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Original article

Sharp weighted bounds for multiple integral operators

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Abstract

Sharp weighted bounds for strong maximal functions, multiple potentials and singular integrals are derived in terms of Muckenhoupt type characteristics of weights.

Keywords: Strong maximal functions; Integral operators with product kernels; Boundedness; One-weight inequality; Sharp bound

1. Introduction

In this paper, we establish sharp weighted bounds for strong maximal functions and multiple integral operators. Our derived results involve, in particular, Buckley-type estimates for strong Hardy–Littlewood and fractional maximal functions, potentials and singular integrals with product kernels, and their one-sided analogs.

One of the main problems in Harmonic Analysis is to characterize a weight w for which a given integral operator is bounded in L^p_w (one-weight inequality). An important class of such weights is the well-known A_p class. It is known that A_p condition is necessary and sufficient for the boundedness of Hardy–Littlewood and singular integral operators (see, e.g., [1–3]); however, the sharp dependence of the corresponding L^p_w norms in terms of A_p characteristic of w is known only for some operators. The interest in the sharp weighted norm, for example, for singular integral operators is motivated by applications in partial differential equations (see e.g., [4–7]).

Strong maximal operator different from the usual one is defined with respect to parallelepipeds with sides parallel to the co-ordinate axes; the operators with product kernels, such as multiple singular and potential operators have singularities not only at a single point but on the hyperplanes. That is why to study mapping properties for such

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operators became more complicated; however from the one weight viewpoint it is possible to get one-weight boundedness results as well as sharp weighted bounds by deducing the problem to the single variable result and using repeatedly the latter one uniformly with respect to other variables. In this direction Proposition 2.1 is one of the keys to get the main results. One of the important aspects of this paper is that this point enables us to get sharp one-weight results for a quite large class of multiple operators including one-sided cases.

Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{X\to Y}$ which is defined in the standard way i.e. $\|T\|_{X\to Y} := \sup_{\|f\|_X \le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_X$.

An almost everywhere positive locally integrable function (i.e. weight) w defined on $\mathbb{R}^n$ is said to satisfy $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

In 1972 B. Muckenhoupt [3] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy$$

is bounded in $L^p_w(\mathbb{R}^n)$.

S. Buckley [8] investigated the sharp A_p bound for the operator M and established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w\in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

To explain better the point of sharp estimates for multiple operators, let us discuss, for example, the strong Hardy–Littlewood maximal operator $M^{(s)}$ defined on $\mathbb{R}^2$. Denote by $A^{(s)}_p(\mathbb{R}^2)$ the Muckenhoupt class taken with respect to the rectangles with sides parallel to the co-ordinate axes (see Section 2 for the definitions). Let $\|w\|_{A^{(s)}_p(\mathbb{R}^2)}$ be $A^{(s)}_p$ characteristic of w. There arises a natural question regarding the sharp bound in the inequality

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\|w\|^{\beta}_{A^{(s)}_p(\mathbb{R}^2)}. \tag{1.2}$$

We show that the following estimate is sharp

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\left(\|w\|_{A_p(x_1)} \|w\|_{A_p(x_2)} \right)^{1/(p-1)}, \tag{1.3}$$

where $\|w\|_{A_p(x_i)}$ is the characteristic of the weight w defined with respect to the ith variable uniformly to another one $i = 1, 2$ (see e.g., [9–11], Ch. IV for the one-weight theory for multiple integral operators). Inequality (1.3) together with the Lebesgue differentiation theorem implies that (1.2) holds for $\beta = \frac{2}{p-1}$; however, unfortunately we do not know whether it is or not sharp.

Under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \le B \le c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$.

Finally we mention that constants (often different constants in one and the same lines of inequalities) will be denoted by c or C. The symbol p' stands for the conjugate number of p: $p' = p/(p-1)$, where $1 < p < \infty$.

2. Strong maximal and multiple integral operators

Let w be a weight function on a domain $\Omega \subseteq \mathbb{R}^n$. We denote by $L^p_w(\Omega)$, $1 < p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

is finite. If $w \equiv$ const, then we denote $L^p_w(\Omega) = L^p(\Omega)$.

In this section, we give sharp weighted bounds for strong maximal and multiple integral operators. Given an operator $T_{\mathbb{R}}$ acting on function in $\mathbb{R}$, by T^k, $k = 1 \cdots n$, we denote the operators defined on class of functions acting on $\mathbb{R}^n$ by letting $T_{\mathbb{R}}$ acting on the kth variable and keeping rest of $n-1$ variable fixed. Formally, for every $x \in \mathbb{R}^n$,

$$(T^k f)(x) = T_{\mathbb{R}}(f(x_1, x_2, \ldots, x_{k-1}, \cdot, x_k, \ldots, x_n))(x_k). \tag{2.1}$$

Remark 2.1. It can be easily verified (see [11], pg. 450–451) that if $T_{\mathbb{R}}$ is bounded, then T^k is also bounded and further

$$\|T^k f\|_{L^p(\mathbb{R}^n)} \le c\|T_{\mathbb{R}}\|\|f\|_{L^p(\mathbb{R}^n)},$$

holds.

Definition 2.1. A weight function w satisfies $A^{(s)}_p(\mathbb{R}^n)$ condition ($w \in A^{(s)}_p(\mathbb{R}^n)$), $1 < p < \infty$, if

$$\|w\|_{A^{(s)}_p(\mathbb{R}^n)} := \sup_P \left(\frac{1}{|P|}\int_P w(x)dx\right)\left(\frac{1}{|P|}\int_P w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Definition 2.2. Let $1 < p < \infty$. A weight function $w = w(x_1, \ldots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy A_p condition in x_i uniformly with respect to other variables ($w \in A_p(x_i)$) if

$$\|w\|_{A_p(x_i)} := \operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} \sup_I \left(\frac{1}{|I|}\int_I w(x_1,\ldots,x_n)dx_i\right) \times \left(\frac{1}{|I|}\int_I w^{1-p'}(x_1,\ldots,x_n)dx_i\right)^{p-1} < \infty,$$

where by I we denote a bounded interval in $\mathbb{R}$.

Remark 2.2. $w(x_1, \ldots, x_n) \in A^{(s)}_p(\mathbb{R}^n) \Leftrightarrow w \in \bigcap_{i=1}^n A_p(x_i)$ (see e.g., pp. 453–454 of [11,10]).

Proposition 2.1. *Let T^k be the operators given by the formula (2.1) and let T be an operator defined for functions on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$,*

$$(Tf)(x) \le (T^1 \circ \cdots \circ T^n)(f)(x)$$

and

$$\|T^k\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\gamma(p)}_{A_p(x_k)} \quad k = 1, \ldots, n, \tag{2.2}$$

holds, where $\gamma(p)$ is a constant depending only on p. Then the following estimate

$$\|T\|_{L^p_w(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)})^{\gamma(p)}$$

holds.

Proof. For simplicity we give proof for $n = 2$ the proof general case is the same. Suppose that $f \geq 0$. Using (2.1) two times and Fubini's theorem we have,

$$\begin{aligned}
\|Tf\|^p_{L^p_w(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} (Tf(x_1, x_2))^p w(x_1, x_2) dx_1 dx_2 \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^1(T^2 f(\cdot, x_2)))(x_1)^p w(x_1, x_2) dx_1 \right) dx_2 \\
&\leq c\|w\|^{p\gamma(p)}_{A_p(x_1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^2 f(x_1, x_2))^p w(x_1, x_2) dx_1 \right) dx_2 \\
&= c\|w\|^{p\gamma(p)}_{A_p(x_1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (T^2 f(x_1, x_2))^p w(x_1, x_2) dx_2 \right) dx_1 \\
&= c(\|w\|_{A_p(x_1)} \|w\|_{A_p(x_2)})^{p\gamma(p)} \|f\|^p_{L^p_w(\mathbb{R}^2)}. \quad \square
\end{aligned}$$

2.1. *Strong Hardy–Littlewood maximal functions and multiple singular integrals*

The following theorem is due to S. Buckley [8].

Theorem A. *If $w \in A_p(\mathbb{R}^n)$, then $\|Mf\|_{L^p_w(\mathbb{R}^n)} \leq c_{n,p}\|w\|^{1/(p-1)}_{A_p(\mathbb{R}^n)} \|f\|_{L^p_w(\mathbb{R}^n)}$. The exponent $1/(p-1)$ is best possible.*

Let f be a locally integrable function on $\mathbb{R}^n$. Then we define strong Hardy–Littlewood maximal operator as

$$\big(M^{(s)} f\big)(x) = \sup_{P \ni x} \frac{1}{|P|} \int_P |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all parallelepipeds $P \ni x$ in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Theorem 2.3. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^n$ such that $w \in A^{(s)}_p(\mathbb{R}^n)$. Then there exists a constant c depending only on n and p such that the following inequality*

$$\|M^{(s)} f\|_{L^p_w(\mathbb{R}^n)} \leq c \left(\prod_{i=1}^n \|w\|_{A_p(x_i)} \right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{2.3}$$

holds, for all $f \in L^p_w(\mathbb{R}^n)$. Further, the exponent $1/(p-1)$ in estimate (2.3) is sharp.

Proof. For every $x \in \mathbb{R}^n$ we can estimate $M^{(s)}$ as follows

$$\big(M^{(s)} f\big)(x) \leq \big(M^1 \circ M^2 \circ \cdots \circ M^n\big) f(x),$$

where

$$\begin{aligned}
\big(M^k f\big)(x_1, \ldots, x_n) &= M(f(x_1, x_2, \ldots, x_{k-1}, \cdot, x_k, \ldots, x_n))(x_k) \\
&= \sup_{I_k \ni x_k} \frac{1}{|I_k|} \int_{I_k} |f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)| dt.
\end{aligned}$$

Now by Theorem A and Proposition 2.1 (for $\gamma(p) = \frac{1}{p-1}$) we find that

$$\|M^{(s)} f\|_{L^p_w(\mathbb{R}^n)} \leq c \left(\prod_{i=1}^n \|w\|_{A_p(x_i)} \right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)}.$$

For sharpness we consider the case for $n = 2$. Observe that when w is of product type, i.e. $w(x_1, x_2) = w_1(x_1) w_2(x_2)$, then

$$\|w\|_{A_p(x_1)} = \|w_1\|_{A_p(\mathbb{R})}, \qquad \|w\|_{A_p(x_2)} = \|w_2\|_{A_p(\mathbb{R})}. \tag{2.4}$$

Let us take $0 < \epsilon < 1$. Suppose that $w(x_1, x_2) = |x_1|^{(1-\epsilon)(p-1)}|x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A_{p(x_1)}}\|w\|_{A_{p(x_2)}})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for

$$f(x_1, x_2) = {x_1}^{\epsilon-1}\chi_{(0,1)}(x_1){x_2}^{\epsilon-1}\chi_{(0,1)}(x_2),$$

we have $\|f\|^p_{L^p_w} \approx \frac{1}{\epsilon^2}$. Now let $0 < x_1, x_2 < 1$. Then we find that the following estimate

$$\big(M^{(s)}f\big)(x_1, x_2) \geq \frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f(t,\tau)dtd\tau = \frac{1}{\epsilon^2}f(x,y)$$

holds. Finally

$$\|M^{(s)}f\|_{L^p_w} \geq \frac{1}{\epsilon^2}\|f\|_{L^p_w}.$$

Thus we have the sharpness in (2.3). □

Now we present the sharp weighted estimates for multiple singular integrals. S. Buckley, in his celebrated paper [8] showed that for $1 < p < \infty$, convolution Calderón–Zygmund singular operator satisfies

$$\|T\|_{L^p_w(\mathbb{R}^n)} \leq c\|w\|^{\frac{p}{p-1}}_{A_p(\mathbb{R}^n)}$$

and the best possible exponent is at least $\max\{1, \frac{1}{p-1}\}$. S. Petermichl [6,7] proved that the estimate

$$\|S\|_{L^p_w(\mathbb{R}^n)} \leq c\|w\|^{\max\left\{1, \frac{1}{p-1}\right\}}_{A_p(\mathbb{R}^n)}$$

is sharp, where S is either the Hilbert transform or one of the Riesz transforms in $\mathbb{R}^n$

$$R_jf(x) = c_np\cdot v\cdot\int_{\mathbb{R}^n}\frac{x_j - y_j}{|x-y|^{n+1}}f(y)dy.$$

S. Petermichl obtained the results for $p = 2$. The general case $p \neq 2$ then follows by the sharp version of the Rubio de Francia extrapolation theorem given by O. Dragičević, L. Grafakos, C. Pereyra and S. Petermichl [12] (see also, T. Hytönen [13] regarding the A_2 conjecture for Calderón–Zygmund operators which, in fact, implies appropriate estimate for all exponents $1 < p < \infty$ by applying a sharp version of the Rubio de Francia's extrapolation theorem).

Let us denote by $\mathcal{H}^{(n)}$ the Hilbert transform with product kernels (or n-dimensional Hilbert transform) defined by

$$\big(\mathcal{H}^{(n)}f\big)(x) = \lim_{\substack{\epsilon_1\to 0\\ \cdots\\ \epsilon_n\to 0}}\int_{|x_1-t_1|>\epsilon_1}\cdots\int_{|x_n-t_n|>\epsilon_n}\frac{f(t_1,\ldots,t_n)}{(x_1-t_1)\cdots(x_n-t_n)}dt_1\cdots dt_n.$$

We denote $\mathcal{H}^{(1)} =: \mathcal{H}$. Notice that for each $x \in \mathbb{R}^n$, we can write

$$\big(\mathcal{H}^{(n)}f\big)(x) = \big(\mathcal{H}^1\circ\cdots\circ\mathcal{H}^n\big)f(x) \tag{2.5}$$

where,

$$\begin{aligned}\big(\mathcal{H}^kf\big)(x) &= \mathcal{H}(f(x_1, x_2, \ldots, x_{k-1}, \cdot, x_k, \ldots, x_n))(x_k)\\ &= \lim_{\epsilon_k\to 0}\int_{|x_k-y_k|>\varepsilon_k}\frac{f(x_1,\ldots,y_k,\ldots,x_n)}{x_k-y_k}dy_k.\end{aligned}$$

The following theorem is due to S. Petermichl [6].

Theorem B. *Let $1 < p < \infty$. Then there exists a positive constant c depending only on p such that for all weights $w \in A_p(\mathbb{R})$ we have*

$$\|\mathcal{H} f\|_{L^p_w(\mathbb{R})} \le c \|w\|^{\beta}_{A_p(\mathbb{R})} \|f\|_{L^p_w(\mathbb{R})}, \quad f \in L^p_w(\mathbb{R}), \tag{2.6}$$

where $\beta = \max\{1, p'/p\}$. Moreover, the exponent β in this estimate is sharp.

Theorem 2.4. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^n$ such that $w \in A^{(s)}_p(\mathbb{R}^n)$. Then there exists a constant c depending only on n and p such that the following inequality*

$$\|\mathcal{H}^{(n)} f\|_{L^p_w(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)})^{\max\{1, p'/p\}} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{2.7}$$

holds for all $f \in L^p_w$. Further the exponent $\max\{1, p'/p\}$ in estimate (2.7) is sharp.

Proof. Using representation (2.5), Proposition 2.1 and Theorem B, we have that

$$\|\mathcal{H}^{(n)} f\|_{L^p_w(\mathbb{R}^n)} \ll \left(\|w\|_{A_p(x_1)} \cdots \|w\|_{A_p(x_n)}\right)^{\max\{1, p'/p\}} \|f\|_{L^p_w(\mathbb{R}^n)}.$$

Let $n = 2$. For sharpness we observe that when w is of product type i.e. $w(x_1, x_2) = w_1(x_1) w_2(x_2)$, then inequality (2.4) holds. Let us first derive sharpness for $p = 2$. Let us take $0 < \epsilon < 1$ and let $w(x_1, x_2) = w_1(x_1) w_2(x_2)$, where $w_1(x_1) = |x_1|^{1-\epsilon}$ and $w_2(x_2) = |x_2|^{1-\epsilon}$. Then it is easy to check that (2.4) holds. Observe also that for

$$f(x_1, x_2) = {x_1}^{\epsilon-1} \chi_{(0,1)}(x_1) {x_2}^{\epsilon-1} \chi_{(0,1)}(x_2), \tag{2.8}$$

$\|f\|^2_{L^2_w} \approx \frac{1}{\epsilon}$. Now let $0 < x_1, x_2 < 1$. Then we find that

$$\|\mathcal{H}^{(2)} f\|_{L^2_w(\mathbb{R}^2)} \ge 4\epsilon^{-3}.$$

Letting $\epsilon \to 0$ we have sharpness in (2.7) for $p = 2$ i.e., the estimate

$$\|\mathcal{H}^{(2)}\|_{L^2_w(\mathbb{R}^2)} \ll \|w\|_{A_2(x_1)} \|w\|_{A_2(x_2)}$$

is sharp.

Let $1 < p < 2$. Suppose that $0 < \epsilon < 1$ and that $w(x_1, x_2) = |x_1|^{(1-\epsilon)(p-1)} |x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A_{p(x_1)}} \|w\|_{A_{p(x_2)}})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for the function defined by (2.8) the relation $\|f\|_{L^p_w} \approx (\frac{1}{\epsilon^2})^{\frac{1}{p}}$ holds. Now let $0 < x_1, x_2 < 1$. Then we find that following estimates

$$\|\mathcal{H}^{(2)} f\|_{L^p_w(\mathbb{R}^2)} \ge \frac{1}{\epsilon^2} \|f\|_{L^p_w(\mathbb{R}^2)} \approx \left(\|w\|_{A_p(x_1)} \|w\|_{A_p(x_2)}\right)^{p'/p} \|f\|_{L^p_w(\mathbb{R}^2)}$$

are fulfilled. Thus we have sharpness in (2.7) for $1 < p < 2$. Using the fact that n-dimensional Hilbert transform is essentially self-adjoint and applying duality argument together with the obvious equality

$$\|u^{1-p'}\|_{A_{p'}} = \|u\|^{1/(p-1)}_{A_p}, \quad u \in A_p,$$

we have sharpness for $p > 2$. This completes the proof. $\square$

Let $x = (x^{(1)}, \ldots, x^{(n)}) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$, where $d_1, d_2, \ldots, d_n \in \mathbb{N}$. Suppose that $x^{(k)}_{j_k}$ are components of $x^{(k)}$, $k = 1, \ldots, n$, $1 \le j_k \le d_k$. Then we define n-fold Riesz transform

$$\left(R^{(n)}_{(j_1,\ldots,j_n)} f\right)(x) = p.v. \int_{\mathbb{R}^{d_1}} \cdots \int_{\mathbb{R}^{d_n}} \prod_{k=1}^{n} \frac{(x^{(k)}_{j_k} - y^{(k)}_{j_k})}{|x^{(k)} - y^{(k)}|^{d_k+1}} f(y^{(1)}, \ldots, y^{(n)}) dy^{(1)} \cdots dy^{(n)},$$

where $1 \le j_k \le d_k$, $k = 1, \ldots, n$. It can be noticed that

$$\left(R^{(n)}_{(j_1,\ldots,j_n)} f\right)(x) = \left(R^1_{j_1} \circ \cdots \circ R^n_{j_n} f\right)(x)$$

where

$$\big(R^k_{(j_1,\ldots,j_n)}f\big)(x) = p.v.\int_{\mathbb{R}^{d_k}} \frac{x^{(k)}_{j_k} - y^{(k)}_{j_k}}{|x^{(k)} - y^{(k)}|^{d_k+1}} f(x^{(1)},\ldots,x^{(k-1)},y^{(k)},x^{(k+1)}\cdots,x^{(n)})dy^{(k)}.$$

Theorem 2.5. *Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n}$ satisfy the condition $w \in A^{(s)}_p(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_n})$. Then there exists a constant c independent of $f \in L^p_w(\mathbb{R}^d)$ and w such that the following inequality*

$$\|R^{(n)}_{(j_1,\ldots,j_n)}f\|_{L^p_w(\mathbb{R}^d)} \le c(\|w\|_{A_p(x^{(1)})}\cdots\|w\|_{A_p(x^{(n)})})^{\max\{1,p'/p\}}\|f\|_{L^p_w(\mathbb{R}^d)} \tag{2.9}$$

holds for all $1 \le j_k \le d_k$, $k = 1\cdots n$, where $d = d_1 + \cdots, +d_n$. Further, the exponent $\max\{1, p'/p\}$ in estimate (2.9) is sharp.

Proof of this statement is similar to that of the previous one; we need to apply Proposition 2.1 and the results of [7]. □

Example 2.1. Let $-1 < \gamma < p-1$. It is known that $w(x) = |x|^\gamma$ belongs to $A^{(s)}_p(\mathbb{R}^n)$. Let $\overline{w}(t) = |t|^\gamma$, $t \in \mathbb{R}$. We set $b_\gamma := \max\{2^{\frac{\gamma}{2}}, 1\}$ and $d_\gamma := \max\{2^{\frac{-\gamma}{2}-p+1}, 1\}$.

(i) It follows from Theorem 2.3 that

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^n)} \le C^n C_\gamma^{\frac{n}{p-1}}\left(1 + \|\overline{w}\|_{A_p(\mathbb{R})}\right)^{\frac{n}{p-1}} \tag{2.10}$$

where C is the constant from the Buckley's estimate (see (1.1)) and

$$C_\gamma = \begin{cases} b_\gamma, & 0 \le \gamma < p-1, \\ d_\gamma, & -1 < \gamma < 0. \end{cases} \tag{2.11}$$

It is known (see [14] pp 287–289) that C in (2.10) can be taken as $C = 3^{p+p'}2^{p'-p}p'24^{\frac{2}{p}}p^{\frac{1}{p-1}}$.

(ii) It follows from Theorem 2.4 that

$$\|\mathcal{H}^{(n)}\|_{L^p_w(\mathbb{R}^n)} \le c^n C_\gamma^{n\max\{1,p'/p\}}\left(1 + \|\overline{w}\|_{A_p(\mathbb{R})}\right)^{n\max\left\{1,\frac{p'}{p}\right\}}$$

holds, where c is the constant from (2.6) and C_γ is defined in (2.11).

Following [14], pp 285–286, it can be verified that

$$\|\overline{w}\|_{A_p(\mathbb{R})} \le \max\left\{2^{|\gamma|}, \frac{4^p}{(\gamma+1)(\gamma(1-p')+1)^{p-1}}\right\}.$$

We can get also another type of estimate of the norms in L^p_w. By using the same arguments as in [14], pp 285–286, we find that

$$\|w\|_{A_p(x_i)} \le \begin{cases} \Gamma_\gamma, & 0 \le \gamma < p-1, \\ G_\gamma, & -1 < \gamma < 0; \end{cases} \qquad i = 1, 2,$$

where

$$\Gamma_\gamma = \max\left\{((4/3)^2+1)^{\gamma/2}(2/3)^\gamma, b_\gamma 4^p\left((\gamma+1)^{-1}(\gamma(1-p')+1)^{1-p}+1\right)\right\},$$

$$G_\gamma = \max\left\{((4/3)^2+1)^{-\gamma/2}(2/3)^\gamma, d_0 d_\gamma 4^p\left((\gamma+1)^{-1}(\gamma(1-p')+1)^{1-p}+1\right)\right\}.$$

Consequently, using directly Theorems 2.3 and 2.4 we have the following estimate

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^n)} \le C^n \begin{cases} \Gamma_\gamma^{n/(p-1)}, & 0 \le \gamma < p-1, \\ G_\gamma^{n/(p-1)}, & -1 < \gamma < 0; \end{cases}$$

$$\|\mathcal{H}^{(n)}\|_{L^p_w(\mathbb{R}^n)} \le c^n \begin{cases} \Gamma_\gamma^{n\max\{1,p'/p\}}, & 0 \le \gamma < p-1, \\ G_\gamma^{n\max\{1,p'/p\}}, & -1 < \gamma < 0; \end{cases}$$

where C and c are constants in (1.1) and (2.6) respectively.

2.2. *Strong fractional maximal functions and Riesz potentials with product kernels*

In this subsection, we state and prove sharp weighted norm estimates for strong fractional maximal and Riesz potential with product kernels. To get the main results we use the ideas of the previous subsection.

In 1974 B. Muckenhoupt and R. Wheeden [15] found necessary and sufficient condition for the one-weight inequality; namely, they proved that the Riesz potential I_α (resp the fractional maximal operator M_α) is bounded from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$, where $1 < p < \infty$, $0 < \alpha < n/p$, $q = \frac{np}{n-\alpha p}$ if and only if w satisfies the so called $A_{p,q}(\mathbb{R}^n)$ condition (see the definition below). Moreover, from their result it follows that there is a positive constant c depending only on p and α such that

$$\|T_\alpha\|_{L^p_{w^p}(\mathbb{R}^n) \to L^q_{w^q}(\mathbb{R}^n)} \le c\|w\|^\beta_{A_{p,q}(\mathbb{R}^n)}, \tag{2.12}$$

for some positive exponent β, where T_α is I_α (resp. M_α), and $\|w\|_{A_{p,q}(\mathbb{R}^n)}$ is the $A_{p,q}$ characteristic of w:

$$\|w\|_{A_{p,q}(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}(x)dx\right)^{q/p'}.$$

In their paper M. Lacey, K. Moen, C. Perez and R. Torres [16] proved that the best possible value of β in (2.12) is $(1-\alpha/n)\max\{1, p'/q\}$ for I_α (resp. $p'/q(1-\alpha/n)$) for M_α (see also [17] for this and other sharp results).

Definition 2.3. A weight function w satisfies $A^{(s)}_{p,q}$ condition ($w \in A^{(s)}_{p,q}$), $1 < p < \infty$ if

$$\|w\|_{A^{(s)}_{p,q}} := \sup_{P \ni x}\left(\frac{1}{|P|}\int_P w^q(x)dx\right)^{1/q}\left(\frac{1}{|P|}\int_P w(x)^{-p'}dx\right)^{1/p'} < \infty,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes.

Definition 2.4. Let $1 < p \le q < \infty$. A weight function $w = w(x_1, \dots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy $A_{p,q}$ condition in x_i uniformly with respect to other variables ($w \in A_{p,q}(x_i)$) if

$$\|w\|_{A_{p,q}(x_i)} := \operatorname*{ess\,sup}_{(x_1,\dots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}} \sup_I \left(\frac{1}{|I|}\int_I w^q(x_1,\dots,x_n)dx_i\right)^{1/q} \times \left(\frac{1}{|I|}\int_I w^{-p'}(x_1,\dots,x_n)dx_i\right)^{1/p'} < \infty,$$

where I is a bounded interval.

Remark 2.6. Like $A^{(s)}_p(\mathbb{R}^n)$ weights for given $w(x_1, \dots, x_n) \in A^{(s)}_{p,q} \Leftrightarrow w \in \bigcap_{i=1}^n A_{p,q}(x_i)$.

Proposition 2.2. *Let $1 < p \le q < \infty$. Suppose that operators T^k are defined by the formula (2.1) and that T is an operator defined for functions on $\mathbb{R}^n$. Suppose that weight w belongs to the class $A^{(s)}_{p,q}$. Let*

$$\|T^k\|_{L^p_{w^p}(\mathbb{R}) \to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\gamma(p,q)}_{A_{p,q}(x_k)} \quad k = 1, \dots, n, \tag{2.13}$$

hold, where $\gamma(p,q)$ is a constant depending only on p and q. Then

$$\|T\|_{L^p_{w^p}(\mathbb{R}^n)\to L^q_{w^q}(\mathbb{R}^n)} \le c(\|w\|_{A_p(x_1)}\cdots\|w\|_{A_p(x_n)})^{\gamma(p,q)}.$$

Proof is similar to that of Proposition 2.1; therefore it is omitted.

The following theorem is from [16].

Theorem C. *Suppose that $0<\alpha<n$, $1<p<n/\alpha$ and q is defined by the relationship $1/q=1/p-\alpha/n$. If $w\in A_{p,q}(\mathbb{R}^n)$, then*

$$\|wM_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\|w\|_{A_{p,q}(\mathbb{R}^n)}^{\frac{p'}{q}(1-\alpha/n)}\|wf\|_{L^p(\mathbb{R}^n)}.$$

Furthermore, the exponent $\frac{p'}{q}(1-\alpha/n)$ is sharp.

Let f be a locally integrable function and let $0<\alpha<1$. The strong fractional maximal operator is defined by

$$\big(M_\alpha^{(s)}f\big)(x)=\sup_{P\ni x}\frac{1}{|P|^{1-\alpha}}\int_P|f(y)|dy,$$

where the supremum is taken over all parallelepipeds P in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes. It is easy to see that

$$\big(M_\alpha^{(s)}f\big)(x)\le\big(M_\alpha^1\circ M_\alpha^2\circ\cdots\circ M_\alpha^n f\big)(x), \tag{2.14}$$

where

$$\begin{aligned}\big(M_\alpha^k f\big)(x_1,\dots,x_n)&=M_\alpha(f(x_1,x_2,\dots,x_{k-1},\cdot,x_k,\dots,x_n))(x_k)\\&=\sup_{I_k\ni x_k}\frac{1}{|I_k|^{1-\alpha}}\int_{I_k}|f(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)|dt,\end{aligned}$$

where I_k are intervals in $\mathbb{R}$ such that $P=I_1\times\cdots\times I_k$.

Theorem 2.7. *Let $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $q=\frac{p}{1-\alpha p}$ and w be a weight function on $\mathbb{R}^n$ such that $w\in A_{p,q}^{(s)}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|wM_\alpha^{(s)}f\|_{L^q(\mathbb{R}^n)}\le c\left(\prod_{i=1}^n\|w\|_{A_{p,q}(x_i)}\right)^{\frac{p'}{q}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{2.15}$$

holds, for all $f\in L^p_{w^p}(\mathbb{R}^n)$. Further, the exponent $\frac{p'}{q}(1-\alpha)$ in estimate (2.15) is sharp.

Proof. Using estimate (2.14), Theorem C and Proposition 2.2 we get easily (2.15). The main “difficulty” here is to derive sharpness. Let, for simplicity, $n=2$. Let us take $0<\epsilon<1$. Suppose that w is of product type $w(x_1,x_2)=w_1(x_1)w_2(x_2)$, where $w_1(x_1)=|x_1|^{(1-\epsilon)/p'}$ and $w_2(x_2)=|x_2|^{(1-\epsilon)/p'}$. Then it is easy to see that

$$\|w\|_{A_{p,q}(x_1)}=\|w_1\|_{A_{1+q/p'}(\mathbb{R})}\approx\epsilon^{-q/p'};\qquad \|w\|_{A_{p,q}(x_2)}=\|w_2\|_{A_{1+q/p'}(\mathbb{R})}\approx\epsilon^{-q/p'}.$$

Further, if

$$f(t_1,t_2)=|t_1|^{\epsilon-1}\chi_{(0,1)}(t_1)|t_2|^{\epsilon-1}\chi_{(0,1)}(t_2),$$

then $\|wf\|_{L^p(\mathbb{R}^2)}\approx\frac{1}{\epsilon^{2/p}}$. Let $0<x_1,x_2<1$. Then we find that

$$M_\alpha^{(s)}f(x_1,x_2)\ge\frac{1}{|x_1|^{1-\alpha}|x_2|^{1-\alpha}}\int_0^{x_1}\int_0^{x_2}f(t_1,t_2)dt_1dt_2\approx c\frac{|x_1|^{\epsilon-1+\alpha}|x_2|^{\epsilon-1+\alpha}}{\epsilon^2}.$$

Finally we conclude that,

$$\|wM_\alpha^{(s)}f\|_{L^q(\mathbb{R}^2)}\ge\epsilon^{-2-2/q}. \tag{2.16}$$

Thus letting $\epsilon\to0$ we have sharpness. □

Let $0 < \alpha < 1$. We define Riesz potential with product kernels on $\mathbb{R}^n$ as follows:

$$(I_\alpha^{(n)} f)(x) = \int_{\mathbb{R}^n} \frac{f(t_1, \ldots, t_n)}{\prod_{i=1}^n |x_i - t_i|^{1-\alpha}} dt_1 \cdots dt_n, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

When $n = 1$ we use the symbol I_α for $I_\alpha^{(1)}$. The following theorem is from [16].

Theorem D. *Let $0 < \alpha < n$, $1 < p < n/\alpha$. We put $q = \frac{np}{n-\alpha p}$. Suppose that $w \in A_{p,q}(\mathbb{R}^n)$. Then*

$$\|w I_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\|w\|_{A_{p,q}(\mathbb{R}^n)}^{(1-\alpha/n)\max\{1, p'/q\}} \|wf\|_{L^p(\mathbb{R}^n)}.$$

Further, the exponent $(1 - \alpha/n) \max\{1, p'/q\}$ is sharp.

Our result regarding $I_\alpha^{(n)}$ reads as follows:

Theorem 2.8. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We put $q = \frac{p}{1-\alpha p}$. Let w be a weight function on $\mathbb{R}^n$ such that $w \in A_{p,q}^{(s)}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|w I_\alpha^{(n)} f\|_{L^q(\mathbb{R}^n)} \le c \left(\prod_{i=1}^n \|w\|_{A_{p,q}(x_i)} \right)^{\max\{1, \frac{p'}{q}\}(1-\alpha)} \|wf\|_{L^p(\mathbb{R}^n)} \tag{2.17}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. Further, the exponent $\max\{1, \frac{p'}{q}\}(1 - \alpha)$ in estimate (2.17) is sharp.

Proof of this statement follows using the same arguments as in the proof of Theorem 2.7 together with Theorem D.

3. One-sided operators

In 1986 E. Sawyer proved the following inequality for the right maximal operator M^+:

$$\|M^+ f\|_{L^p_w(\mathbb{R})} \le C_p \|w\|_{A_p^+(\mathbb{R})}^\beta \|f\|_{L^p_w(\mathbb{R})}, \quad f \in L^p_w(\mathbb{R}), \tag{3.1}$$

with some positive exponent β, where $\|w\|_{A_p^+(\mathbb{R})}$ is A_p^+ characteristic of a weight w defined by

$$\|w\|_{A_p^+(\mathbb{R})} := \sup_{x \in \mathbb{R}, h>0} \left(\frac{1}{h} \int_{x-h}^x w(t)dt \right) \left(\frac{1}{h} \int_x^{x+h} w^{1-p'}(t)dt \right)^{p-1},$$

and

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)|dt.$$

The authors of this work in [18] showed that the best possible exponent in (3.1) is $\beta = \frac{1}{p-1}$.

In their celebrated work [19], K. Andersen and E. Sawyer completely characterized the one-weight boundedness for one-sided fractional operators. In particular, they proved that if $1 < p < \infty$, $0 < \alpha < 1/p$, $q = \frac{p}{1-\alpha p}$, then

$$\|w \mathcal{N}_\alpha^+ f\|_{L^q(\mathbb{R})} \le C_{p,\alpha} \|w\|_{A_{p,q}^+(\mathbb{R})}^\beta \|wf\|_{L^p(\mathbb{R})}, \quad f \in L^p_{w^p}(\mathbb{R}), \tag{3.2}$$

for some positive β, where $\mathcal{N}_\alpha^+$ is either the Weyl transform $\mathcal{W}_\alpha$ or the right fractional maximal operator M_α^+ defined by:

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^\alpha} \int_x^{x+h} |f(t)|dt \qquad \mathcal{W}_\alpha(f)(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad 0 < \alpha < 1,$$

and $\|w\|_{A^+_{p,q}(\mathbb{R}^n)}$ is the right $A^+_{p,q}$ characteristic of a weight w given by

$$\|w\|_{A^+_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int_{x-h}^{x} w^q(t)dt\right)\left(\frac{1}{h}\int_{x}^{x+h} w^{-p'}(t)dt\right)^{q/p'}.$$

In [18] the authors proved that the best possible exponent β in (3.2) is $\frac{p'}{q}(1-\alpha)$ for M^+_α, and is $(1-\alpha)\max\{1, \frac{p'}{q}\}$ for $\mathcal{W}_\alpha$.

Now we list these and related results from [18].

Theorem 3.1. *Let $1 < p < \infty$. Then*

(i)

$$\|M^+\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\frac{1}{p-1}}_{A^+_p(\mathbb{R})}$$

holds and the exponent $\frac{1}{p-1}$ is best possible, where $A^+_p(\mathbb{R})$.

(ii)

$$\|M^-\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\frac{1}{p-1}}_{A^-_p(\mathbb{R})}$$

holds and the exponent $\frac{1}{p-1}$ is best possible, where $A^-_p(\mathbb{R})$ is the left Muckenhoupt characteristic of weight:

$$\|w\|_{A^-_p(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int_{x}^{x+h} w(t)dt\right)\left(\frac{1}{h}\int_{x+h}^{x} w^{1-p'}(t)dt\right)^{p-1}.$$

Theorem 3.2. *Suppose that $0 < \alpha < 1$, $1 < p < 1/\alpha$ and that q is such that $1/p - 1/q - \alpha = 0$. Then*

(i) *there exists a positive constant c depending only on p and α such that*

$$\|M^+_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^+_{p,q}(\mathbb{R})}. \tag{3.3}$$

Moreover, the exponent $\frac{p'}{q}(1-\alpha)$ is best possible.

(ii) *there exists a positive constant c depending only on p and α such that*

$$\|M^-_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^-_{p,q}(\mathbb{R})}, \tag{3.4}$$

where

$$\|w\|_{A^-_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int_{x}^{x+h} w^q(t)dt\right)\left(\frac{1}{h}\int_{x+h}^{x} w^{-p'}(t)dt\right)^{q/p'}.$$

Moreover, the exponent $\frac{p'}{q}(1-\alpha)$ is best possible,

Theorem 3.3. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$ and let q satisfy $q = \frac{p}{1-\alpha p}$. Then*

(a) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{R}_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^-_{p,q}(\mathbb{R})}. \tag{3.5}$$

Furthermore, this estimate is sharp;

(b) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{W}_\alpha\|_{L^p_{w^p}(\mathbb{R})\to L^q_{w^q}(\mathbb{R})} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^+_{p,q}(\mathbb{R})}. \tag{3.6}$$

Moreover, this estimate is sharp.

One of our aims is to apply known results to give sharp estimates for multiple operators.

3.1. Strong one-sided maximal operators

Let f be locally integrable function on $\mathbb{R}^n$. We define one-sided strong fractional maximal operators as

$$M^{+(s)}f(x_1,\ldots,x_n)=\sup_{h_1,\ldots,h_n>0}\frac{1}{\prod_{i=1}^{n}h_i}\int_{x_1}^{x_1+h_1}\cdots\int_{x_n}^{x_n+h_n}|f(y_1,\ldots,y_n)|dy_1\cdots dy_n, \tag{3.7}$$

$$M_\alpha^{-(s)}f(x_1,\ldots,x_n)=\sup_{h_1,\ldots,h_n>0}\frac{1}{\prod_{i=1}^{n}h_i}\int_{x_1-h_1}^{x_1}\cdots\int_{x_n-h_n}^{x_n}|f(y_1,\ldots,y_n)|dy_1\cdots dy_n. \tag{3.8}$$

Let $1<p<\infty$. We say that a weight function w belongs to the class $A_p^{-(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_p^{-(s)}(\mathbb{R}^n)}:=\sup_{\substack{h_1,\ldots,h_n>0\\ x_1,\ldots,x_n\in\mathbb{R}}}\left(\frac{1}{h_1\cdots h_n}\int_{x_1}^{x_1+h_1}\cdots\int_{x_n}^{x_n+h_n}w(t_1,\ldots,t_n)dt_1\cdots dt_n\right)$$
$$\times\left(\frac{1}{h_1\cdots h_n}\int_{x_1-h_1}^{x_1}\cdots\int_{x_n-h_n}^{x_n}w^{1-p'}(t_1,\ldots,t_n)dt_1\cdots dt_n\right)^{p-1}<\infty;$$

further, $w\in A_p^{-(s)}(\mathbb{R}^n)$ if

$$\|w\|_{A_p^{+(s)}(\mathbb{R}^n)}:=\sup_{\substack{h_1,\ldots,h_n>0\\ x_1,\ldots,x_n\in\mathbb{R}}}\left(\frac{1}{h_1\cdots h_n}\int_{x_1-h_1}^{x_1}\cdots\int_{x_n-h_n}^{x_n}w(t_1,\ldots,t_n)dt_1\cdots dt_n\right)$$
$$\times\left(\frac{1}{h_1\cdots h_n}\int_{x_1}^{x_1+h_1}\cdots\int_{x_n}^{x_n+h_n}w^{1-p'}(t_1,\ldots,t_n)dt_1\cdots dt_n\right)^{p-1}<\infty.$$

Definition 3.1. Let $1<p<\infty$. A weight function $w=w(x_1,\ldots,x_n)$ defined on $\mathbb{R}^n$ is said to satisfy A_p^- condition in x_i uniformly with respect to other variables ($w\in A_p^-(x_i)$) if

$$\|w\|_{A_p^-(x_i)}:=\operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}}\ \sup_{h_i>0}\left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i}w(x_1,\ldots,x_{i-1},t,x_{i-1},\ldots,x_n)dt\right)$$
$$\times\left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i}w(x_1,\ldots,x_{i-1},t,x_{i-1},\ldots,x_n)^{-1/(p-1)}dt\right)^{p-1}<\infty.$$

Further, $w\in A_p^+(x_i)$ if

$$\|w\|_{A_p^+(x_i)}:=\operatorname*{ess\,sup}_{(x_1,\ldots,x_{i-1},x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}}\ \sup_{h_i>0}\left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i}w(x_1,\ldots,x_{i-1},t,x_{i-1},\ldots,x_n)dt\right)$$
$$\times\left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i}w(x_1,\ldots,x_{i-1},t,x_{i-1},\ldots,x_n)^{-1/(p-1)}dt\right)^{p-1}<\infty.$$

Remark 3.4. It is known that (see [20], Ch. 5) that $w(x_1,\ldots,x_n)\in A_p^{\pm(s)}(\mathbb{R}^n)\Leftrightarrow w\in\bigcap_{i=1}^n A_p^{\pm}(x_i)$.

Theorem 3.5. *Let* $1<p<\infty$.

(i) *Suppose that a weight function* w *on* $\mathbb{R}^n$ *belongs to the class* $A_p^{+(s)}(\mathbb{R}^n)$. *Then there exists a constant* c *depending only on* n *and* p *such that the following inequality*

$$\|M^{+(s)}f\|_{L_w^p(\mathbb{R}^n)}\le c\left(\prod_{i=1}^{n}\|w\|_{A_p^+(x_i)}\right)^{1/(p-1)}\|f\|_{L_w^p(\mathbb{R}^n)} \tag{3.9}$$

holds for all $f\in L_w^p(\mathbb{R}^n)$. *Further, the exponent* $1/(p-1)$ *in estimate* (3.9) *is sharp.*

(ii) *Let $w \in A_p^{-(s)}(\mathbb{R}^n)$. Then there exists a constant c depending only on n and p such that the following inequality*

$$\|M^{-(s)} f\|_{L^p_w(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A^-_p(x_i)}\right)^{1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)} \tag{3.10}$$

holds for all $f \in L^p_w(\mathbb{R}^n)$. Further, the exponent $1/(p-1)$ in estimate (3.10) is sharp.

Proof. We show (i). The proof of (ii) is similar. Since the proof of inequality (3.9) follows in the same way as in the case of $M^{(s)}$ (see Theorem 2.3), we show only sharpness. Let $n = 2$. We take $0 < \epsilon < 1$. Let $w(x_1, x_2) = |1 - x_1|^{(1-\epsilon)(p-1)}|1 - x_2|^{(1-\epsilon)(p-1)}$. Then it is easy to check that

$$(\|w\|_{A^+_p(x_1)}\|w\|_{A^+_p(x_2)})^{1/(p-1)} \approx \frac{1}{\epsilon^2}.$$

Observe also that for

$$f(x_1, x_2) = (1 - x_1)^{\epsilon(p-1)-1}\chi_{(0,1)}(x_1)(1 - x_2)^{\epsilon(p-1)-1}\chi_{(0,1)}(x_2),$$

we have $\|f\|_{L^p_w} \approx \frac{1}{\epsilon^2}$. Now let $0 < x_1, x_2 < 1$. Then

$$M^{+(s)} f(x_1, x_2) \ge \frac{1}{(1 - x_1)(1 - x_2)} \int_{x_1}^1 \int_{x_2}^1 f(t, \tau)dtd\tau = c\frac{1}{\epsilon^2} f(x_1, x_2).$$

Finally

$$\|M^{+(s)} f\|_{L^p_w(\mathbb{R}^2)} \ge c\frac{1}{\epsilon^2}\|f\|_{L^p_w}.$$

Thus we have the sharpness in (3.9). □

3.2. One-sided multiple fractional integrals

Now we discuss sharp bounds for one-sided strong maximal potential operators with product kernels.

Let f be a locally integrable function on $\mathbb{R}^n$ and let $0 < \alpha < 1$. We define one-sided strong fractional maximal operators as

$$M_\alpha^{+(s)} f(x_1, \dots, x_n) = \sup_{h_1,\dots,h_n>0} \frac{1}{\prod_{i=1}^n h_i^{1-\alpha}} \int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} |f(y_1, \dots, y_n)|dy_1 \cdots dy_n, \tag{3.11}$$

$$M_\alpha^{-(s)} f(x_1, \dots, x_n) = \sup_{h_1,\dots,h_n>0} \frac{1}{\prod_{i=1}^n h_i^{1-\alpha}} \int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} |f(y_1, \dots, y_n)|dy_1 \cdots dy_n. \tag{3.12}$$

Let $1 < p \le q < \infty$. We say that a weight function w belongs to the class $A^{-(s)}_{p,q}(\mathbb{R}^n)$ if

$$\|w\|_{A^{-(s)}_{p,q}(\mathbb{R}^n)} := \sup_{\substack{h_1,\dots,h_n>0\\ x_1,\dots,x_n\in\mathbb{R}}} \left(\frac{1}{h_1\cdots h_n}\int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w^q(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/q}$$
$$\times \left(\frac{1}{h_1\cdots h_n}\int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w^{-p'}(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/p'} < \infty;$$

further, $w \in A^{+(s)}_{p,q}(\mathbb{R}^n)$ if

$$\|w\|_{A^{+(s)}_{p,q}(\mathbb{R}^n)} := \sup_{\substack{h_1,\dots,h_n>0\\ x_1,\dots,x_n\in\mathbb{R}}} \left(\frac{1}{h_1\cdots h_n}\int_{x_1-h_1}^{x_1} \cdots \int_{x_n-h_n}^{x_n} w^q(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/q}$$
$$\times \left(\frac{1}{h_1\cdots h_n}\int_{x_1}^{x_1+h_1} \cdots \int_{x_n}^{x_n+h_n} w^{-p'}(t_1, \dots, t_n)dt_1 \cdots dt_n\right)^{1/p'} < \infty.$$

Definition 3.2. Let $1 < p \le q < \infty$. A weight function $w = w(x_1, \dots, x_n)$ defined on $\mathbb{R}^n$ is said to satisfy $A^-_{p,q}$ condition in x_i uniformly with respect to other variables ($w \in A^+_{p,q}(x_i)$) if

$$\|w\|_{A^+_{p,q}(x_i)} := \operatorname*{ess\,sup}_{(x_1,\dots,x_{i-1},x_{i+1}\dots,x_n)\in\mathbb{R}^{n-1}} \sup_{h_i>0} \left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i} w^q(x_1,\dots,x_{i-1},t,x_{i+1}\dots,x_n)dt\right)^{1/q} \times \left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i} w^{-p'}(x_1,\dots,x_{i-1},t,x_{i+1}\dots,x_n)dt\right)^{1/p'} < \infty,$$

further, $w \in A^-_{p,q}(x_i)$ if

$$\|w\|_{A^-_{p,q}(x_i)} \equiv \sup_{\substack{(x_1,\dots,x_{i-1},\\ x_{i+1}\cdots,x_n)\in\mathbb{R}^{n-1}}} \sup_{h_i>0} \left(\frac{1}{h_i}\int_{x_i-h_i}^{x_i} w^q(x_1,\dots,x_{i-1},t,x_{i+1}\dots,x_n)dt\right)^{1/q} \times \left(\frac{1}{h_i}\int_{x_i}^{x_i+h_i} w^{-p'}(x_1,\dots,x_{i-1},t,x_{i+1}\dots,x_n)dt\right)^{1/p'} < \infty.$$

Remark 3.6. It is easy to check that $w(x_1,\dots,x_n) \in A^{\pm(s)}_{p,q}(\mathbb{R}^n) \Leftrightarrow w \in \bigcap_{i=1}^n A^{\pm}_{p,q}(x_i)$.

Theorem 3.7. *Let $0 < \alpha < 1$, $1 < p < 1/\alpha$. We put $q = \frac{p}{1-\alpha p}$. Suppose that w is a weight function defined on $\mathbb{R}^n$ such that $w \in A^{+(s)}_{p,q}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following inequality*

$$\|wM^{+(s)}_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\big(\|w\|_{A^+_{p,q}(x_1)}\cdots\|w\|_{A^+_{p,q}(x_n)}\big)^{\frac{p'}{q}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{3.13}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. Further, the exponent $\frac{p'}{q}(1-\alpha)$ in estimate (3.13) is sharp.

Proof. Estimate (3.13) follows in the same way as in the previous cases. For sharpness we take $n = 2$ and $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, where $w_1(x_1) = |1-x_1|^{(1-\epsilon)p'}$; $w_2(x_2) = |1-x_2|^{(1-\epsilon)p'}$, $0 < \epsilon < 1$. Then

$$\|w\|_{A^+_{p,q}(x_1)}\|w\|_{A^+_{p,q}(x_2)} = \|w_1\|_{A^+_{p,q}(\mathbb{R})}\|w\|_{A^+_{p,q}(\mathbb{R})} = \|w_1^q\|_{A^+_{1+q/p'}(\mathbb{R})}\|w_2\|_{A^+_{1+q/p'}(\mathbb{R})} \approx \varepsilon^{2q/p'}.$$

If

$$f(t_1,t_2) = (1-t_1)^{\epsilon-1}\chi_{(0,1)}(t_1)(1-t_2)^{\epsilon-1}\chi_{(0,1)}(t_2),$$

then $\|wf\|_{L^p(\mathbb{R})^2} \approx \frac{1}{\epsilon^{2/p}}$. Now let $0 < x < 1$. Then we find that the following estimate

$$M^{+(s)}_\alpha f(x_1,x_2) \ge \frac{1}{|1-x_1|^{1-\alpha}|1-x_2|^{1-\alpha}}\int_{x_1}^1\int_{x_2}^1 f(t_1,t_2)dt_1dt_2 \approx \frac{|1-x_1|^{\epsilon-1+\alpha}|1-x_2|^{\epsilon-1+\alpha}}{\epsilon^2}$$

holds. Finally

$$\|wM^{+(s)}_\alpha f\|_{L^q(\mathbb{R}^2)} \ge \epsilon^{-2-2/q}. \tag{3.14}$$

Thus, letting $\epsilon \to 0$ we are done. □

The next statement can be proved analogously. Details are omitted.

Theorem 3.8. *Let α, p and q satisfy the condition of Theorem 3.7. Let w be a weight function on $\mathbb{R}^n$ such that $w \in A^{-(s)}_{p,q}(\mathbb{R}^n)$. Then there exists a constant c depending only on n, p and α such that the following* *inequality*

$$\|wM_{\alpha}^{-(s)}f\|_{L^q(\mathbb{R}^n)} \le c(\|w\|_{A^-_{p,q}(x_1)}\cdots\|w\|_{A^+_{p,q}(x_n)})^{\frac{p'}{q}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{3.15}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\frac{p'}{q}(1-\alpha)$ *in estimate* (3.15) *is sharp.*

Let f be a measurable function on $\mathbb{R}^n$ and let $0 < \alpha < 1$. We define one-sided potentials $\mathcal{R}_\alpha^{(n)}$ and $\mathcal{W}_\alpha^{(n)}$ with product kernels

$$\mathcal{R}_\alpha^{(n)} f(x_1,\ldots,x_n) = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n} \frac{f(t_1,\ldots,t_n)}{(x_1-t_1)^{1-\alpha}\cdots(x_n-t_n)^{1-\alpha}}dt_1\cdots dt_n,$$
$$\mathcal{W}_\alpha^{(n)} f(x_1,\ldots,x_n) = \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} \frac{f(t_1,\ldots,t_n)}{(t_1-x_1)^{1-\alpha}\cdots(t_n-x_n)^{1-\alpha}}dt_1\cdots dt_n,$$

where $x_i \in \mathbb{R}$, $i = 1,\ldots,n$.

Finally we formulate the "sharp result" for one-sided potentials with product kernels. We do not repeat the arguments using above, and therefore omit the proof of the next statement.

Theorem 3.9. *Let* α, p *and* q *satisfy the conditions of Theorem* 3.7. *Suppose that* w *be a weight function on* $\mathbb{R}^n$ *such that* $w \in A^{-(s)}_{p,q}(\mathbb{R}^n)$. *Then*

(i) *there exists a constant* c *depending only on* n, p *and* α *such that the following inequality*

$$\|w\mathcal{R}_\alpha^{(n)}f\|_{L^q(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A^-_{p,q}(x_i)}\right)^{\max\{1,\frac{p'}{q}\}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{3.16}$$

holds for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\max\{1, \frac{p'}{q}\}(1-\alpha)$ *in estimate* (3.16) *is sharp.*

(ii) *There is a constant* c *depending only on* n, p *and* α *such that*

$$\|w\mathcal{W}_\alpha^{(n)}f\|_{L^q(\mathbb{R}^n)} \le c\left(\prod_{i=1}^n \|w\|_{A^+_{p,q}(x_i)}\right)^{\max\{1,\frac{p'}{q}\}(1-\alpha)}\|wf\|_{L^p(\mathbb{R}^n)} \tag{3.17}$$

for all $f \in L^p_{w^p}(\mathbb{R}^n)$. *Further, the exponent* $\max\{1, \frac{p'}{q}\}(1-\alpha)$ *in estimate* (3.17) *is sharp.*

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