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Grand Bochner–Lebesgue space and its associate space

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Abstract

Our aim is to introduce the grand Bochner–Lebesgue space in the spirit of Iwaniec–Sbordone spaces, also known as grand Lebesgue spaces, and prove some of its properties. We will also deal with the associate space for grand Bochner–Lebesgue spaces.

Keywords: Bochner spaces; Grand Bochner–Lebesgue space; Associate space

1. Introduction

In 1992 T. Iwaniec and C. Sbordone [17], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces, $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [15]. Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), were intensively studied during last years due to various

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applications, we mention e.g. [3,6,9–12,18,22,25] and continue to attract attention of various researchers. For example, in the theory of PDE's it turned out that these are the right spaces in which some nonlinear equations have to be considered (see [13,15]).

Also noteworthy to mention the extension of the ideas regarding grand Lebesgue spaces into the framework of the so-called grand Morrey spaces, e.g. [19–21,23,24].

2. Preliminaries

2.1. Bochner integral

In this subsection we want to recall some basic properties of integrals of vector-valued functions with respect to scalar measures. We will follow, almost verbatim, [7]. From now on $(\mathrm{X}, \mathscr{A}, \mu)$ will stand for a finite measure space and B for a Banach space.

Definition 2.1. A function $s : \mathrm{X} \to \mathrm{B}$ is *simple* if there exist $x_1, x_2, \dots, x_n \in \mathrm{B}$ and $E_1, E_2, \dots, E_n \in \mathscr{A}$ such that $s = \sum_{k=1}^{n} x_k \chi_{E_k}$, where χ_E stands for the characteristic function of the set E, and we will denote this set as $\mathrm{S}(\mathrm{X}, \mu, \mathrm{B})$. For the simple function s we define the integral as $\int_{\mathrm{X}} s(t)\,\mathrm{d}\mu(t) := \sum_{k=1}^{n} x_k \mu(E_k)$.

A function $f : \mathrm{X} \to \mathrm{B}$ is called *μ-measurable* (sometimes also referred as *strong measurable*) if there exists a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \|s_n - f\| = 0$ μ-almost everywhere.

Definition 2.2. A μ-measurable function $f : \mathrm{X} \to \mathrm{B}$ is called *Bochner integrable* if there exists a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \int_{\mathrm{X}} \|s_n - f\|\,\mathrm{d}\mu = 0.$$

The Bochner integral was introduced by S. Bochner [2]. It is also worth mentioning that equivalent definitions were given by T. Hildebrandt [16] and N. Dunford (the D_0-integral) [8]. For this reason, the Bochner integral is sometimes called *Dunford's first integral*.

We will cite some theorems that will be used, for the proofs, see e.g. [7].

Theorem 2.3 *(Dominated Convergence Theorem). Let $(\mathrm{X}, \mathscr{A}, \mu)$ be a finite measure space and (f_n) be a sequence of Bochner integrable B-valued functions on X. If $\lim f_n = f$ in μ-measure (i.e. $\lim_n \mu(\{\omega \in \mathrm{X} \mid \|s_n - f\| \geqslant \varepsilon\}) = 0$ for every $\varepsilon > 0$) and if there exists a real-valued Lebesgue integrable function g on X with $\|f_n\| \leqslant g$ μ-almost everywhere, then f is Bochner integrable and $\lim_n \int_E f_n\,\mathrm{d}\mu = \int_E f\,\mathrm{d}\mu$ for each $E \in \mathscr{A}$. In fact, $\lim_n \int_{\mathrm{X}} \|f - f_n\|\,\mathrm{d}\mu = 0$.*

Since we have the failure of the Radon–Nikodým Theorem for the Bochner integral, we will introduce the following property

Definition 2.4 *(Radon–Nikodým property).* A Banach space B has the *Radon–Nikodým property with respect to* $(\mathrm{X}, \mathscr{A}, \mu)$ if for each μ-continuous vector measure $G : \mathscr{A} \to \mathrm{B}$ of bounded variation there exists $g \in L^1(\mu, \mathrm{X})$ such that $G(E) = \int_E g\,\mathrm{d}\mu$ for all $E \in \mathscr{A}$. A Banach space B has the *Radon–Nikodým property* if B has the Radon–Nikodým property with respect to every finite measure space.

Lemma 2.5 *(Exhaustion Lemma). Let $G : \mathscr{A} \to \mathrm{B}$ be a vector measure. Suppose P is a property of G such that*

(a) *G has P on every μ-null set*;
(b) *if G has property P on $E \in \mathscr{A}$, then G has property P on every $A \in \mathscr{A}$ contained in E*;
(c) *if G has property P on E_1 and E_2 (both members of $\mathscr{A}$), then G has property P on $E_1 \cup E_2$; and*
(d) *every set $A \in \mathscr{A}$ of positive μ-measure contains a set $B \in \mathscr{A}$ of positive μ-measure such that G has property P on B.*

Then there exists a sequence (A_n) of disjoint members of $\mathscr{A}$ such that $\mathrm{X} = \bigcup_{n=1}^{\infty} A_n$ and such that G has property P on each A_n.

3. Grand and small Bochner–Lebesgue spaces

In this section we will introduce grand and small Bochner–Lebesgue spaces as well as some of their most important properties with respective proofs.

Definition 3.1 *(Grand Bochner–Lebesgue spaces).* Let $(\mathrm{X}, \mathscr{A}, \mu)$ be a finite measure space, $1 < p < \infty$ and $\varphi : (0, p-1) \to \mathbb{R}_+$ be a finite non-decreasing function with $\lim_{t\to 0} \varphi(t) = 0$. The *grand Bochner–Lebesgue space*, denoted by $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$, is the set of all Banach-valued measurable functions for which

$$\|f\|_{L^{p),\varphi}(\mathrm{X};\mathrm{B})} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\left(\int_{\mathrm{X}} \|f(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

From now on, all of the above conditions will be tacitly assumed whenever we speak of grand Bochner–Lebesgue spaces.

Remark 3.2. Recently, in [4], it was shown that the assumption on φ is essentially optimal in the sense that even if φ is non-decreasing, the space constructed by φ can be characterized by a new function Φ, which is non-decreasing.

Remark 3.3. Almost all papers where grand spaces are dealt with, the definition is somewhat different, the integral $\int$ is replaced by the integral average $⨍$, but since we are working on a finite measure space, there is no essential difference. Taking

$$\varphi(x) = x^{\frac{1}{p-x}} + \chi_{[0,1]^{\complement}}\left(1 - x^{\frac{1}{p-x}}\right) \tag{1}$$

the induced Lebesgue measure in a bounded subset of the Euclidean space and B as the one-dimensional Euclidean space, we recover the space introduced by T. Iwaniec and C. Sbordone in [17] and we get the space introduced in [15] when $\varphi(x) = x^{\frac{\theta}{p-x}} + \chi_{[0,1]^{\complement}}(1 - x^{\frac{\theta}{p-x}})$.

The grand Bochner–Lebesgue space has a very interesting property, let us call it *the nesting property*, namely

Lemma 3.4 *(Nesting property). If $1 < p < \infty$, for all $\varepsilon \in (0, p-1)$ we have*

$$L^p(\mathrm{X}, \mu, \mathrm{B}) \subset L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B}) \subset L^{p-\varepsilon}(\mathrm{X}, \mu, \mathrm{B}). \tag{2}$$

Proof. The embedding $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B}) \subset L^{p-\varepsilon}(\mathrm{X}, \mu, \mathrm{B})$ simply follows from the definition of supremum, the fact that $(\mathrm{X}, \mathscr{A}, \mu)$ is a finite measure space and the condition on φ.

To see the other inclusion, we note that by Hölder's inequality we have

$$\varphi(\varepsilon)\left(\fint_{\mathrm{X}} \|f(x)\|_{\mathrm{B}}^{p-\varepsilon}\, \mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} \leqslant \varphi(\varepsilon)\left(\fint_{\mathrm{X}} \|f(x)\|_{\mathrm{B}}^{p}\, \mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$

where $\fint_E f\, \mathrm{d}x := \frac{1}{\mu(E)} \int_E f\, \mathrm{d}x$ stands for the integral average of the function f in E, and we are done. □

Remark 3.5. The left inclusion in (2) can be strict depending on the measure space. For example, on the interval $[0, 1]$, the function $f(x) = x^{-\frac{1}{p}} \in L^{p),\phi}([0, 1])$ (with ϕ defined in (1)) but does not belong to $L^p([0, 1])$.

Theorem 3.6. *The space $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$ is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$. Then for fixed η and arbitrary $\varepsilon \in (0, p-1)$, exists $N(\eta)$ such that, whenever $n, m > N(\eta)$ we have

$$\varphi(\varepsilon)\left(\int_{\mathrm{X}} \|f_n(x) - f_m(x)\|_{\mathrm{B}}^{p-\varepsilon}\, \mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} < \eta/3.$$

Therefore $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{p-\varepsilon}(\mathrm{X}, \mu, \mathrm{B})$ for arbitrary $\varepsilon \in (0, p-1)$, as a result we define f_∞ as its limit in $L^{p-\varepsilon}(\mathrm{X}, \mu, \mathrm{B})$ for every $\varepsilon \in (0, p-1)$ (suppose the contrary that there are ε_1 and ε_2, $\varepsilon_1 < \varepsilon_2$, such that f_n converges to f_1 in $L^{p-\varepsilon_1}$ and f_n converges to f_2 in $L^{p-\varepsilon_2}$. If $f_1 \neq f_2$, then we have that f_n has two different limits in $L^{p-\varepsilon_2}$ because $L^{p-\varepsilon_1} \hookrightarrow L^{p-\varepsilon_2}$). Taking $n > N(\eta)$ and using the definition of the supremum, there exists positive $\varepsilon_0(n)$ less than $p-1$ such that

$$\begin{aligned} &\|f_\infty - f_n\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} \\ &\quad \leqslant \varphi\big(\varepsilon_0(n)\big)\left(\int_{\mathrm{X}} \|f_\infty(x) - f_n(x)\|_{\mathrm{B}}^{p-\varepsilon_0(n)}\, \mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon_0(n)}} + \eta/3. \end{aligned}$$

We also have that there exists $N_1 \in \mathbb{N}$ such that for $m > N_1$ we have

$$\varphi\big(\varepsilon_0(n)\big)\left(\int_{\mathrm{X}} \|f_\infty(x) - f_m(x)\|_{\mathrm{B}}^{p-\varepsilon_0(n)}\, \mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon_0(n)}} \leqslant \eta/3.$$

Collecting the previous results, we get

$$\begin{aligned}
&\|f_\infty - f_n\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} \\
&\leqslant \varphi\big(\varepsilon_0(n)\big)\Bigg(\int_{\mathrm{X}} \|f_n(x) - f_m(x)\|_{\mathrm{B}}^{p-\varepsilon_0(n)}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon_0(n)}} + \eta/3 \\
&\quad + \varphi\big(\varepsilon_0(n)\big)\Bigg(\int_{\mathrm{X}} \|f_\infty(x) - f_m(x)\|_{\mathrm{B}}^{p-\varepsilon_0(n)}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon_0(n)}} \leqslant \eta
\end{aligned}$$

for $n > M$ and $m > N_1$. From this we obtain the desired result. □

Theorem 3.7. *Let* $\mathrm{S}(\mathrm{X},\mu,\mathrm{B}) =: \mathrm{S}$ *be the set of simple functions* (*see* Definition 2.1). *Its closure* $\overline{[\mathrm{S}]}_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}$ *consists of functions* $f \in L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})$ *such that*

$$\lim_{\varepsilon\to 0} \varphi(\varepsilon) \int_{\mathrm{X}} \|f(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x) = 0. \tag{3}$$

Proof. Taking an element from the closure, there exists a sequence (s_n) of functions belonging to $\mathrm{S}(\mathrm{X},\mu,\mathrm{B})$ such that $\|f - s_n\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} \to 0$ when $n \to \infty$. Then, for fixed $\delta > 0$, we can choose $n(\delta)$ for which $\|f - s_{n(\delta)}\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} < \delta/2$. By Hölder's inequality we have that

$$\varphi(\varepsilon)\Bigg(\fint_{\mathrm{X}} \|s_{n(\delta)}(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon}} \leqslant \varphi(\varepsilon)\Bigg(\fint_{\mathrm{X}} \|s_{n(\delta)}(x)\|_{\mathrm{B}}^{p}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p}} \to 0$$

when $\varepsilon \to 0$. We can now take an $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$ we have

$$\varphi(\varepsilon)\Bigg(\fint_{\mathrm{X}} \|s_{n(\delta)}(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon}} < \delta/2.$$

Defining $I := \varphi(\varepsilon)(\fint_{\mathrm{X}} \|f(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x))^{\frac{1}{p-\varepsilon}}$ and gathering all, we have

$$\begin{aligned}
I &\leqslant \varphi(\varepsilon)\Bigg(\fint_{\mathrm{X}} \|f(x) - s_{n(\delta)}(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon}} + \varphi(\varepsilon)\Bigg(\fint_{\mathrm{X}} \|s_{n(\delta)}(x)\|_{\mathrm{B}}^{p-\varepsilon}\,\mathrm{d}\mu(x)\Bigg)^{\frac{1}{p-\varepsilon}} \\
&\leqslant \|f - s_{n(\delta)}\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} + \delta/2 \\
&\leqslant \delta
\end{aligned}$$

when $0 < \varepsilon < \varepsilon_0$, obtaining the result. □

Remark 3.8. The previous theorem has been proved by L. Greco in [14] for the case of finite measurable sets in $\mathbb{R}^n$, see also M. Carozza and C. Sbordone [5]. In this case it is possible to prove that the closure of simple functions is not dense, for example, in $L^{p),\phi}([0,1])$ with ϕ as in (1). Let us take the function $f(x) = x^{-\frac{1}{p}}$ in $L^{p),\phi}[(0,1)]$ and we obtain that $f \in L^{p),\phi} \backslash \overline{[\mathrm{S}]}_{L^{p)}}$, since $(\varepsilon \int_0^1 |f(t)|^{p-\varepsilon}\, \mathrm{d}t)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p-\varepsilon}} \nrightarrow 0$ as $\varepsilon \to 0$.

3.1. Associate space

By $[\mathrm{X}]'$ we denote the associate space of X understood in the sense of Banach function space theory, see [1] for the notion of associate space.

It is a well-known fact that the associate space for L^p space is isometrically isomorphic to $L^{p'}$, where p' is the conjugate exponent. The same is also true in the more general case when $L^p = L^p(\mathrm{X}, \mathscr{A}, \mu, \mathrm{B})$ represents the Lebesgue space of the functions with the values in the separable B-space and $L^{p'} = L^{p'}(\mathrm{X}, \mathscr{A}, \mu, \mathrm{B}^*)$ is the Lebesgue space of the functions with the values in the separable space B^* dual with B.

Our aim is to extend these results to the grand Lebesgue space framework. To deal with the associate space for grand Bochner–Lebesgue spaces, we first introduce the auxiliary Banach space $L^{(p',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$.

Definition 3.9. Let $(\mathrm{X}, \mathscr{A}, \mu)$ be a finite measure space and $1 < p < \infty$ and $\varphi : (0, p-1) \to \mathbb{R}_+$ be a non-decreasing function with $\lim_{t\to 0} \varphi(t) = 0$. By $L^{(p',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ we denote the set of all functions $g \in \mathrm{M}_0(\mathrm{X}, \mu, \mathrm{B}^*)$ as the set of all B^*-valued μ-measurable functions for which the $\|\cdot\|_{\mathrm{B}^*}$-value is finite a.e. in X which can be represented in the form $g(x) = \sum_{k=1}^{\infty} g_k(x)$ (convergence a.e.) and such that the following norm

$$\|g\|_{L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \frac{1}{\varphi(\varepsilon)} \left(\int_{\mathrm{X}} \|g_k(x)\|_{\mathrm{B}^*}^{(p-\varepsilon)'} \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}} \right\},$$

is finite.

If $f : \mathrm{X} \to \mathrm{B}$ and $g : \mathrm{X} \to \mathrm{B}^*$, we define $\langle f, g\rangle(x)$ on X by

$$\langle f, g\rangle(x) = g(x)\big(f(x)\big), \quad \text{for } x \in \mathrm{X}. \tag{4}$$

With the previous definition taken into account, we note that a Hölder type inequality is valid, namely

Theorem 3.10 *(Hölder's type inequality). Let $f \in L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$ and $g \in L^{(p',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$, then*

$$\int_{\mathrm{X}} \langle f, g\rangle(x)\, \mathrm{d}\mu(x) \leqslant \|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} \|g\|_{L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}. \tag{5}$$

Proof. Let us take a decomposition of $g = \sum_{k=1}^{\infty} g_k$. For each k and for each $0 < \varepsilon < p-1$ we have

$$
\begin{aligned}
\left|\int_X \langle f, g_k\rangle(x)\,\mathrm{d}\mu(x)\right| &\leqslant \int_X \left|\langle f, g_k\rangle(x)\right|\mathrm{d}\mu(x) \\
&\leqslant \int_X \left\|f(x)\right\|_B \left\|g_k(x)\right\|_{B^*}\mathrm{d}\mu(x) \\
&\leqslant \left(\int_X \left\|f(x)\right\|_B^{p-\varepsilon}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} \left(\int_X \left\|g_k(x)\right\|_{B^*}^{(p-\varepsilon)'}\mathrm{d}\mu(x)\right)^{\frac{1}{(p-\varepsilon)'}} \\
&\leqslant \varphi(\varepsilon)\left(\int_X \left\|f(x)\right\|_B^{p-\varepsilon}\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} \frac{1}{\varphi(\varepsilon)}\left(\int_X \left\|g_k(x)\right\|_{B^*}^{(p-\varepsilon)'}\mathrm{d}\mu(x)\right)^{\frac{1}{(p-\varepsilon)'}} \\
&\leqslant \|f\|_{L^{p),\varphi}(X,\mu,B)} \frac{1}{\varphi(\varepsilon)}\left(\int_X \left\|g_k(x)\right\|_{B^*}^{(p-\varepsilon)'}\mathrm{d}\mu(x)\right)^{\frac{1}{(p-\varepsilon)'}}
\end{aligned}
$$

which entails

$$
\int_X \langle f, g_k\rangle(x)\,\mathrm{d}\mu(x) \leqslant \inf_{0<\varepsilon<p-1} \frac{1}{\varphi(\varepsilon)}\left(\int_X \left\|g_k(x)\right\|_{B^*}^{(p-\varepsilon)'}\mathrm{d}\mu(x)\right)^{\frac{1}{(p-\varepsilon)'}} \|f\|_{L^{p),\varphi}(X,\mu,B)}
$$

from which we get

$$
\begin{aligned}
\int_X \langle f, g\rangle(x)\,\mathrm{d}\mu(x) &\leqslant \int_X \left\|f(x)\right\|_B \left\|\sum_{k=1}^{\infty} g_k(x)\right\|_{B^*}\mathrm{d}\mu(x) \\
&\leqslant \sum_{k=1}^{\infty}\int_X \left\|f(x)\right\|_B \left\|g_k(x)\right\|_{B^*}\mathrm{d}\mu(x) \\
&\leqslant \sum_{k=1}^{\infty}\inf_{0<\varepsilon<p-1} \frac{1}{\varphi(\varepsilon)}\left(\int_X \left\|g_k(x)\right\|_{B^*}^{(p-\varepsilon)'}\mathrm{d}\mu(x)\right)^{\frac{1}{(p-\varepsilon)'}} \|f\|_{L^{p),\varphi}(X,\mu,B)}
\end{aligned}
$$

and we obtain (5). □

We can obtain the following inclusions

$$
L^{p'+\varepsilon}(X,\mu,B) \subset L^{(p',\varphi}(X,\mu,B) \subset L^{p'}(X,\mu,B)
$$

for all $\varepsilon > 0$, giving in particular that $L^{\infty}(X,\mu,B) \subset L^{(p',\varphi}(X,\mu,B)$.

After introducing the auxiliary function space $L^{(p',\varphi}(X,\mu,B)$, we introduce the notion of *small Bochner–Lebesgue spaces*, namely

Definition 3.11 *(Small Bochner–Lebesgue spaces).* The *small Bochner–Lebesgue space* is defined by

$$L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}) := \{g \in \mathrm{M}_0 \mid \|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B})} < \infty\}$$

where

$$\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B})} = \sup_{\substack{0<\|\psi\|_{\mathrm{B}}\leqslant\|g\|_{\mathrm{B}} \\ \psi\in L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B})}} \|\psi\|_{L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B})}.$$

The next theorem follows from Theorem 3.10

Theorem 3.12. *Let $1 < p < \infty$. The following Hölder inequality holds*

$$\int_{\mathrm{X}} |\langle f, g\rangle(x)|\,\mathrm{d}\mu(x)(x) \leqslant \|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} \tag{6}$$

for all $f \in L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})$ and $g \in L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$.

Proof. For $f \in L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})$ and any $g \in \mathrm{M}_0(\mathrm{X},\mu,\mathrm{B}^*)$ we have

$$\begin{aligned}
\int_{\mathrm{X}} |\langle f, g\rangle|(x)\,\mathrm{d}\mu(x) &= \sup_{\substack{0<\|\psi\|_{\mathrm{B}^*}\leqslant\|g\|_{\mathrm{B}^*} \\ \psi\in L^{\infty}(\mathrm{X},\mu,\mathrm{B}^*)}} \int_{\mathrm{X}} \langle f, \psi\rangle(x)\,\mathrm{d}\mu(x) \\
&\leqslant \sup_{\substack{0<\|\psi\|_{\mathrm{B}^*}\leqslant\|g\|_{\mathrm{B}^*} \\ \psi\in L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}} \int_{\mathrm{X}} \langle f, \psi\rangle(x)\,\mathrm{d}\mu(x) \\
&\leqslant \sup_{\substack{0<\|\psi\|_{\mathrm{B}^*}\leqslant\|g\|_{\mathrm{B}^*} \\ \psi\in L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}} \|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}\|\psi\|_{L^{(p',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} \\
&\leqslant \|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}
\end{aligned}$$

and we have (6). □

We will first show that $L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ is isometrically contained in the space $[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'$ and then we will show that they coincide whenever we impose a certain restriction on B^*.

Theorem 3.13. *For $1 < p < \infty$ we have that $L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ is isometrically contained in $[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'$.*

Proof. Let $g \in L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ and (g_n) be a sequence of simple functions in $L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ converging a.e. to g. Taking $\langle f, g\rangle(x)$ as in (4) and $f \in L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})$ we have that $\langle f, g_n\rangle(x)$ is measurable for each n and also that $\lim_n \langle f, g_n\rangle(x) = \langle f, g\rangle(x)$, which shows that $\langle f, g\rangle(x)$ is measurable, see [7]. By Hölder's type inequality (6), we get that

$$\int_{\mathrm{X}} |\langle f, g\rangle(x)|\,\mathrm{d}\mu(x)(x) \leqslant \|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)},$$

showing that the function g belongs to $[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'$ whose norm is not greater than $\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}$.

We will show that the reverse inequality is true ($\|g\|_{[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'} \geqslant \|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)}$). Let $\varepsilon > 0$ and suppose first that $g(\cdot) = \sum_{i\geqslant 1} x_i^*(\cdot)\chi_{E_i}(\cdot)$, where (x_i^*) is a sequence in B^* and (E_i) is a countable partition of X by elements of $\mathscr{A}$ with $\mu(E_i) > 0$ for all i.

We choose $h \geqslant 0$ in $L^{p),\varphi}(\mathrm{X},\mu,\mathbb{R})$ such that $0 < \|h\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathbb{R})} \leqslant 1$ and such

$$\|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} - \varepsilon/2 < \int_{\mathrm{X}} \|g(x)\|_{\mathrm{B}^*} h(x)\,\mathrm{d}\mu(x). \tag{7}$$

We will now take $x_i \in \mathrm{B}$ with $\|x_i\|_{\mathrm{B}} = 1$ and such that

$$\|x_i^*\|_{\mathrm{B}^*} - \varepsilon/2\|h\|_1 < x_i^*(x_i). \tag{8}$$

Taking f as $f(\cdot) = \sum_{i=1}^{\infty} x_i h \chi_{E_i}(\cdot)$ we have that $f \in L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})$ with $\|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} = \|h\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathbb{R})} \leqslant 1$. We now have

$$\begin{aligned}
\int_{\mathrm{X}} \langle f, g\rangle(x)\,\mathrm{d}\mu(x) &= \int_{\mathrm{X}} h(x) \sum_{i=1}^{\infty} x_i^*(x_i)\chi_{E_i}(x)\,\mathrm{d}\mu(x) \\
&\geqslant \int_{\mathrm{X}} h(x) \sum_{i=1}^{\infty} \left(\|x_i^*\|_{\mathrm{B}^*} - \frac{\varepsilon}{2\|h\|_1} \right) \chi_{E_i}(x)\,\mathrm{d}\mu(x) \\
&\geqslant \int_{\mathrm{X}} h(x) \|g(x)\|_{\mathrm{B}^*}\,\mathrm{d}\mu(x) - \frac{\varepsilon}{2} \frac{\int_{\mathrm{X}} h(x)\,\mathrm{d}\mu(x)}{\|h\|_1} \\
&\geqslant \|g\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} - \varepsilon,
\end{aligned}$$

where we took (7) and (8) into account. Therefore $\|g\|_{[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'} = \|g\|_{L^{p)',\varphi}}(\mathrm{X},\mu,\mathrm{B}^*)$ when $g \in L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ is countably-valued. By a limiting argument we get the result for general $g \in L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$ (see [7, p. 98]). □

We now deal with the associate space of grand Bochner–Lebesgue space, namely

Theorem 3.14. *Let* $(\mathrm{X},\mathscr{A},\mu)$ *be a finite measure space,* $1 < p < \infty$ *and* B *be a Banach space. Then* $[L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]' = L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)$*, if and only if* B^* *has the Radon–Nikodým property with respect to* μ.

We would like to stress that the condition on B^* in Theorem 3.14 steams from the fact that Radon–Nikodým property fails, in general, for the Bochner integral.

Proof of Theorem 3.14. *Sufficiency*. We have $L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*) \subset [L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'$ isometrically. We now assume that B^* has the Radon–Nikodým property. For $F \in [L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})]'$ we define $G : \mathscr{A} \to \mathrm{B}^*$ by

$$G(E)(x) = F(x\chi_E)$$

for $E \in \mathscr{A}$.

Since $|F(x\chi_E)| \leqslant \|F\| \|x\chi_E\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} = \|F\| \|x\|_{\mathrm{B}} \|\chi_E\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}$ we have that G is countably additive and has its values in B^*. To see that $|G|(\mathrm{X}) < \infty$, let $\{E_1, \dots, E_n\}$ be a partition and $x_1, \dots, x_n$ be in the closed unit ball of B. Then

$$\begin{aligned} \left|\sum_{i=1}^n G(E_i)(x_i)\right| &= \left|F\left(\sum_{i=1}^n x_i \chi_{E_i}\right)\right| \\ &\leqslant \|F\| \left\|\sum_{i=1}^n x_i \chi_{E_i}\right\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})} \\ &\leqslant \|F\| \left\|\sum_{i=1}^n \chi_{E_i}\right\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}, \quad \text{since } \|x_i\| \leqslant 1, \\ &\leqslant \|F\| M \end{aligned}$$

where $M = (1 + \mu(\mathrm{X})) \cdot \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)$. It follows that G is of bounded variation after taking appropriate suprema.

Since B^* has the Radon–Nikodým property, there is a Bochner integrable function $g : \mathrm{X} \to \mathrm{B}^*$ such that $G(E) = \int_E g \,\mathrm{d}\mu(x)$ for all $E \in \mathscr{A}$. If $f \in L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$ is a simple function, then $F(f) = \int_{\mathrm{X}} \langle f, g\rangle(x) \,\mathrm{d}\mu(x)$. Take an expanding sequence (E_n) in $\mathscr{A}$ such that $\bigcup_n E_n = \mathrm{X}$ and such that g is bounded on each E_n. Fix n_0 and note that $\int_{E_{n_0}} \langle \cdot, g\rangle(x) \,\mathrm{d}\mu(x)$ is a bounded linear functional on $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$ which agrees with F on all simple functions supported on E_{n_0}. We have that

$$F(f\chi_{E_{n_0}}) = \int_{\mathrm{X}} \langle f, g\chi_{E_{n_0}}\rangle(x) \,\mathrm{d}\mu(x)$$

for all $f \in L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$. Moreover, since $g\chi_{E_{n_0}}$ is bounded, one has $g\chi_{E_{n_0}} \in L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ and $\|g\chi_{E_{n_0}}\|_{L^{p)',\varphi}(\mathrm{X},\mu,\mathrm{B}^*)} \leqslant \|F\|$. By the fact that this last inequality is obtained for each n_0, the Monotone Convergence Theorem guarantees that $g \in L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$. With the above considerations and the Hölder inequality, we have that

$$F(f) = \lim_n \int_{\mathrm{X}} \langle f\chi_{E_n}, g\rangle(x) \,\mathrm{d}\mu(x) = \int_{\mathrm{X}} \langle f, g\rangle(x) \,\mathrm{d}\mu(x),$$

for all $f \in L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$. Therefore the space $[L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})]'$ coincides with $L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ proving the sufficiency.

Necessity. Suppose that $[L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})]' = L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ and let $G : \mathscr{A} \to \mathrm{B}^*$ be a μ-continuous vector measure of bounded variation. We shall show that if $E_0 \in \mathscr{A}$ has positive μ-measure, then G has a Bochner integrable Radon–Nikodým derivative on a set $D \in \mathscr{A}$,

$D \subseteq E_0$ with $\mu(D) > 0$. Invoking the Exhaustion Lemma 2.5 will then complete the proof. Thus let $E_0 \in \mathscr{A}$ have positive μ-measure. Applying the Hahn decomposition theorem to the scalar measures $|G|$ and $k\mu$ for a sufficiently large positive integer k produces a subset D of E_0, $D \in \mathscr{A}$, $\mu(D) > 0$ such that $|G|(E) \leqslant k\mu(E)$ for all $E \in \mathscr{A}$ with $E \subseteq D$. Define for a simple function $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $x_i \in \mathrm{B}$, $E_i \in \mathscr{A}$, and $E_i \cap E_j = \emptyset$ for $i \neq j$,

$$F(f) = \sum_{i=1}^{n} G(E_i \cap D)(x_i).$$

Then

$$\begin{aligned} |F(f)| &= \left| \sum_{i=1}^{n} G(E_i \cap D)(x_i) \right| = \left| \sum_{i=1}^{n} \frac{G(E_i \cap B)}{\mu(E_i \cap B)} \big(\mu(E_i \cap D)x_i\big) \right| \\ &\leqslant \sum_{i=1}^{n} k \big\| \mu(E_i \cap D)x_i \big\| \leqslant k\|f\|_1 \\ &\leqslant k\mathcal{C}\big(1 + \mu(\mathrm{X})\big)\|f\|_{L^{p),\varphi}(\mathrm{X},\mu,\mathrm{B})}, \end{aligned}$$

where the last inequality comes from Hölder's inequality and the definition of grand spaces, where we can take $\mathcal{C} = 1/\varphi((p-1)/2) < +\infty$.

Since F is linear on the simple functions in $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$, this shows that F is continuous on the simple functions in $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$ and therefore has a bounded linear extension to all of $L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B})$. By hypothesis, there is $g \in L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ such that

$$F(f) = \int_{\mathrm{X}} \langle f, g\rangle(x)\, \mathrm{d}\mu(x), \quad \text{for all } f \in L^{p),\varphi}(\mathrm{X}, \mu, \mathrm{B}).$$

But one has $G(E \cap D)(x) = F(x\chi_E) = \int_E \langle x, g\rangle(x)\, \mathrm{d}\mu(x)$ for all $x \in \mathrm{B}$ and $E \in \mathscr{A}$. Since each $g \in L^{p)',\varphi}(\mathrm{X}, \mu, \mathrm{B}^*)$ is Bochner integrable, we get

$$G(E \cap D)(x) = \left(\int_E g\, \mathrm{d}\mu(x) \right)(x), \quad \text{for all } x \in \mathrm{B} \text{ and } E \in \mathscr{A}.$$

As a result, $G(E \cap D) = \int_E g\, \mathrm{d}\mu(x)$, for all $E \in \mathscr{A}$. □

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