Kokilashvili et al. *Journal of Inequalities and Applications* 2013, **2013**:173
http://www.journalofinequalitiesandapplications.com/content/2013/1/173

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Weighted kernel operators in variable exponent amalgam spaces

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Abstract

The paper is devoted to weighted inequalities for positive kernel operators in variable exponent amalgam spaces. In particular, a characterization of a weight v governing the boundedness/compactness of the weighted kernel operators K_v and $\mathcal{K}_v$, defined on $\mathbb{R}_+$ and $\mathbb{R}$, respectively, under the log-Hölder continuity condition on exponents of spaces is established. These operators involve, for example, weighted variable parameter fractional integrals. The results are new even for constant exponent amalgam spaces.

MSC: 46E30; 47B34

Keywords: variable exponent amalgam spaces; positive kernel operator; boundedness; compactness

1 Introduction

In the paper, we derive necessary and sufficient conditions on a weight function v governing the boundedness/compactness of the positive kernel operators

$$K_v f(x) = v(x) \int_0^x k(x,t) f(t)\, dt, \quad x > 0,$$

$$(\mathcal{K}_v f)(x) = v(x) \int_{-\infty}^x k(x,t) f(t)\, dt, \quad x \in \mathbb{R}$$

in variable exponent amalgam spaces (VEAS) under the log-Hölder continuity condition on exponents of spaces. It should be emphasized that the results are new even for constant exponent amalgam spaces.

Historically, the boundedness problem for the two-weighted Hardy transform $(H_{v,w}f)(x) = v(x)\int_0^x f(t)w(t)\,dt$ from $L^{p(\cdot)}$ to $L^{q(\cdot)}$ was studied in the papers [1, 2] in different terms on weights (see also [3] for related topics). In [1], the authors explored also the compactness problem for $H_{v,w}$. The boundedness for fractional integral operators in (weighted) variable exponent Lebesgue spaces defined on Euclidean spaces was investigated by many authors (see, *e.g.*, the papers [4–14], *etc.*). The compactness (resp. non-compactness) of fractional and singular integrals in weighted $L^{p(\cdot)}$ spaces was studied in [15]. We refer also to the monograph [16] for related topics.

The space $L^{p(\cdot)}$ is a special case of the Musielak-Orlicz space (see [17, 18]). The first systematic study of modular spaces is due to Nakano [19].

Variable exponent Lebesgue and Sobolev spaces arise, *e.g.*, in the study of mathematical problems related to applications to mechanics of the continuum medium (see [16, 20] and references cited therein).

The manuscript consists of four sections. In Section 2, we recall some well-known facts about variable exponent Lebesgue spaces $L^{p(\cdot)}$. In Section 3, we recall the definition, history and some essential properties of amalgam spaces with a constant exponent, and also known results about the boundedness of some integral operators in these spaces; boundedness criteria for the operators K_ν and $\mathcal{K}_\nu$ in VEAS are also established. The compactness of positive kernel operators in VEAS is studied in Section 4.

Throughout the paper, constants (often different constants in the same series of inequalities) will mainly be denoted by c or C; by the symbol $p'(x)$, we denote the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$; the relation $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 a \leq b \leq c_2 a$.

2 Preliminaries

We begin this section by the definition and essential properties of variable exponent Lebesgue spaces.

Let E be a measurable set in $\mathbb{R}$ with positive measure. We denote

$$p_-(E) := \inf_E p, \qquad p_+(E) := \sup_E p$$

for a measurable function p on E. Suppose that $1 < p_-(E) \leq p_+(E) < \infty$. Denote by ρ a weight function on E (*i.e.*, ρ is an almost everywhere positive measurable function). We say that a measurable function f on E belongs to $L^{p(\cdot)}_\rho(E)$ (or to $L^{p(x)}_\rho(E)$) if

$$S_{p(\cdot),\rho}(f) = \int_E \left|f(x)\right|^{p(x)} \rho(x)\,dx < \infty.$$

It is a Banach space with respect to the norm (see, *e.g.*, [21–24])

$$\|f\|_{L^{p(\cdot)}_\rho(E)} = \inf\bigl\{\lambda > 0 : S_{p(\cdot),\rho}(f/\lambda) \leq 1\bigr\}.$$

If $\rho \equiv \text{const}$, then we use the symbol $L^{p(\cdot)}(E)$ (resp. $S_{p(\cdot)}$) instead of $L^{p(\cdot)}_\rho(E)$ (resp. $S_{p(\cdot),\rho}$). It is clear that $\|f\|_{L^{p(\cdot)}_\rho(E)} = \|f(\cdot)\rho^{1/p(\cdot)}(\cdot)\|_{L^{p(\cdot)}(E)}$.

In the sequel, we will denote by $\mathbb{Z}$ and $\mathbb{Z}_-$ the set of all integers and the set of non-positive integers, respectively.

To prove the main results, we need some known statements.

Proposition A ([22–24]) *Let E be a measurable subset of $\mathbb{R}$. Suppose that $1 < p_-(E) \leq p_+(E) < \infty$. Then*

(i)

$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \leq S_p(f\chi_E) \leq \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \qquad \|f\|_{L^{p(\cdot)}(E)} \leq 1;$$
$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \leq S_p(f\chi_E) \leq \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \qquad \|f\|_{L^{p(\cdot)}(E)} \geq 1;$$

(ii) *Hölder's inequality*

$$\left|\int_E f(x)g(x)\,dx\right| \le \left(\frac{1}{p_-(E)} + \frac{1}{(p_+(E))'}\right)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$, $g \in L^{p'(\cdot)}(E)$.

Proposition B ([22–24]) *Let* $1 \le r(x) \le p(x)$ *and let* E *be a bounded subset of* $\mathbb{R}$. *Then the following inequality*

$$\|f\|_{L^{r(\cdot)}(E)} \le \bigl(|E|+1\bigr)\|f\|_{L^{p(\cdot)}(E)}$$

holds.

Definition 2.1 We say that p satisfies the weak Lipschitz (log-Hölder continuity) condition on $E \subset \mathbb{R}$ ($p \in \mathrm{WL}(E)$), if there is a positive constant A such that for all x and y in E with $0 < |x-y| < 1/2$ the inequality

$$\bigl|p(x)-p(y)\bigr| \le A/\bigl(-\ln|x-y|\bigr)$$

holds.

Lemma A ([25]) *Let* I *be an interval in* $\mathbb{R}$. *Then* $p \in \mathrm{WL}(I)$ *if and only if there exists a positive constant* c *such that*

$$|J|^{p_-(J)-p_+(J)} \le c$$

for all intervals $J \subseteq I$ *with* $|J| > 0$. *Moreover, the constant* c *does not depend on* I.

For the next statement we refer to [2] in the case of finite interval, and [26] for infinite interval.

Proposition C *Let* p *and* q *be measurable functions on* $I := (a,b)$ $(-\infty < a < b \le +\infty)$ *satisfying the condition* $1 < p_-(I) \le p(x) \le q(x) < q_+(I) < \infty$, $x \in I$. *Let* $p, q \in \mathrm{WL}(I)$. *Suppose also that if* $b = \infty$, *then* $p(x) \equiv p_c \equiv \mathrm{const}$, $q(x) \equiv q_c \equiv \mathrm{const}$ *outside some large interval* (a,d). *Then there is a positive constant* c *depending only on* p *and* q *such that for all* $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ *and all sequences of intervals* $S_k := [x_{k-1}, x_{k+1})$, *where* $[x_k, x_{k+1})$ *are disjoint intervals satisfying the condition* $\bigcup_k [x_k, x_{k+1}) = I$, *the inequality*

$$\sum_k \|f\chi_{S_k}\|_{L^{p(\cdot)}(I)}\|g\chi_{S_k}\|_{L^{q'(\cdot)}(I)} \le cC_{a,b}\|f\|_{L^{p(\cdot)}(I)}\|g\|_{L^{q'(\cdot)}(I)}$$

holds. Moreover, the value of $C_{a,b}$ *is defined as follows:* $C_{a,b} = [(b-a)+1]^2$ *if* $b < \infty$ *and* $C_{a,\infty} = [(d-a)+1]^2 + 1$ *if* $b = \infty$.

Let v and w be a.e. positive measurable function on $[a,b)$, $-\infty < a < b \le \infty$, and let

$$\bigl(H^{(a,b)}_{v,w} f\bigr)(x) = v(x)\int_a^x f(t)w(t)\,dt, \quad x \in [a,b).$$

Further, we denote

$$(H_{v,w}f)(x) = v(x)\int_0^x f(t)w(t)\,dt, \quad x > 0,$$
$$(\mathcal{H}_{v,w}f)(x) = v(x)\int_{-\infty}^x f(t)w(t)\,dt, \quad x \in \mathbb{R}.$$

Let us recall the two-weight criterion for the Hardy operator in classical Lebesgue spaces:

Theorem A ([27, 28]) *Let r and s be constants such that $1 < r \le s < \infty$. Suppose that $0 \le a < b \le \infty$. Let v and w be non-negative measurable functions on $[a,b)$. Then the Hardy inequality*

$$\left(\int_a^b v(x)\left(\int_a^x f(t)\,dt\right)^s dx\right)^{1/s} \le c\left(\int_a^b w(t)\big(f(t)\big)^r dt\right)^{1/r}, \quad f \ge 0,$$

holds if and only if

$$A := \sup_{a\le t\le b}\left(\int_t^b v(x)\,dx\right)^{1/s}\left(\int_a^t w^{1-r'}(x)\,dx\right)^{1/r'} < \infty.$$

Moreover, if c is the best constant in the Hardy inequality, then there are positive constants c_1 and c_2 depending only on r and s such that $c_1A \le c \le c_2A$.

For the Hardy inequalities, we also refer the books [29, 30].

The following statement was proved in [2] for finite interval and in [12] for the case of infinite interval, but we give the proof because of the upper and lower bound of the norm of $H_{v,w}$.

Theorem B *Let $-\infty < a < b \le +\infty$ and let p and q be measurable functions on $I := (a,b)$ satisfying the conditions*: $1 < p_-(I) \le p(x) \le q(x) \le q_+(I) < \infty$, $p, q \in \mathrm{WL}(I)$. *We assume that $p \equiv p_c \equiv$ const, $q \equiv q_c \equiv$ const outside some large interval (a,d) if $b = \infty$. Then $H^I_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ if and only if*

$$A_{a,b} \equiv \sup_{a<t<b}\big\|\chi_{(t,b)(\cdot)}v(\cdot)\big\|_{L^{q(\cdot)}(I)}\big\|\chi_{(a,t)(\cdot)}w(\cdot)\big\|_{L^{p'(\cdot)}(I)} < \infty.$$

Moreover, there are positive constants c_1 and c_2 independent of the interval I such that

$$c_1A_{a,b} \le \big\|H^{(a,b)}_{v,w}\big\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \le c_2C_{a,b}A_{a,b},$$

where the constant $C_{a,b}$ is defined in Proposition C.

Proof *Sufficiency*. Let $f \ge 0$. Suppose that $b < \infty$ and that $\int_a^b f(t)\,dt \in [2^{m_0}, 2^{m_0+1})$ for some integer m_0. We construct a sequence $\{x_k\}$ so that

$$\int_a^{x_k} fw = \int_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $(a,b) = \bigcup_k [x_k, x_{k+1})$. Let g be a function satisfying the condition $\|g\|_{L^{q'(\cdot)}([a,b])} \le 1$. Applying Hölder's inequality for variable exponent Lebesgue spaces and Proposition C we have that

$$\begin{aligned}
\int_a^b (H_{v,w}f)g &\le \sum_k \left(\int_{x_k}^{x_{k+1}} gv\right)\left(\int_0^{x_{k+1}} fw\right) \\
&= 4\sum_k \left(\int_{x_k}^{x_{k+1}} gv\right)\left(\int_{x_{k-1}}^{x_k} fw\right) \\
&\le 4\sum_k \left\|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\right\|_{L^{q'(\cdot)}(I)} \left\|\chi_{(x_k,x_{k+1})}(\cdot)v(\cdot)\right\|_{L^{q(\cdot)}(I)} \\
&\quad \times \left\|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\right\|_{L^{p(\cdot)}(I)} \left\|\chi_{(x_{k-1},x_k)}(\cdot)w(\cdot)\right\|_{L^{p'(\cdot)}(I)} \\
&\le 4A_{a,b}\sum_k \left\|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\right\|_{L^{q'(\cdot)}(I)} \left\|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\right\|_{L^{p(\cdot)}(I)} \\
&\le 4C_{a,b}A_{a,b}\left\|f(\cdot)\right\|_{L^{p(\cdot)}(I)}\left\|g(\cdot)\right\|_{L^{q'(\cdot)}(I)},
\end{aligned}$$

where $C_{a,b}$ is the constant defined in Proposition C. Taking now the supremum with respect to g, we have sufficiency for $b < \infty$.

Let now $b = \infty$. Then

$$\begin{aligned}
\left\|H_{v,w}^{(a,\infty)}f\right\|_{L^{q(\cdot)}((a,+\infty))} &\le \left\|v(x)\int_a^x fw\right\|_{L^{q(\cdot)}((a,d))} + \left\|v(x)\int_a^x fw\right\|_{L^{q_c}([d,+\infty))} \\
&:= I_1 + I_2.
\end{aligned}$$

By applying already used arguments, we have that $I_1 \le 4C_{a,\infty}A_{a,+\infty}$, where $C_{a,\infty} = [(d-a)+1]^2$. Further, due to Hölder's inequality and Theorem A, we find that

$$\begin{aligned}
I_2 &\le \left\|v(x)\int_a^d fw\right\|_{L^{q_c}([d,+\infty))} + \left\|v(x)\int_d^x fw\right\|_{L^{q_c}([d,+\infty))} \\
&\le \left\|v(\cdot)\chi_{[d,+\infty)}(\cdot)\right\|_{L^{q(\cdot)}}\left\|w(\cdot)\chi_{[a,d)}(\cdot)\right\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}} \\
&\quad + 4A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)} \le 5A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}$$

To get the lower bound for $\|H_{v,w}^{(a,b)}\|$ is trivial by choosing the appropriate test function $f(x) = \chi_{(a,t)}(x)$, $a < t < b$ in the boundedness of $H_{v,w}^I$ from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. □

Corollary A *Let p and q be defined on $\mathbb{R}_+$ and satisfy the conditions of Theorem* B. *Then for all $n \in \mathbb{Z}$,*

$$\left\|v(x)\int_{2^n}^x f(t)w(t)\,dt\right\|_{L^{q(\cdot)}([2^n,2^{n+1}])} \le D\|f\|_{L^{p(\cdot)}([2^n,2^{n+1}])},$$

where $D = \max\{c(2d+1)^2, 4\}\sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}$, $A_{2^n,2^{n+1}}$ is defined in Theorem B *and the constant c depends only on p and q.*

Proof By the hypothesis, p and q are constant outside some large interval $(0,d)$. Let $d \in [2^{m_0-1}, 2^{m_0})$ for some integer m_0. Then by Theorem B for $n \le m_0$, we have

$$\begin{aligned}\|H_{v,w}^{(2^n,2^{n+1})}\|_{L^{p(\cdot)}([2^n,2^{n+1}))\to L^{q(\cdot)}([2^n,2^{n+1}))} &\le c\bigl(2^n+1\bigr)^2 A_{2^n,2^{n+1}}\\ &\le c\bigl(2^{m_0}+1\bigr)^2 A_{2^n,2^{n+1}}\\ &\le c(2d+1)^2 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}},\end{aligned}$$

where the positive constant c depends only on p and q. If $n > m_0$, then p and q are constants on the intervals $[2^n, 2^{n+1})$. In this case taking the proof of Theorem B into account, we find that

$$\sup_{n>m_0} \|H_{v,w}^{(2^n,2^{n+1})}\|_{L^{p(\cdot)}([2^n,2^{n+1}])\to L^{q(\cdot)}([2^n,2^{n+1}])} \le 4 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}. \qquad \square$$

Theorem C ([5]) *Let $p(x)$ and $q(x)$ be measurable functions on an interval $I \subseteq R_+$. Suppose that $1 < p_-(I) \le p_+(I) < \infty$ and $1 < q_-(I) \le q_+(I) < \infty$. If*

$$\bigl\| \|k(x,y)\|_{L^{p'(y)}(I)} \bigr\|_{L^{q(x)}(I)} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int_I k(x,y) f(y)\,dy$$

is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Lemma B (see, *e.g.* [31]) *Let $1 < q < \bar{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\left\{\sum_{n\in\mathbb{Z}} \bigl(|a_n|u_n\bigr)^q\right\}^{1/q} \le C\left\{\sum_{n\in\mathbb{Z}} \bigl(|a_n|v_n\bigr)^{\bar{q}}\right\}^{1/\bar{q}}$$

holds for all sequences $\{a_n\}$ of real numbers.

(ii) $\{\sum_{n\in\mathbb{Z}} (u_n {v_n}^{-1})^s\}^{1/s} < \infty$.

Lemma C (see *e.g.* [32]) *Let p, q be constants such that $1 < p, q < \infty$. Suppose that $v_k \ge 0$, $w_k > 0$, $k \in \mathbb{Z}$. Then there exists a constant $c > 0$ such that*

$$\left\{\sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} v_n a_k\right)^q\right\}^{1/q} \le c\left(\sum_{n\in\mathbb{Z}} (w_n a_n)^p\right)^{1/p}$$

holds for all non-negative sequence $\{a_k\} \in l^p_{\{v_n^p\}}$, if and only if

(i) *in case $1 < p \le q < \infty$,*

$$A_1 := \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q\right)^{1/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'}\right)^{1/p'} < \infty;$$

(ii) *in case* $1 < q < p < \infty$,

$$A_2 := \left\{ \sum_{m \in \mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q \right)^{r/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'} \right)^{r/q'} w_m^{-p'} \right\}^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$.

Definition 2.2 Let $I = (0, a)$, $0 < a \le \infty$. We say that a kernel $k : \{(x, y) : 0 < y < x < a\} \to (0, \infty)$ belongs to $V(I)$ ($k \in V(I)$) if there exists a constant c_1 such that for all x, y, t with $0 < y < t < x < a$ the inequality

$$k(x, y) \le c_1 k(x, t)$$

holds.

Definition 2.3 Let r be a measurable function on $I = (0, a)$, $0 < a \le \infty$ with values in $(1, +\infty)$. We say a kernel k belongs to $V_{r(\cdot)}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in (0, a)$, the inequality

$$\left\| \chi_{(\frac{x}{2}, x)}(\cdot) k(x, \cdot) \right\|_{L^{r(\cdot)}(I)} \le c_2 x^{\frac{1}{r(x)}} k\left(x, \frac{x}{2}\right)$$

is fulfilled.

Example 2.1 (Lemma 3 of [26]) Let $I := (0, a)$, where $0 < a \le \infty$. Let α be a measurable function on I satisfying the condition $0 < \alpha_-(I) \le \alpha_+(I) \le 1$. Suppose that r is a function on I with values in $(1, +\infty)$ satisfying the condition $r \in \mathrm{WL}(I)$. Suppose that $r(x) \equiv r_0 \equiv \mathrm{const}$ outside some interval $(0, b)$ when $a = +\infty$. Then $k(x, t) = (x - t)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$.

The next examples of kernels can be checked easily:

Example 2.2 Let $I := (0, a)$, where $0 < a \le \infty$. Suppose that α is a measurable function on I satisfying the condition $0 < \alpha_-(I) \le \alpha_+(I) \le 1$. Let r be a function on I with the values in $(1, +\infty)$ satisfying the condition $r, \bar{r} \in \mathrm{WL}(I)$ where $\bar{r}(t) = r(t^{1/\sigma})$. Suppose that $r(x) \equiv r_0 \equiv \mathrm{const}$ outside some interval $(0, b)$ when $a = +\infty$. Then $k(x, y) = (x^\sigma - y^\sigma)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$ and $\sigma > 0$.

Example 2.3 Let $I := (0, a)$, $0 < a \le \infty$. Let r be a function on I with the values in $(1, +\infty)$ satisfying the condition $r \in \mathrm{WL}(I)$ and let r be increasing on I. Suppose that $r(x) \equiv r_0 \equiv \mathrm{const}$ outside some interval $(0, b)$ when $a = +\infty$. Further, let $0 < \alpha_-(I) \le \alpha(x) \le 1$ and $\alpha(x) + \beta(x) > 2 - \frac{1}{r(x)}$. Then $k(x, y) = (x - y)^{\alpha(x)-1} \ln^{\beta(x)-1} \frac{x}{y} \in V(I) \cap V_{r(\cdot)}(I)$.

For other examples of kernel k satisfying the condition $k \in V(I) \cap V_r(I)$, where r is constant, we refer to [33] (see also [34], p.163).

3 Boundedness on VEAS

This section is devoted to the boundedness of weighted kernel operators in VEAS.

3.1 Amalgam spaces

Let I be $\mathbb{R}$ or $\mathbb{R}_+$ and $\alpha = \{I_n; n \in \mathbb{Z}\}$ be a cover of I consisting of disjoint half-open intervals I_n, each of the form $[a_1, a_2)$, whose union is I. Let

$$\|f\|_{(L_u^{p(\cdot)}(I),l^q)_\alpha} := \left(\sum_{n\in\mathbb{Z}} \left\| \chi_{I_n}(\cdot) f(\cdot) \right\|^q_{L_u^{p(\cdot)}(I)}\right)^{1/q},$$

we define the general amalgams with variable exponent

$$\left(L_u^{p(\cdot)}(I), l^q\right)_\alpha = \left\{f : \|f\|_{(L_u^{p(\cdot)}(I),l^q)_\alpha} < \infty\right\}.$$

If $u \equiv \text{const}$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ is denoted by $(L^{p(\cdot)}(I), l^q)_\alpha$.

Let $p \equiv p_c \equiv \text{const}$ and $u \equiv \text{const}$. Then we have the usual irregular amalgam (see [35]); if $I = \mathbb{R}$ and $I_n = [n, n+1)$, then $(L^{p_c}(I), l^q)_\alpha$ is the amalgam space introduced by Wiener (see [36, 37]) in connection with the development of the theory of generalized harmonic analysis.

We call $(L_u^{p(\cdot)}(I), l^q)_\alpha$ irregular weighted amalgam spaces with variable exponent. If $I_n = [n, n+1)$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ will be denoted by $(L_u^{p(\cdot)}(I), l^q)$.

Let $d = \{[2^n, 2^{n+1}); n \in \mathbb{Z}\}$ and $I = \mathbb{R}_+$. We denote weighted dyadic amalgam with variable exponent by $(L_u^{p(\cdot)}(I), l^q)_d$. Some properties regarding general amalgams with variable exponent can be derived in the same way as for usual irregular amalgams $(L_u^p(\mathbb{R}), l^q)_\alpha$, where p is constant. Irregular amalgams were introduced in [38] and studied in [35].

Theorem D *Let p be a measurable function on I with $1 < p_-(I) \le p_+(I) < \infty$ and q be constant with $1 < q < \infty$. The irregular amalgams with variable exponent $(L^{p(\cdot)}(I), l^q)_\alpha$ is a Banach space whose dual space is $(L^{p(\cdot)}(I), l^q)^*_\alpha = (L^{p'(\cdot)}(I), l^{q'})_\alpha$. Further, Hölder's inequality holds in the following form*:

$$\left|\int_I f(t)g(t)\,dt\right| \le \|f\|_{(L^{p(\cdot)}(I),l^q)_\alpha} \|g\|_{(L^{(p(\cdot))'}(I),l^{q'})_\alpha}.$$

Proof Since $L^{p(\cdot)}$ is a Banach space and $(L^{p(\cdot)})^* = L^{p'(\cdot)}$ (see [22]), from general arguments (see [35, 39–41]) we have the desired result. □

The next statement for more general case, *i.e.*, when amalgams are defined with respect to Banach spaces, can be found in [35].

Theorem E *Let p be measurable function on I and $1 \le q_1 \le q_2$, then*

$$\left(L^{p(\cdot)}(I), l^{q_1}\right)_\alpha \subset \left(L^{p(\cdot)}(I), l^{q_2}\right)_\alpha.$$

Other structural properties of amalgams are investigated, *e.g.*, in [41] and [35].

The next statement is a generalization of Theorem 4 in [35] for variable exponent amalgams with weights.

Proposition D *Let p, q be measurable functions on I such that $1 \le q_-(I) \le q(x) < p(x) \le p_+(I)$ and $1 \le r < \infty$. Then the space $(L_w^{p(\cdot)}(I), l^r)_\alpha$ is continuously embedded in $(L_v^{q(\cdot)}(I), l^r)_\alpha$*

if

$$S := \sup_{n\in\mathbb{Z}} \int_{I_n} \left(\frac{v(x)}{w(x)}\right)^{\frac{p(x)}{p(x)-q(x)}} dx < \infty. \tag{3.1}$$

Conversely, if $1 < q_-(I) \le q_+(I) < p_-(I) \le p_+(I) < \infty$, *then condition* (3.1) *is also necessary for the continuous embedding of* $(L^{p(\cdot)}_w(I), l^r)_\alpha$ *into* $(L^{q(\cdot)}_v(I), l^r)_\alpha$.

Proof It is known (see [42]) that the continuous embedding $L^{p(\cdot)}_w(I) \hookrightarrow L^{q(\cdot)}_v(I)$ $(q(x) < p(x))$ holds if and only if

$$\int_I \left(\frac{v(x)}{w(x)}\right)^{\frac{p(x)}{p(x)-q(x)}} dx < \infty.$$

Moreover, the estimate

$$\frac{\|(v/w)^{1/(p(\cdot)-q(\cdot))}\|_{L^{q(\cdot)}_v}}{\|(v/w)^{1/(p(\cdot)-q(\cdot))}\|_{L^{p(\cdot)}_w}} \le \|Id\|_{L^{p(\cdot)}_w \to L^{q(\cdot)}_v} \le c \max\left\{1, \|v/w\|_{L_w^{(p(\cdot)/q(\cdot))'}}\right\} \tag{3.1$'$}$$

holds, where the positive constant c depends only on p and q; Id is the identity operator.

Let condition (3.1) hold. Then

$$\|Id\|_{L^{p(\cdot)}_w(I_n) \to L^{q(\cdot)}_v(I_n)} \le \|Id\|_{L^{p(\cdot)}_w(I) \to L^{q(\cdot)}_v(I)} < \infty.$$

Hence, $(L^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L^{q(\cdot)}, l^r)_\alpha$.

Conversely, let the continuous embedding $(L^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L^{q(\cdot)}, l^r)_\alpha$ hold and let $1 < q_-(I) \le q_+(I) < p_-(I) \le p_+(I) < \infty$. By taking functions supported in I_n we can derive the estimate

$$\sup_{n\in\mathbb{Z}} \|Id\|_{L^{p(\cdot)}(I_n) \mapsto L^{q(\cdot)}_v(I_n)} \le \|Id\|_{(L^{p(\cdot)}(I), l^r)_\alpha \mapsto (L^{q(\cdot)}_v(I), l^r)_\alpha}.$$

By applying the left-hand side inequality of (3.1′) and Proposition A, we conclude that condition (3.1) is satisfied. □

3.2 General operators in VEAS

We begin this subsection by the following definition.

Definition 3.1 ([31]) Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}$. Define a sequence of local operators

$$(T_n f)(x) := T(f\chi_{(n-1,n+2)})(x), \quad x \in (n-1, n+2), n \in \mathbb{Z}.$$

Let us assume that there is a discrete operator T^d satisfying the following conditions:

(i) There exists a positive constant c such that for all non-negative functions f, $x \in (n, n+1)$ and arbitrary $n \in \mathbb{Z}$ the inequality

$$T(f\chi_{(-\infty,n-1)} + f\chi_{(n+2,\infty)})(x) \le cT^d\left(\int_{m-1}^m f\right)(n)$$

holds.

(ii) There is $c > 0$ such that for all sequences $\{a_k\}$ of non-negative real numbers and $n \in \mathbb{Z}$, the inequality

$$T^d\big(\{a_k\}\big)(n) \le cTf(y)$$

holds for all $y \in (n, n+1)$ and all non-negative f, where $\int_{m-1}^{m} f =: a_m$, $m \in \mathbb{Z}$. It is also assumed that T satisfies the conditions

$$Tf = T|f|, \qquad T(\lambda f) = |\lambda| Tf, \qquad T(f+g) \le Tf + Tg, \qquad Tf \le Tg \quad \text{if } f \le g.$$

We will say that an operator T satisfying all the above mentioned conditions is admissible on $\mathbb{R}$.

For example, Hardy operators, Hardy-Littlewood maximal operators, fractional integral operators, fractional maximal operators are admissible on $\mathbb{R}$ (see [31]). Carton-Leburn, Heinig and Hoffmann [32] established two weighted criteria for the Hardy transform $(\mathcal{H}f)(x) = \int_{-\infty}^{x} f(t)\,dt$ in amalgam spaces defined on $\mathbb{R}$ (see also [43, 44] for related topics). In [32], the authors derived some sufficient conditions for the two-weight boundedness of the kernel operator $(\mathcal{K}f)(x) := \int_{-\infty}^{x} k(x,y) f(y)\,dy$ where k is non-decreasing in the second variable and non-increasing in the first one. In the paper [45], the two-weight problem for generalized Hardy-type kernel operators including the fractional integrals of order greater than one (without singularity) was solved.

General type results for the admissible operators read as follows.

Theorem F ([31]) *Let $1 < p, \bar{p}, q, \bar{q} < \infty$, and let w and v be weight functions on $\mathbb{R}$. Suppose that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^p(\mathbb{R}), l^q)} \le c\|wf\|_{(L^{\bar{p}}(\mathbb{R}), l^{\bar{q}})}$$

holds for all measurable f if and only if

(i) *T^d is bounded from $l^{\bar{q}}(\{w_n\})$ to $l^q(\{v_n\})$, where $w_n := (\int_{n-1}^{n} w^{-\bar{p}'})^{\frac{-\bar{q}}{\bar{p}'}}$, $v_n := (\int_{n}^{n+1} v^p)^{\frac{q}{p}}$.*

(ii) (a) *$\sup_{n\in\mathbb{Z}} \|T_n\|_{[L^{\bar{p}}_{w^{\bar{p}}}(n-1,n+2) \to L^p_{v^p}(n-1,n+2)]} < \infty$ for $1 < \bar{q} \le q < \infty$.*

(b) *$\|T_n\|_{[L^{\bar{p}}_{w^{\bar{p}}}(n-1,n+2) \to L^p_{v^p}(n-1,n+2)]} \in l^s$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ for $1 < q < \bar{q} < \infty$.*

Our aim is to establish weighted characterization of the boundedness of kernel operators involving fractional integrals of variable parameter of order less than one in variable exponent amalgam spaces. For the continuous part of amalgam spaces, we take variable exponent Lebesgue spaces defined on I.

It should be emphasized that the following fact holds: by the change of variable $z \to \log_2 x$ it is possible to get appropriate boundedness or compactness results from dyadic amalgams $(L^{p(\cdot)}(\mathbb{R}_+), l^q)_d$ to amalgams defined on $\mathbb{R}$.

Analyzing the proof of Theorem 1 of [31], we can formulate the next statement and give the proof for completeness.

Proposition 3.1 *Let $\bar{p}(\cdot), p(\cdot)$ be measurable functions on $\mathbb{R}$ satisfying $1 < p_-(\mathbb{R}) \le p_+(\mathbb{R}) < \infty$, $1 < \bar{p}_-(\mathbb{R}) \le \bar{p}_+(\mathbb{R}) < \infty$. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$.*

Assume that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \le c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})} \tag{3.2}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}(\{\bar{w}_n\})$ to $l^q(\{\bar{v}_n\})$ where $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}$.*

(ii) (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}}(n-1,n+2)\to L^{p(\cdot)}_{v^{p(\cdot)}}(n-1,n+2)]} < \infty$ *for* $1 < \bar{q} \le q < \infty$.

(b) $\|T_n\|_{[L^{\bar{p}(\cdot)}_w(n-1,n+2)\to L^{p(\cdot)}_v(n-1,n+2)]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (3.2) *hold. Then*

(1) *conditions* (ii) *are satisfied*;

(2) *condition* (i) *is satisfied for* $w \equiv \mathrm{const}$ *or for p and $\bar{p}$ being constants outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.*

Proof Let (i) and (ii) hold. We have

$$\begin{aligned}\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} &\le c\left\{\sum_{n\in\mathbb{Z}} \left\|T\left[f\left(\chi_{(-\infty,n-1)} + \chi_{(n+2,\infty)}\right)\right]v(\cdot)\right\|^q_{L^{p(\cdot)}(n,n+1)}\right\}^{1/q} \\ &\quad + c\left\{\sum_{n\in\mathbb{Z}} \|vT_nf\|_{L^{p(\cdot)}(n,n+1)}\right\}^{1/q} =: S_1 + S_2.\end{aligned}$$

Let $a_m := \int_{m-1}^m f$. By the hypothesis and Hölder's inequality for variable exponents $p(\cdot)$ and $p'(\cdot)$, we have that

$$\begin{aligned}S_1 &\le c\left\{\sum_{n\in\mathbb{Z}} \left(T^d\left(\{a_m\}\right)(n)\right)^q \|\chi_{(n,n+1)}v\|^q_{L^{p(\cdot)}(n,n+1)}\right\}^{1/q} \le c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \|\chi_{(n-1,n)}w^{-1}\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\right\}^{1/\bar{q}} \\ &\le c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^q)}.\end{aligned}$$

Let us estimate S_2. Suppose that $1 < \bar{q} \le q < \infty$. Since the operators T_n are uniformly bounded, we find that

$$\begin{aligned}S_2 &\le c\left\{\sum_{n\in\mathbb{Z}} \|fw\|^q_{L^{\bar{p}(\cdot)}(n-1,n+1)}\right\}^{1/q} \le c\left\{\sum_{n\in\mathbb{Z}} \|fw\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(n-1,n+1)}\right\}^{1/\bar{q}} \\ &\le c\|fw\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})}.\end{aligned}$$

If $1 < q < \bar{q} < \infty$, then by using Hölder's inequality we derive

$$\begin{aligned}S_2 &\le c\left\{\sum_{n\in\mathbb{Z}} \|T_n\|^q_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}}}(n-1,n+2)\to L^{p(\cdot)}_{v^p}(n-1,n+2)]} \|\chi_{(n-1,n+2)}fw\|^q_{L^{\bar{p}(\cdot)}}\right\}^{1/q} \\ &\le c\left[\left\{\sum_{n\in\mathbb{Z}} \|T_n\|^{\frac{q\bar{q}}{\bar{q}-q}}\right\}^{\frac{\bar{q}-q}{q}} \left\{\sum_{n\in\mathbb{Z}} \|\chi_{(n-1,n+2)}fw\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right\}^{\frac{q}{\bar{q}}}\right]^{1/q} \\ &\le c\|fw\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})}.\end{aligned}$$

Conversely, suppose that (3.2) holds. Let $n \in \mathbb{Z}$ and let f be a non-negative function supported in $(n-1, n+2)$. Then

$$\|fw\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})} \leq 3\|fw\chi_{(n-1,n+2)}\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}))}.$$

On the other hand,

$$\begin{aligned}\|\nu Tf\|_{(L^{p(\cdot)},l^{q})} &\geq \|\nu\chi_{(n-1,n+2)}Tf\|_{L^{p(\cdot)}}\\ &\geq \|\nu\chi_{(n-1,n+2)}T_n f\|_{L^{p(\cdot)}}\\ &= \|\nu T_n f\|_{L^{p(\cdot)}}.\end{aligned}$$

Now due to inequality (3.2), we conclude that (a) of (ii) holds. Let us now show that if $1 < q < \bar{q} < \infty$, then (b) of (ii) is satisfied.

Since $\|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}} \to L^{p(\cdot)}_{\nu^{p(\cdot)}}]} = \sup_{\{f:\|wf\|_{L^{\bar{p}(\cdot)}}=1\}} \|\nu T_n f\|_{L^{p(\cdot)}}$, we have that for each n, there exists a non-negative measurable function f_n, with the support in $(n-1, n+2)$ and with $\|w\chi_{(n-1,n+2)}f_n\|_{L^{\bar{p}(\cdot)}} = 1$, such that $\|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}} \to L^{p(\cdot)}_{\nu^{p(\cdot)}}]} < \|\nu T_n f_n\|_{L^{p(\cdot)}} + \frac{1}{2^{|n|}}$. Thus, it is sufficient to prove that $\|\nu T_n f_n\|_{L^{p(\cdot)}} \in l^s$.

Let $\{a_n\}$ be a sequence of non-negative real numbers and $f = \sum_n a_n f_n$. For each $n \in \mathbb{Z}$, $f(x) > a_n f_n(x)$ and then $Tf(x) \geq a_n T_n f_n(x)$ for all $x \in (n-1, n+2)$.

Consequently,

$$\|\nu Tf\|_{(L^{p(\cdot)}(\mathbb{R}),l^{q})} \geq \left\{\sum_{n\in\mathbb{Z}} ca_n^q \|\chi_{(n-1,n+2)}\nu T_n f\|^q_{L^{p(\cdot)}}\right\}^{1/q} = c\left\{\sum_{n\in\mathbb{Z}} a_n^q \|\nu T_n f_n\|^q_{L^{p(\cdot)}}\right\}^{1/q}.$$

Hence, inequality (3.2) yields that

$$\begin{aligned}\left\{\sum_{n\in\mathbb{Z}} a_n^q \|\nu T_n f_n\|^q_{L^{p(\cdot)}}\right\}^{1/q} &\leq c\left\{\sum_{n\in\mathbb{Z}} \|\chi_{(n-1,n+2)} wf\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right\}^{1/\bar{q}}\\ &\leq c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \|\chi_{(n-1,n+2)} wf_n\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right\}^{1/\bar{q}} = c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\right\}.\end{aligned}$$

Finally, by Lemma B, we see that (b) of (ii) holds.

Now let us prove that (i) holds when $w \equiv \text{const}$. If $\{a_m\}$ is a sequence of non-negative real numbers and

$$f = \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)},$$

then $\int_{m-1}^{m} f = a_m$, and $\|\chi_{(n,n+1)}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} = a_n^{\bar{q}}\|\chi_{(n,n+1)}\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} = a_n^{\bar{q}}$ and by the properties of T, we have

$$\begin{aligned}\|\nu Tf\|_{(L^{p(\cdot)},l^{q})} &= \left\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)}\nu Tf\|^q_{L^{p(\cdot)}}\right\}^{1/q}\\ &\geq \left\{\sum_{n\in\mathbb{Z}} \left\|\chi_{(n,n+1)}\nu T^d\left(\int_{m-1}^{m} f\right)\right\|^q_{L^{p(\cdot)}}\right\}^{1/q}\end{aligned}$$

$$\begin{aligned}&\geq c\left\{\sum_{n\in\mathbb{Z}} T^d(a_m)^q(n)\|\chi_{(n,n+1)}v\|^q_{L^{p(\cdot)}}\right\}^{1/q}\\&= \left\|T^d\{a_m\}\right\|_{l^q\{\bar{v}_n^q\}}.\end{aligned}$$

Applying the two-weight inequality, we find that

$$\begin{aligned}\left\|T^d\{a_m\}\right\|_{l^q\{\bar{v}_n^q\}} &\leq c\left\{\sum_{n\in\mathbb{Z}}\|\chi_{(n,n+1)}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right\}^{1/\bar{q}}\\&= c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\right\}^{1/\bar{q}} = \|a_n\|_{l^{\bar{q}}}.\end{aligned}$$

Hence, (i) holds.

Suppose now that w is a general weight and there is a positive integer m_0 such that p, $\bar{p}$ are constants outside $[-m_0, m_0]$. Taking

$$f(x) = \sum_{m\in\mathbb{Z}} a_m\chi_{(m-1,m)}(x)\left(\int_{m-1}^m w^{-\bar{p}'(y)}(y)\,dy\right)^{-1} w^{-\bar{p}'(x)}(x),$$

it is easy to see that $\int_{m-1}^m f = a_m$. Moreover, by virtue of Proposition A and the fact that

$$\int_{m-1}^m w^{-\bar{p}'(y)}(y)\,dy \leq \int_{-m_0}^{m_0} w^{-\bar{p}'(y)}(y)\,dy < \infty, \qquad [m-1,m]\subset[-m_0,m_0],$$

we have for $m \leq m_0 + 1$,

$$\begin{aligned}\|\chi_{(m-1,m)}fw\|_{L^{\bar{p}(\cdot)}} &= a_m\left(\int_{m-1}^m w^{-\bar{p}'(y)}(y)\,dy\right)^{-1}\left\|\chi_{(m-1,m)}w^{(1-\bar{p}'(\cdot))}\right\|_{L^{\bar{p}(\cdot)}}\\&\leq ca_m\left(\int_{m-1}^m w^{-\bar{p}'(y)}(y)\,dy\right)^{-1/\bar{p}_+([m-1,m))},\end{aligned}$$

where the positive constant c depends on m_0. Since

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \geq C\left\|\bar{v}_n\left(T^d\{a_m\}\right)(n)\right\|_{l^q},$$

using again Proposition A, we find that

$$\begin{aligned}\left\|\bar{v}_n\left(T^d\{a_m\}\right)(n)\right\|_{l^q} &\leq C\left[\sum_m\|\chi_{(m-1,m)}fw\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(\mathbb{R})}\right]^{1/\bar{q}}\\&\leq c\left[\sum_m a_m^{\bar{q}}\left(\int_{m-1}^m w^{-\bar{p}'(y)}(y)\,dy\right)^{-\bar{q}/\bar{p}_+([m-1,m))}\right]^{1/\bar{q}} = \|a_m\bar{w}_m\|_{l^{\bar{q}}}.\end{aligned}$$

□

Definition 3.2 Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}_+$. We say that an operator T is admissible on $\mathbb{R}_+$ if the conditions of Definition 3.1 are satisfied replacing n by 2^n, $n\in\mathbb{Z}$.

The next statement can be obtained in the similar manner as Proposition 3.1 was proved; therefore, we omit the proof.

Proposition 3.2 *Let $\bar{p}(\cdot)$, $p(\cdot)$ be measurable functions on $\mathbb{R}_+$ satisfying $1 < p_-(\mathbb{R}_+) \le p_+(\mathbb{R}_+) < \infty$, $1 < \bar{p}_-(\mathbb{R}_+) \le \bar{p}_+(\mathbb{R}_+) < \infty$. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$. Suppose also that w and v are weight functions on $\mathbb{R}_+$ and that T is an admissible operator on $\mathbb{R}_+$.*

Then the inequality

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d} \le c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d} \tag{3.3}$$

holds if

(i) T^d *is bounded from* $l^{\bar{q}}(\{\bar{w}_n\})$ *to* $l^q(\{\bar{v}_n\})$ *where* $\bar{w}_n := \|\chi_{(2^{n-1},2^n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}$, $\bar{v}_n := \|\chi_{(2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}$.

(ii) (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}}(2^{n-1},2^{n+2})\to L^{p(\cdot)}_{v^{p(\cdot)}}(2^{n-1},2^{n+2})]} < \infty$ *for* $1 < \bar{q} \le q < \infty$.

(b) $\|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}}(2^{n-1},2^{n+2})\to L^{p(\cdot)}_{v^{p(\cdot)}}(2^{n-1},2^{n+2})]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, if (3.3) *holds, then*

(1) *conditions* (ii) *are satisfied*;

(2) *condition* (i) *is also satisfied but for* $w \equiv \text{const}$ *or for* p *and* $\bar{p}$ *satisfying the condition* $p \equiv \text{const}$, $\bar{p} \equiv \text{const}$ *outside some large interval* $[0, 2^{m_0}]$, $m_0 \in \mathbb{Z}$.

Proposition 3.2 gives criteria for the boundedness of $H_{v,w}$ in dyadic amalgams on $\mathbb{R}_+$ but by the next statement we prove the two-weight inequality under slightly different conditions.

Proposition 3.3 *Let $I := \mathbb{R}_+$ and let $1 < \bar{p}_-(I) \le \bar{p}(\cdot) \le p(\cdot) \le p_+(I) < \infty$. Let $1 < \bar{q}, q < \infty$. Suppose that $p, \bar{p} \in \mathrm{WL}(\mathbb{R}_+)$ and that $p \equiv p_c \equiv$ const outside some large interval $(0, b)$. Then the inequality*

$$\|H_{v,w}f\|_{(L^{p(\cdot)}(I),l^q)_d} \le c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}$$

with a positive constant independent of f holds if

(i) *in the case* $1 < \bar{q} \le q < \infty$,

(a)

$$\sup_{m\in\mathbb{Z}} \left\{ \sum_{n=m}^{\infty} \left\| \chi_{[2^n,2^{n+1})(\cdot)} v(\cdot) \right\|^q_{L^{p(\cdot)}} \right\}^{1/q} \left\{ \sum_{n=-\infty}^{m} \left\| \chi_{[2^{n-1},2^n)(\cdot)} w(\cdot) \right\|^{\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right\}^{1/\bar{q}'} < \infty,$$

(b)

$$\sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \left\| \chi_{[2^{n+\alpha},2^{n+1})(\cdot)} v(\cdot) \right\|_{L^{p(\cdot)}} \left\| w(\cdot) \chi_{(2^n,2^{n+\alpha})(\cdot)} \right\|_{L^{\bar{p}'(\cdot)}} < \infty;$$

(ii) *in the case* $1 < q < \bar{q} < \infty$,

(a) $\{C_n\} \in l^s$, *where*

$$C_n = \sup_{\beta\in(0,1)} \left\| \chi_{[2^{n+\beta},2^{n+1})} v(\cdot) \right\|_{L^{p(\cdot)}} \left\| w(\cdot) \chi_{[2^n,2^{n+\beta})} \right\|_{L^{\bar{p}(\cdot)}},$$

(b)

$$\left\{\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}^{\infty}\|\chi_{[2^k,2^{k+1})}v(\cdot)\|_{L^{p(\cdot)}}^{q}\right)^{s/q}\left(\sum_{k=-\infty}^{n}\|\chi_{[2^{k-1},2^k)}(\cdot)w(\cdot)\|_{L^{\bar{p}'(\cdot)}}^{1-\bar{q}'}\right)^{s/\bar{q}'}\right.$$
$$\left.\times\|\chi_{[2^n,2^{n+1})}v(\cdot)\|_{L^{p(\cdot)}}^{q}\right\}^{1/s}<\infty,$$

where $\frac{1}{s}=\frac{1}{\bar{q}}-\frac{1}{q}$.

Proof Let $1<\bar{q}\le q<\infty$. Suppose that $f\ge 0$. We represent:

$$\begin{aligned}(H_{v,w}f)(x) &= v(x)\int_0^{2^n}f(t)w(t)\,dt+v(x)\int_{2^n}^{x}f(t)w(t)\,dt\\ &=: \bigl(H_{v,w}^{(1)}f\bigr)(x)+\bigl(H_{v,w}^{(2)}f\bigr)(x),\quad x\in\bigl[2^n,2^{n+1}\bigr].\end{aligned}\tag{3.4}$$

We have

$$\begin{aligned}\|(H_{v,w}f)\chi_{[2^n,2^{n+1})}(\cdot)\|_{L^{p(\cdot)}} &\le \|v(\cdot)\chi_{[2^n,2^{n+1})(\cdot)}\|_{L^{p(\cdot)}}\left(\int_0^{2^n}f(t)w(t)\,dt\right)\\ &\quad+\left\|v(x)\int_{2^n}^{x}f(t)w(t)\,dt\right\|_{L^{p(\cdot)}([2^n,2^{n+1}))}\\ &=: S_1^{(n)}+S_2^{(n)}.\end{aligned}$$

Let $a_k:=\int_{2^{k-1}}^{2^k}fw$. Then by the discrete Hardy inequality (see Lemma C) and Hölder's inequality with respect to the exponents $\bar{p}(\cdot)$ and $(\bar{p}(\cdot))'$ we derive

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}\bigl(S_1^{(n)}\bigr)^q\right)^{1/q} &= \left[\sum_{n\in\mathbb{Z}}\|v(\cdot)\chi_{[2^n,2^{n+1})}(\cdot)\|_{L^{p(\cdot)}}^{q}\left(\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k}f(t)w(t)\,dt\right)^{q}\right]^{1/q}\\ &\le c\left[\sum_{n\in\mathbb{Z}}\left(\int_{2^{n-1}}^{2^n}f(t)w(t)\,dt\right)^{\bar{q}}\|w(\cdot)\chi_{[2^{n-1},2^n)}(\cdot)\|_{L^{\bar{p}'(\cdot)}}^{-\bar{q}}\right]^{1/q}\\ &\le c\left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^{n-1},2^n)}(\cdot)f(\cdot)\|_{L^{\bar{p}(\cdot)}}^{\bar{q}}\right]^{1/\bar{q}}=c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}.\end{aligned}$$

Further, by Corollary A and Theorem E, we have that

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}\bigl(S_2^{(n)}\bigr)^q\right)^{1/q} &= \left[\sum_{n\in\mathbb{Z}}\left\|v(x)\int_{2^n}^{x}f(t)w(t)\,dt\right\|_{L^{p(\cdot)}(2^n,2^{n+1})}^{q}\right]^{1/q}\\ &\le c\left[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}^{q}\right]^{1/q}\\ &\le c\left[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}^{\bar{q}}\right]^{1/\bar{q}}\\ &= c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}.\end{aligned}$$

Let $1 < q < \bar{q} < \infty$. Using representation (3.4), we derive

$$\begin{aligned}\left\|(H_{v,w}f)\right\|_{(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d} &\le \left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^n,2^{n+1})}H^{(1)}_{v,w}f\right\|^q_{L^{p(\cdot)}}\right]^{1/q}\\ &\quad+\left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^n,2^{n+1})}H^{(2)}_{v,w}f\right\|^q_{L^{p(\cdot)}}\right]^{1/q}\\ &=: S_1+S_2.\end{aligned}$$

We estimate S_1 and S_2.

$$\begin{aligned}S_1 &= \left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\right\|^q_{L^{p(\cdot)}}\left(\int_0^{2^n}fw\right)^q\right]^{1/q}\\ &= \left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\right\|^q_{L^{p(\cdot)}}\left(\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k}fw\right)^q\right]^{1/q}.\end{aligned}$$

By the two-weight inequality for the discrete Hardy transform (see Lemma C), we have

$$\begin{aligned}S_1 &\le c\left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\right\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\left(\int_{2^{n-1}}^{2^n}fw\right)^{\bar{q}}\right]^{1/\bar{q}}\\ &\le c\left[\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\right\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\left\|\chi_{[2^{n-1},2^n)}f\right\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\left\|\chi_{[2^{n-1},2^n)}w\right\|^{\bar{q}}_{L^{\bar{p}'(\cdot)}}\right]^{1/\bar{q}}\\ &\le c\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.\end{aligned}$$

Now we estimate S_2. Using Corollary A for intervals $(2^n,2^{n+1}]$ and Hölder's inequality, we find that

$$\begin{aligned}S_2 &\le c\left\{\sum_{n\in\mathbb{Z}}C_n^q\left\|\chi_{[2^n,2^{n+1})}f\right\|^q_{L^{\bar{p}(\cdot)}}\right\}^{1/q}\\ &\le c\left\{\left(\sum_{n\in\mathbb{Z}}\left\|\chi_{[2^n,2^{n+1})}f\right\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right)^{q/\bar{q}}\left(\sum_{n\in\mathbb{Z}}C_n^{\frac{q\bar{q}}{\bar{q}-q}}\right)^{\frac{\bar{q}-q}{q}}\right\}^{1/q}\\ &\le c\left(\sum_{n\in\mathbb{Z}}C_n^s\right)^{1/s}\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.\end{aligned}$$

□

3.3 Kernel operators on amalgams $(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d$ and $(L^{p(\cdot)}(\mathbb{R}),l^q)$

The conditions of general-type statements (see Propositions 3.1 and 3.2) are not easily verifiable for general kernel operators as well as for some concrete fractional integral operators such as the Riemann-Liouville fractional integral transform with variable parameter. That is why we investigate mapping properties of general kernel operators independently from general-type statements.

Let

$$(K_vf)(x) = v(x)\int_0^x f(t)k(x,t)\,dt, \quad x>0.$$

One of our aims is to characterize a class of weights v governing the boundedness of K_v from $(L^{\bar{p}(\cdot)},l^{\bar{q}})_d$ to $(L^{p(\cdot)},l^q)_d$.

We will use the notation:

$$B_1 := \sup_{m\in\mathbb{Z}}\left[\sum_{n=m}^{\infty}\left\|\chi_{(2^n,2^{n+1}]}(x)k\left(x,\frac{x}{2}\right)v(x)\right\|_{L^{p(\cdot)}}^{q}\right]^{1/q}\left[\sum_{n=-\infty}^{m}\|\chi_{(2^{n-1},2^n]}\|_{L^{\bar{p}'(\cdot)}}^{\bar{q}'}\right]^{1/\bar{q}'}; \tag{3.5}$$

$$B_2 := \sup_{n\in\mathbb{Z}}\sup_{0<\alpha<1}\left\|\chi_{(2^{n+\alpha},2^{n+1}]}k(x,x/2)v(x)\right\|_{L^{p(\cdot)}}\|\chi_{(2^n,2^{n+\alpha}]}\|_{L^{\bar{p}'(\cdot)}}. \tag{3.6}$$

Theorem 3.1 *Let $I := \mathbb{R}_+$, $1 < \bar{p}_-(I) \le \bar{p}(\cdot) \le p(\cdot) \le p_+(I) < \infty$ and let $\bar{p}, p \in \mathrm{WL}(I)$. Suppose that $\bar{q}$ and q are constants such that $1 < \bar{q} \le q < \infty$. Let $p(x) \equiv p_c \equiv \mathrm{const}$ and $\bar{p}(x) \equiv \bar{p}_c \equiv \mathrm{const}$ outside some large interval $(0, 2^{m_0})$, $m_0 \in \mathbb{Z}$. Let $k \in V(I) \cap V_{\bar{p}'(\cdot)}(I)$. Then K_v is bounded from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ if and only if $B < \infty$, where $B = \max\{B_1, B_2\}$.*

Proof Sufficiency. Using the representation:

$$\begin{aligned}(K_v f)(x) &= v(x)\int_0^{x/2} k(x,t)f(t)\,dt + v(x)\int_{x/2}^{x} k(x,t)f(t)\,dt\\ &=: \left(K_v^{(1)}f\right)(x) + \left(K_v^{(2)}f\right)(x)\end{aligned}$$

we have that

$$\|K_v f\|_{(L^{p(\cdot)},l^q)_d} \le \left\|K_v^{(1)}f\right\|_{(L^{p(\cdot)},l^q)_d} + \left\|K_v^{(2)}f\right\|_{(L^{p(\cdot)},l^q)_d}.$$

Further, taking Proposition 3.3 and the condition $k \in V(I)$ into account, we find that

$$\begin{aligned}\left\|K_v^{(1)}f\right\|_{(L^{p(\cdot)}(I),l^q)_d} &\le c\left\|v(x)k\left(x,\frac{x}{2}\right)\int_0^x f(t)\,dt\right\|_{(L^{p(\cdot)},l^q)_d}\\ &\le cB\|f\|_{(L^{\bar{p}(\cdot)}(I),l^q)_d}.\end{aligned}$$

Now observe that by the condition $k \in V_{\bar{p}'(\cdot)}(I)$, Proposition A and Lemma A we obtain

$$\begin{aligned}&\left\|K_v^{(2)}f\right\|_{(L^{p(\cdot)}([0,2^{m_0}+1)),l^q)_d}\\ &\quad\le \left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)\left(\int_{x/2}^{x} f(t)k(x,t)\,dt\right)\right\|_{L^{p(x)}}^{q}\right]^{1/q}\\ &\quad\le \left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)\left\|\chi_{(x/2,x)}(\cdot)f(\cdot)\right\|_{L^{\bar{p}(\cdot)}}\left\|\chi_{(x/2,x)}k(x,\cdot)\right\|_{L^{\bar{p}'(\cdot)}}\right\|_{L^{p(x)}}^{q}\right]^{1/q}\\ &\quad\le \left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)x^{\frac{1}{\bar{p}'(x)}}k(x,x/2)\right\|_{L^{p(x)}}^{q}\left\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\right\|_{L^{\bar{p}(\cdot)}}^{q}\right]^{1/q}\\ &\quad\le c\left[\sum_{k=-\infty}^{+\infty}2^{kq/(\bar{p})'(2^k)}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)k(x,x/2)\right\|_{L^{p(x)}}^{q}\left\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\right\|_{L^{\bar{p}(\cdot)}}^{q}\right]^{1/q}\\ &\quad\le c\bar{B}_1\left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^{k-1},2^k)}(\cdot)f(\cdot)\right\|_{L^{\bar{p}(\cdot)}}^{q}\right]^{1/q} + c\bar{B}_1\left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1})}(\cdot)f(\cdot)\right\|_{L^{\bar{p}(\cdot)}}^{q}\right]^{1/q}\\ &\quad\le c\bar{B}_1\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d},\end{aligned}$$

where

$$\bar{B}_1 := \sup_{n\in\mathbb{Z}} \left\| \chi_{(2^n,2^{n+1}]}(x) k\left(x, \frac{x}{2}\right) v(x) \right\|_{L^{p(x)}} 2^{1/(\bar{p}_n)'},$$

$$\bar{p}_n := \begin{cases} \bar{p}(2^n), & n \le m_0, \\ \bar{p}_c, & n > m_0. \end{cases}$$

Let us now observe that by Proposition A and Lemma A, $\bar{B}_1 \approx \bar{A} \le cB_1$, where

$$\bar{A} := \sup_{k\in\mathbb{Z}} \left\| v(\cdot)k(x,x/2)\chi_{(2^k,2^{k+1}]} \right\|_{L^{p(\cdot)}} \left\| \chi_{(2^{k-1},2^k]}(\cdot) \right\|_{L^{\bar{p}'(\cdot)}}. \tag{3.7}$$

Necessity. Let $\bar{p}_n$ be the sequence defined above. Considering the test function $f_n = \chi_{(2^n,2^{n+1}]} 2^{-n/\bar{p}_n}$ in the boundedness of K_v from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ and taking the condition $k \in V(I)$ into account we have that

$$I_n := \left\| \chi_{(2^n,2^{n+1}]}(x) v(x) k(x,x/2) \right\|_{L^{p(x)}} \le c2^{-n/(\bar{p}_n)'}. \tag{3.8}$$

It is easy to see that

(i)

$$\sum_{n=m}^{\infty} I_n \le c\left(2^{-m/\bar{p}'(0)} + 2^{-m_0/\bar{p}'_c}\right) \tag{3.9}$$

for $m \le m_0$;

(ii)

$$\sum_{n=m}^{\infty} I_n \le c2^{-m_0/\bar{p}'_c} \tag{3.10}$$

for $m \ge m_0 + 1$.

Denoting $S_m := [\sum_{n=m}^{\infty} I_n^q]^{1/q} [\sum_{n=-\infty}^{m-1} \| \chi_{(2^n,2^{n+1}]} \|_{L^{\bar{p}'(\cdot)}}^{\bar{q}}]^{1/\bar{q}}$ and taking (3.9), Proposition A and Lemma A into account we have for $m \le m_0$,

$$\begin{aligned} S_m &\le \left[\sum_{n=m}^{\infty} I_n^q \right]^{1/q} 2^{m/\bar{p}'(0)} \le \left[2^{-m/\bar{p}'(0)} + 2^{-m_0/\bar{p}'_c}\right] 2^{m/\bar{p}'(0)} \\ &\le 1 + 2^{m/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} \le 1 + 2^{m_0/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} < \infty. \end{aligned}$$

Similarly if $m \ge m_0 + 1$, then by (3.10),

$$\begin{aligned} S_m &\le \left[\sum_{n=m}^{\infty} I_n^q \right]^{1/q} \left[2^{m_0/\bar{p}'(0)} + 2^{m/\bar{p}'_c}\right] \le 2^{-m/\bar{p}'_c} \left[2^{m_0/\bar{p}'(0)} + 2^{m/\bar{p}'_c}\right] \\ &\le 1 + 2^{m_0/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} < \infty. \end{aligned}$$

Hence, $B_1 < \infty$.

Let now f be a function supported in $(2^m, 2^{m+1}]$. Then due to the boundedness of K_v from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ and the condition $k \in V(I)$ we have that

$$\left\| \chi_{(2^m,2^{m+1}]} v(x) k(x, x/2) \left(\int_{2^m}^{x} f(y)\, dy \right) \right\|_{(L^{p(\cdot)}(I), l^q)_d} \leq c \| \chi_{(2^m,2^{m+1}]} f \|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d},$$

where the positive constant c does not depend on n. Using Theorem B with respect to the intervals $[2^m, 2^{m+1})$ and the weight pair $(\bar{v}, w)$, where $\bar{v}(x) = v(x)k(x, x/2)\ \chi_{(2^m,2^{m+1}]}$ and $\bar{w} \equiv \text{const}$, it follows that $B_2 < \infty$. □

Remark 3.1 We have noticed in the proof of Theorem 3.1 that $B_1 \approx \bar{A}$, where $\bar{A}$ is defined in the same proof.

Now we formulate the boundedness criteria for the kernel operator

$$(\mathcal{K}_v f) = v(x) \int_{-\infty}^{x} k(x, t) f(t)\, dt, \quad x \in \mathbb{R},$$

on amalgams defined on $\mathbb{R}$.

Let $k(x, y)$ be a kernel on $\{(x, y) : y < x\}$ and $v, p, \bar{p}$ be defined on $\mathbb{R}$. For the next statement we define $\tilde{k}$, $\tilde{v}$, p_0 and $\bar{p}_0$ as follows:

$$\begin{aligned}
&\tilde{k}(x, t) := \left(\frac{t^{-1/\bar{p}'(\log_2 t)}}{x^{1/p(\log_2 x)}} \right) k(\log_2 x, \log_2 t), \\
&\tilde{v}(x) := v(\log_2 x), \\
&\bar{p}_0(x) := \bar{p}(\log_2 x), \qquad p_0(x) := p(\log_2 x).
\end{aligned}$$

Theorem 3.2 *Let* $1 < \bar{p}_-(\mathbb{R}) \leq \bar{p}(x) \leq p(x) \leq p_+(\mathbb{R}) < \infty$ *and let* $\bar{p}_0, p_0 \in \mathrm{WL}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *are constants such that* $1 < \bar{q} \leq q < \infty$. *Assume that* $\bar{p}(x) \equiv \bar{p}_c \equiv \text{const}$ *and* $p(x) \equiv p_c \equiv \text{const}$ *outside some large interval* $(-\infty, b)$. *Let* $\tilde{k} \in V(\mathbb{R}_+) \cap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$. *Then* $\mathcal{K}_v$ *is bounded from* $(L^{\bar{p}(\cdot)}(\mathbb{R}), l^{\bar{q}})$ *to* $(L^{p(\cdot)}(\mathbb{R}), l^q)$ *if and only if*

$$\begin{aligned}
D_1 &:= \sup_{m \in \mathbb{Z}} \left[\sum_{n=m}^{\infty} \left\| \chi_{[2^n,2^{n+1})}(x) \tilde{k}\left(x, \frac{x}{2}\right) \tilde{v}(x) \right\|_{L^{p_0(\cdot)}(\mathbb{R}_+)}^{q} \right]^{1/q} \\
&\quad \times \left[\sum_{n=-\infty}^{m} \| \chi_{[2^{n-1},2^n)} \|_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)}^{\bar{q}'} \right]^{1/\bar{q}'} < \infty, \\
D_2 &:= \sup_{n \in \mathbb{Z}} \sup_{0 < \alpha < 1} \left\| \chi_{[2^{n+\alpha},2^{n+1})} \tilde{k}\left(x, \frac{x}{2}\right) \tilde{v}(x) \right\|_{L^{p_0(\cdot)}(\mathbb{R}_+)} \| \chi_{[2^n,2^{n+\alpha})} \|_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)} < \infty.
\end{aligned}$$

Proof The proof follows from Theorem 3.1 by the change of variable $z \to \log_2 t$. □

Let

$$(\mathcal{R}_{\alpha(\cdot)} f)(x) = v(x) \int_{-\infty}^{x} \frac{2^t f(t)}{(x - t)^{1-\alpha(x)}}\, dt,$$

where $0 < \inf \alpha \leq \sup \alpha < 1$ and $x \in \mathbb{R}_+$.

By virtue of Theorem 3.2 and Example 2.1 we can easily deduce the next statement.

Corollary 3.1 *Let p, $\bar{p}$, q and $\bar{q}$ be constants. Suppose that α is a measurable function on $\mathbb{R}$ and that $1 < \bar{p} \le p < \infty$, $1 < \bar{q} \le q < \infty$, $\frac{1}{\bar{p}} < \alpha(x) < 1$. Then the operator $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $(L^{\bar{p}}, l^{\bar{q}})$ to (L^p, l^q) if and only if*

$$\tilde{D}_1 = \sup_{m\in\mathbb{Z}} \left[\sum_{n=m}^{\infty} \left(\int_n^{n+1} (2^u)^{\frac{p}{\bar{p}}} v^p(u)\,du \right)^{q/p} \right]^{1/q} 2^{m/\bar{p}'} < \infty,$$

$$\tilde{D}_2 = \sup_{n\in\mathbb{Z}} \sup_{0<\beta<1} \left(\int_{n+\beta}^{n+1} (2^u)^{\frac{p}{\bar{p}}} v^p(u)\,du \right)^{1/p} \bigl(2^n(2^\beta - 1)\bigr)^{1/\bar{p}'} < \infty.$$

Moreover, there are positive constants c_1 and c_2 depending on p, $\bar{p}$, q, $\bar{q}$ and α such that $c_1 \max\{\tilde{D}_1, \tilde{D}_2\} \le \|\mathcal{R}_{\alpha(\cdot)}\| \le c_2 \max\{\tilde{D}_1, \tilde{D}_2\}$.

4 Compactness of kernel operators on VEAS

In this section, we derive compactness necessary and sufficient conditions for kernel operators on VEAS. Since for the amalgam norm we have the property $\|f_n\|_{(L^{p(\cdot)}(I),l^q)_\alpha} \downarrow 0$ when $f_n \downarrow 0$ a.e. ($f_n \in (L^{p(\cdot)}(I), l^q)_\alpha$), therefore, the following statement holds (see [46], Chap. XI).

Proposition 4.1 *Let p, $\bar{p}$ be measurable functions on I such that $1 < \bar{p}, p < \infty$. Let q, $\bar{q}$ be constants satisfying the condition $1 < q, \bar{q} < \infty$. Then the set of all functions of the form*

$$k_n(s,t) \equiv \sum_{i=1}^{n} \eta_i(s)\lambda_i(t), \quad s,t \in I,$$

is dense in the mixed norm space $(L^{p(\cdot)}(I), l^q)_\alpha [(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha]$, where $\lambda_i \equiv \chi_{B_i}$, $\chi_{B_i} \in (L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$ (B_i are measurable disjoint sets of I) and $\eta_i \in (L^{p(\cdot)}(I), l^q)_\alpha \cap L^\infty(I)$.

The next statement gives sufficient condition for the kernel operator to be compact on amalgams defined on $\mathbb{R}_+$.

Proposition 4.2 *Let $p(x)$ and $q(x)$ be measurable functions on an interval $I \subseteq \mathbb{R}_+$. Suppose that $1 < p_-(I) \le p_+(I) < \infty$, $1 < \bar{p}_-(I) \le \bar{p}_+(I) < \infty$. Let q, $\bar{q}$ be constants such that $1 < \bar{q}$, $q < \infty$. If*

$$M := \bigl\| \|k(x,y)\|_{(L^{(\bar{p}(y))'}(I), l^{(\bar{q})'})_\alpha} \bigr\|_{(L^{p(x)}(I), l^q)_\alpha} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int_I k(x,y) f(y)\,dy$$

is compact from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$ to $(L^{p(\cdot)}(I), l^q)_\alpha$.

Proof By Proposition 4.1 the set of functions

$$k_m(s,t) = \sum_{i=1}^{m} \eta_i(s)\lambda_i(t), \quad s,t \in I,$$

is dense in $(L^{p(\cdot)}(I), l^q)_\alpha[(L^{\bar{p}'(\cdot)}(I), l^{\bar{q}'})_\alpha]$. By Hölder's inequality for amalgam spaces (see Theorem D), we have

$$\left|Kf(x)\right| = \left|\int_I k(x,y)f(y)\,dy\right| \le \|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha} \left\|k(x,y)\right\|_{(L^{(\bar{p}(\cdot))'}(I),l^{(\bar{q})'})_\alpha}.$$

Hence,

$$\|Kf\|_{(L^{p(\cdot)}(I),l^q)_\alpha} \le \left\| \left\|k(x,y)\right\|_{(L^{(\bar{p}(y))'}(I),l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I),l^q)_\alpha} \|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha} \le M\|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha}.$$

This means that $\|K\| \le M$.

Now we prove the compactness of K. For each $n \in \mathbb{N}$, let

$$(K_n\phi)(x) = \int_I k_n(x,y)\phi(y)\,dy.$$

Note that

$$(K_n\phi)(x) = \int_I k_n(x,y)\phi(y)\,dy = \sum_{i=1}^n \eta_i(x)\int_I \lambda_i(y)\phi(y)\,dy =: \sum_{i=1}^n \eta_i(x)b_i,$$

where

$$b_i = \int_I \lambda_i(y)\phi(y)\,dy.$$

This means that K_n is a finite rank operator, *i.e.*, it is compact. Further, let $\epsilon > 0$. Using the above-mentioned arguments, we have that there is $N_0 \in \mathbb{N}$ such that for $n > N_0$,

$$\|K - K_n\| \le \left\| \left\|k(x,y) - k_n(x,y)\right\|_{(L^{(\bar{p}(y))'}(I),l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I),l^q)_\alpha} < \epsilon.$$

Thus K can be represented as a limit of finite rank operators. Hence, K is compact. □

Theorem 4.1 *Let* $1 < \bar{p}_-(\mathbb{R}_+) \le \bar{p}(x) \le p(x) \le p_+(\mathbb{R}_+) < \infty$ *and let* $\bar{p}, p \in \mathrm{WL}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \le q < \infty$. *Assume that* $k \in V(\mathbb{R}_+) \cap V_{(\bar{p}(\cdot))'}(\mathbb{R}_+)$. *Suppose that* $\bar{p}(x) \equiv \bar{p}_c \equiv \mathrm{const}$ *and* $p(x) \equiv p_c \equiv \mathrm{const}$ *outside some large interval* $(0, 2^{m_0})$. *Then* K_v *is compact from* $(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d$ *to* $(L^{p(\cdot)}, l^q)_d$ *if and only if*

(i) $B_1 < \infty$; $\quad B_2 < \infty$,

(ii) $\lim_{m\to-\infty} B_1(m) = \lim_{m\to+\infty} \mathbb{B}_1(m) = 0$,

(iii) $\lim_{n\to-\infty} B_2(n) = \lim_{n\to+\infty} B_2(n) = 0$,

where, B_1 *and* B_2 *are defined by* (3.5) *and* (3.6), *respectively, and*

$$B_1(m) := \left\|\chi_{[2^m,2^{m+1})}k(x,x/2)v(x)\right\|_{L^{p(\cdot)}} 2^{m/\bar{p}'(0)};$$

$$\mathbb{B}_1(m) := \left[\sum_{n=m}^{\infty} \left\|\chi_{[2^n,2^{n+1})}k(x,x/2)v(x)\right\|_{L^{p(\cdot)}}^q\right]^{1/q} \left[\sum_{n=-\infty}^{m} \left\|\chi_{[2^{n-1},2^n)}(\cdot)\right\|_{L^{(\bar{p}(\cdot))'}}^{(\bar{q})'}\right]^{1/(\bar{q})'},$$

$$B_2(n) := \sup_{0<\alpha<1} \left\|\chi_{[2^{n+\alpha},2^{n+1})}(x)v(x)k(x,x/2)\right\|_{L^{p(\cdot)}} \left\|\chi_{(2^n,2^{n+\alpha})}(\cdot)\right\|_{L^{(\bar{p}(\cdot))'}}.$$

Proof Sufficiency. Let k_0, n_0 be integers such that $k_0 < m_0 < n_0$. Then we represent K_ν as follows:

$$\begin{aligned}(K_\nu f)(x) &= \chi_{[0,2^{k_0}]}(x)K_\nu(f\chi_{[0,2^{k_0})})(x) + \chi_{(2^{k_0},2^{n_0})}(x)K_\nu(f\chi_{[0,2^{n_0})})(x)\\ &\quad + \chi_{[2^{n_0},\infty)}(x)K_\nu(f\chi_{[0,2^{n_0-1})})(x) + \chi_{[2^{n_0},\infty)}(x)K_\nu(f\chi_{(2^{n_0-1},\infty)})(x)\\ &=: \big(K_\nu^{(1)}f\big)(x) + \big(K_\nu^{(2)}f\big)(x) + \big(K_\nu^{(3)}f\big)(x) + \big(K_\nu^{(4)}f\big)(x).\end{aligned}$$

It is clear that

$$\big(K_\nu^{(2)}f\big)(x) = \int_{\mathbb{R}_+} k_2(x,y)f(y)\,dy,$$

where $k_2(x,y) = \nu(x)\chi_{(2^{k_0},2^{n_0})}(x)k(x,y)$ if $y < x$ and $k_2(x,y) = 0$ if $y \geq x$. Then

$$\begin{aligned}&\big\|\|k_2(x,y)\|_{(L^{(\bar{p})'(y)}(I),l^{(\bar{q})'})_d}\big\|_{(L^{p(x)}([2^{k_0},2^{m_0})),l^q)_d}\\ &\quad = \left\{\sum_{m=k_0}^{n_0-1}\left\|\chi_{(2^m,2^{m+1})}(x)\nu(x)\left(\sum_{n=-\infty}^{m}\|\chi_{(2^n,2^{n+1})}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\right)^{1/(\bar{q})'}\right\|_{L^{p(x)}}^q\right\}^{1/q} =: J(x).\end{aligned}$$

Denoting $I(x) := \sum_{n=-\infty}^{m}\|\chi_{(2^n,2^{n+1})}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}$, $x \in [2^m, 2^{m+1})$, $k_0 \leq m \leq n_0 - 1$, we represent $I(x)$ as

$$\begin{aligned}I(x) &= \sum_{n=-\infty}^{m-2}\|\chi_{(2^n,2^{n+1})}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\\ &\quad + \|\chi_{(2^{m-1},2^m)}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} + \|\chi_{(2^m,x)}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\\ &=: I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

Now we estimate $I_1(x)$, $I_2(x)$ and $I_3(x)$ separately

$$\begin{aligned}I_1(x) &\leq ck^{(\bar{q})'}\left(x,\frac{x}{2}\right)\sum_{n=-\infty}^{m-2}\|\chi_{[2^n,2^n+1)}(y)\|_{L^{(\bar{p})'(\cdot)}}^{(\bar{q})'}\\ &\leq ck^{(\bar{q})'}\left(x,\frac{x}{2}\right)\left[\sum_{n=-\infty}^{m_0}\|\chi_{[2^n,2^n+1)}(\cdot)\|_{L^{(\bar{p})'(\cdot)}}^{(\bar{q})'} + \sum_{n=m_0+1}^{m-2}\|\chi_{[2^n,2^n+1)}(y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\right]\\ &\leq ck^{\bar{q}'}\left(x,\frac{x}{2}\right)\left[\sum_{n=-\infty}^{m_0}\big(2^n\big)^{(\bar{q})'/(\bar{p})'(0)} + \sum_{m_0+1}^{n_0}\big(2^n\big)^{(\bar{q})'/(\bar{p})'_c}\right]\\ &\leq ck^{(\bar{q})'}\left(x,\frac{x}{2}\right)\big[\big(2^{m_0}\big)^{(\bar{q})'/(\bar{p})'(0)} + \big(2^{n_0}\big)^{(\bar{q})'/(\bar{p})'_c}\big].\end{aligned}$$

Further,

$$\begin{aligned}I_2(x) + I_3(x) &\leq 2\|\chi_{(0,x)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\\ &\leq c\|\chi_{(0,x/2)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} + c\|\chi_{(x/2,x)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\\ &\leq k^{(\bar{q})'}\left(x,\frac{x}{2}\right)\big[\|\chi_{(0,2^m)}(y)\|_{L^{\bar{p}(y)}}^{(\bar{q})'} + x^{(\bar{q})'/(\bar{p})'(x)}\big].\end{aligned}$$

Considering separately the cases $m \le m_0$ and $m > m_0$, by using Proposition A and Lemma A we find that

$$I_2(x) + I_3(x) \le ck^{(\bar{q})'}\left(x, \frac{x}{2}\right)\left[\left(2^m\right)^{(\bar{q})'/(\bar{p})'(0)} + \left(2^m\right)^{(\bar{q})'/(\bar{p})'_c}\right].$$

Consequently, since $k_0 \le m < n_0 - 1$, we have

$$I(x) \le ck^{(\bar{q})'}\left(x, \frac{x}{2}\right)\left[\left(2^{n_0}\right)^{(\bar{q})'/(\bar{p})'(0)} + \left(2^{n_0}\right)^{(\bar{q})'/(\bar{p})'_c}\right] =: ck^{(\bar{q})'}\left(x, \frac{x}{2}\right)B_{n_0}.$$

Since $B_1 < \infty$ we find that

$$J(x) \le B_{n_0}^{1/(\bar{q})'}\left[\sum_{m=k_0}^{n_0-1} \left\| \chi_{[2^n,2^{n+1})} k(x,x/2)v(x)\right\|_{L^{p(\cdot)}}^q\right]^{1/q} < \infty.$$

So, by Proposition 4.2, we conclude that $K_v^{(2)}$ is a compact operator. Further, write $K_v^{(3)}$ as follows:

$$K_v^{(3)} f(x) = \int_{\mathbb{R}_+} k_3(x,y) f(y)\,dy,$$

where $k_3(x,y) = k(x,y)\chi_{(0,2^{n_0-1})}(y)\chi_{[2^{n_0},\infty)}(x)v(x)$ if $y < x$ and $k_3(x,y) = 0$ if $y \ge x$. Then we have

$$\begin{aligned}
&\left\| \left\| k_3(x,y)\right\|_{(L^{(\bar{p})'(y)}(I),l^{(\bar{q})'})_d}\right\|_{(L^{p(x)}(I),l^q)_d} \\
&\quad = \left\{\sum_{m=n_0}^{\infty}\left\| \chi_{(2^m,2^{m+1})}(x)v(x)\left(\sum_{n=-\infty}^{n_0-2}\left\|\chi_{(2^n,2^{n+1})}(y)k(x,y)\right\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\right)^{1/(\bar{q})'}\right\|_{L^{p(x)}}^q\right\}^{1/q} \\
&\quad \le \left\{\sum_{m=n_0}^{\infty}\left\|\chi_{(2^m,2^{m+1})}(x)v(x)k(x,x/2)\right\|_{L^{p(x)}}^q\right\}^{1/q}\left(\sum_{n=-\infty}^{n_0-2}\left\|\chi_{(2^n,2^{n+1})}(y)\right\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'}\right)^{1/(\bar{q})'} \\
&\quad =: G.
\end{aligned}$$

Denoting $F := (\sum_{n=-\infty}^{n_0-1}\|\chi_{(2^n,2^{n+1})}(y)\|_{L^{\bar{p}(\cdot)}}^{(\bar{q})'})^{1/(\bar{q})'}$ and considering both cases when $m_0 \le n_0 - 2$ and $m_0 > n_0 - 2$ separately, we derive as previously that

$$F \le c\left[\left(2^{m_0}\right)^{(\bar{q})'/(\bar{p})'(0)} + \left(2^{n_0}\right)^{(\bar{q})'/(\bar{p})'_c}\right]^{1/(\bar{q})'} =: B_{n_0,m_0},$$

and since $B_1 < \infty$ we have

$$G \le B_{n_0,m_0}\left[\sum_{m=n_0}^{\infty}\left\|\chi_{[2^n,2^{n+1})}(x)k(x,x/2)v(x)\right\|_{L^{p(x)}}^q\right]^{1/q} < \infty.$$

Hence, by Proposition 4.2, $K_v^{(3)}$ is compact.

Let us denote

$$I_m := \left\|\chi_{[2^m,2^{m+1})}(x)k(x,x/2)v(x)\right\|_{L^{p(\cdot)}}. \tag{4.1}$$

Following the proofs of Theorems 3.1, 3.2 and applying Proposition A and Lemma A, we have that

$$\begin{aligned}
&\big\|K_\nu^{(1)}\big\|_{(L^{\bar p(\cdot)}(I),l^{\bar q})\to(L^{p(\cdot)}(I),l^q)_d} \\
&\quad\le \max\left\{\sup_{n\le k_0}\left[\sum_{m=n}^{k_0} I_m^q\right]^{1/q}\left[\sum_{m=-\infty}^{n}\big\|\chi_{[2^{m-1},2^m)}(\cdot)\big\|_{L^{(\bar p(\cdot))'}}^{(\bar q)'}\right]^{1/(\bar q)'}, \sup_{m\le k_0} B_2(m)\right\} \\
&\quad\le c\max\left\{\left[\sup_{m\le k_0} I^m 2^{m/\bar p'(0)}\right]\sup_{n\le k_0}\left[\sum_{m=n}^{\infty}2^{-m/\bar p'(0)}\right]\left[\sum_{m=-\infty}^{n}2^{m/\bar p'(0)}\right], \sup_{m<k_0} B_2(m)\right\} \\
&\quad\le c\max\left\{\sup_{m\le k_0} I^m 2^{m/\bar p'(0)}, \sup_{m<k_0} B_2(m)\right\} \\
&\quad\to 0
\end{aligned}$$

as $k_0 \to 0$ because $\lim_{m\to-\infty} B_1(m) = \lim_{m\to-\infty} B_2(m) = 0$. Further, applying Theorem 3.1, we find that

$$\big\|K_\nu^{(4)}\big\|_{(L^{\bar p(\cdot)}(I),l^{\bar q})\to(L^{p(\cdot)}(I),l^q)} \le \max\left\{\sup_{m\ge n_0}\mathbb{B}_1(m), \sup_{m\ge n_0} B_2(m)\right\} \to 0$$

as $n_0 \to +\infty$.

Hence,

$$\big\|K_\nu f - K_\nu^{(2)} f - K_\nu^{(3)} f\big\| \le \big\|K_\nu^{(1)} f\big\| + \big\|K_\nu^{(4)} f\big\| \to 0$$

as $\mathbb{B}_1(m) \to 0$, $B_i(m) \to 0$, $i = 1, 2$. Hence K_ν is compact, since it is the limit of compact operators.

Necessity. First we show that $\lim_{m\to-\infty} B_1(m) = 0$. Let $f_n = \chi_{(2^{n-1},2^{n+1})} 2^{-n/\bar p_n}$, where $\bar p_n$ is defined in the proof of Theorem 3.1. Then $f_n \to 0$ weakly in $(L^{\bar p(\cdot)}(I), l^{\bar q})_d$ as $n \to -\infty$. Indeed, let $\phi \in (L^{(\bar p(\cdot))'}(I), l^{(\bar q)'})_d$. Then

$$\begin{aligned}
\left|\int_0^\infty f_n(y)\phi(y)\,dy\right| &\le \left(\|\chi_{(2^{n-1},2^n]}\|_{L^{\bar p(\cdot)}}^{\bar q} + \|\chi_{(2^n,2^{n+1}]}\|_{L^{\bar p(\cdot)}}^{\bar q}\right)^{1/\bar q} 2^{-n/\bar p_c} \\
&\quad\times \left(\|\phi\chi_{(2^{n-1},2^n]}\|_{L^{(\bar p(\cdot))'}}^{\bar q} + \|\phi\chi_{(2^{n-1},2^n]}\|_{L^{(\bar p(\cdot))'}}^{\bar q}\right)^{1/\bar q} \\
&\to 0
\end{aligned}$$

as $n \to -\infty$.

Observe now that

$$\|K_\nu f_n\|_{(L^{\bar p(\cdot)}(I),l^{\bar q})_d} \ge \big\|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)\big\|_{L^{p(\cdot)}} 2^{n/\bar p_n'}, \quad n \in \mathbb{Z}. \tag{4.2}$$

Hence, $\lim_{n\to-\infty} B_1(n) \to 0$ because K_ν is compact and $\bar p_n = \bar p(0)$ if $n < m_0$.

Further, (4.2) implies that

$$\big\|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)\big\|_{L^{p(\cdot)}} 2^{n/(\bar p_c)'} \to 0$$

as $n \to +\infty$.

To show that $\lim_{n\to+\infty}\mathbb{B}_1(n)\to 0$ we represent $\mathbb{B}_1(n)$ as follows:

$$\begin{aligned}\mathbb{B}_1(n) &= \left(\sum_{m=n}^{\infty} I_m^q\right)^{1/q}\left(\sum_{m=-\infty}^{n-1}\|\chi_{(2^m,2^{m+1}]}\|^{\bar{q}}_{L^{(\bar{p}(\cdot))'}}\right)^{1/\bar{q}}\\ &\le \left(\sum_{m=n}^{\infty} I_m^q\right)^{1/q}\left(\sum_{m=-\infty}^{m_0-1}2^{m\bar{q}/(\bar{p}(0))'}\right)^{1/\bar{q}} + \left(\sum_{m=n}^{\infty} I_m^q\right)^{1/q}\left(\sum_{m=m_0}^{n-1}2^{m\bar{q}/(\bar{p}_c)'}\right)^{1/\bar{q}}\\ &=: J_n^{(1)}+J_n^{(2)},\end{aligned}$$

where $n\ge m_0$ and I_m is defined by (4.1). Observe now that

$$J_n^{(1)} = \left(\sum_{m=n}^{\infty} I_m^q\right)^{1/q} 2^{m_0/(\bar{p}(0))'}\to 0$$

as $n\to+\infty$ because $(\sum_{m=n}^{\infty} I_m^q)^{1/q}\to 0$ as $n\to+\infty$. The latter convergence follows from the convergence of the series.

Further,

$$\begin{aligned}J_n^{(2)} &\le c\sup_{m\ge n}\left(I_m 2^{m/(\bar{p}_c)'}\right)2^{-n/(\bar{p}_c)'}2^{n/(\bar{p}_c)'}\\ &\le c\sup_{m\ge n} I_m 2^{m/(\bar{p}_c)'}\to 0\end{aligned}$$

as $n\to+\infty$ because $I_m 2^{m/(\bar{p}_c)'}\to 0$ as $m\to+\infty$ (see (4.2)). Hence, $\lim_{m\to+\infty}\mathbb{B}_1(m)=0$.

Further, it is easy to see that for $0<\alpha<1$ and f_n,

$$\begin{aligned}\|K_\nu f_n\|_{(L^{p(\cdot)},l^q)_d} &\ge 2^{-n/\bar{p}_n}\left\|\chi_{(2^n,2^{n+1})}(x)\nu(x)k(x,x/2)x\right\|_{L^{p(\cdot)}}\\ &\ge 2^{n/(\bar{p}_n)'}\left\|\chi_{(2^n,2^{n+1})}(x)\nu(x)k(x,x/2)\right\|_{L^{p(\cdot)}}\\ &\ge c\left(2^n\left(2^\alpha-1\right)\right)^{1/(\bar{p}_n)'}\left\|\chi_{(2^{n+\alpha},2^{n+1})}(x)\nu(x)k(x,x/2)\right\|_{L^{p(\cdot)}}.\end{aligned}$$

Hence,

$$\|K_\nu f_n\|_{(L^{p(\cdot)},l^q)_d} \ge \sup_{0<\alpha<1}\left(2^n\left(2^\alpha-1\right)\right)^{1/(\bar{p}_n)'}\left\|\chi_{(2^{n+\alpha},2^{n+1})}(x)\nu(x)k(x,x/2)\right\|_{L^{p(\cdot)}}\to 0$$

as $n\to+\infty$ or $n\to-\infty$.

The conditions $B_1<\infty$ and $B_2<\infty$ follow from the fact that every compact operator is bounded. □

Now we formulate the compactness criteria for the kernel operator $\mathcal{K}_\nu$ defined on $\mathbb{R}$.

Theorem 4.2 *Let $1<\bar{p}_-(\mathbb{R})\le\bar{p}(x)\le p(x)\le p_+(\mathbb{R})<\infty$ and let $\bar{p}_0, p_0\in \mathrm{WL}(\mathbb{R}_+)$. Let $\bar{q}$ and q be constants such that $1<\bar{q}\le q<\infty$. Assume that $\bar{p}(x)\equiv\bar{p}_c\equiv\mathrm{const}$ and $p(x)\equiv p_c\equiv\mathrm{const}$ outside some large interval $(-\infty,2^{m_0})$. Let $\tilde{k}\in V(\mathbb{R}_+)\cap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$. Then $\mathcal{K}_\nu$ is compact from $(L^{\bar{p}(\cdot)},l^{\bar{q}})$ to $(L^{p(\cdot)},l^q)$ if and only if*

(i) $\quad D_1=\sup_{m\in\mathbb{Z}}\mathbb{D}_1(m)<\infty;\qquad D_2=\sup_{n\in\mathbb{Z}}D_2(n)<\infty,$

$$\text{(ii)} \qquad \lim_{m\to-\infty} D_1(m) = \lim_{m\to\infty} D_1(m) = 0,$$

$$\text{(iii)} \qquad \lim_{n\to-\infty} D_2(n) = \lim_{n\to\infty} D_2(n) = 0,$$

where

$$D_1(m) := \left\| \chi_{[2^m,2^{m+1})}\tilde{k}(x,x/2)\tilde{v}(x)\right\|_{L^{p_0(\cdot)}} 2^{m/\bar{p}_0'(0)};$$

$$\mathbb{D}_1(m) := \left[\sum_{n=m}^{\infty}\left\| \chi_{[2^n,2^{n+1})}\tilde{k}(x,x/2)\tilde{v}(x)\right\|^q_{L^{p_0(\cdot)}}\right]^{1/q} \times \left[\sum_{n=-\infty}^{m}\left\| \chi_{[2^{n-1},2^n)}(\cdot)\right\|^{(\bar{q})'}_{L^{\bar{p}_0'(\cdot)}}\right]^{1/(\bar{q})'};$$

$$D_2(n) := \sup_{0<\alpha<1}\left\| \chi_{[2^{n+\alpha},2^{n+1})}(x)\tilde{k}(x,x/2)\tilde{v}(x)\right\|_{L^{p_0(\cdot)}}\left\| \chi_{(2^n,2^{n+\alpha})}(\cdot)\right\|_{L^{(\bar{p}_0(\cdot))'}};$$

$\tilde{k}$, $\tilde{v}$ *and* p_0 *and* $\bar{p}_0$ *are defined in Section* 3.

Proof The proof follows from Theorem 4.1 by the change of variable $z \to \log_2 t$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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Acknowledgements
The first and second authors were supported by the Shota Rustaveli National Science Foundation grant (Contract No. D/13-23). The part of this work is carried out at Abdus Salam School of Mathematical Sciences, GC University, Lahore. The second and third authors are thankful to the Higher Education Commission, Pakistan for the financial support. The authors are grateful to the anonymous referees for their remarks and suggestions.

Received: 28 August 2012 Accepted: 1 April 2013 Published: 16 April 2013

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doi:10.1186/1029-242X-2013-173
Cite this article as: Kokilashvili et al.: **Weighted kernel operators in variable exponent amalgam spaces.** *Journal of Inequalities and Applications* 2013 **2013**:173.