 q_2}{p_2}, \sigma_2}(Y)} \le C \Big\| v(y) \| u(x) f(x,y) \|_{L_u^{p_1), \theta_1, \eta_1}} \Big\|_{L_v^{p_2), \theta_2, \eta_2}},$$

with a positive constant C independent of (f,g), constants σ_i and η_i, $i=1,2$ satisfying the condition

$$\frac{1}{p_1-\eta_1} - \frac{1}{q_1-\sigma_1} = \frac{1}{p_2-\eta_2} - \frac{1}{q_2-\sigma_2} = \frac{1}{p_0} - \frac{1}{q_0},$$

where σ_1, σ_2, η_1 and $\eta_2 > 0$ are positive numbers such that $u^{q_1} \in A_{1+(q_1-\sigma_1)/(p_1-\eta_2)'}(X)$, $v^{q_1} \in A_{1+(q_2-\sigma_2)/(p_2-\eta_2)'}(Y)$.

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