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Original article

Generalized singular integral on Carleson curves in weighted grand Lebesgue spaces

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Abstract

This paper studies the mapping properties of the integral operator generated by that singular integral which arises in the theory of I. Vekua generalized analytic functions. Boundedness problems are explored in weighted grand Lebesgue spaces.
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The theory of generalized analytic functions was developed by L. Bers and I. Vekua. We refer to their books [1–3]. Generalized analytic functions of the class $U_{r,2}(A, B; E)$, $r > 2$, in the sense of I. Vekua, are regular solutions of the equation

$$\partial_{\overline{z}}\Phi(z) + A(z)\Phi(z) + B(z)\overline{\Phi}(z) = 0, \tag{1}$$

where $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $A(z), B(z) \in L^{r,2}(E)$, $r > 2$. Here E denotes the plane. The set of functions f defined on E is called the class $L^{r,2}(E)$ if

$$f(z) \in L^r(U), \quad f_0(z) = z^2 f\left(\frac{1}{z}\right) \in L^r(U), \quad U = \{z : |z| < 1\}.$$

Let Γ be a simple, rectifiable curve of the complex plane.

Let $f \in L^1(\Gamma)$. It is known [3,4] that the integral

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,\tau) f(\tau)\, d\tau - \Omega_2(z,\tau)\overline{f}(\tau)\, d\overline{\tau}, \tag{2}$$

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where Ω_1 and Ω_2 are the so-called basic normalized kernels of the class $U_{r,2}(A, B; E)$, is a regular solution of (1) (see [2,3] for details). The integral (2) is called the generalized Cauchy type integral. The corresponding generalized singular integral is introduced as

$$\widetilde{S}_\Gamma f(t) = \frac{1}{2\pi i} \int_\Gamma \Omega_1(t, \tau) f(\tau)\, d\tau - \Omega_2(t, \tau)\overline{f}(\tau)\, d\overline{\tau}. \tag{3}$$

The kernels Ω_1 and Ω_2 have the following structures

$$\Omega_1(z, \tau) = \frac{1}{t - z} + \frac{m_1(z, t)}{|t - z|^\alpha} \quad \text{and} \quad \Omega_2(z, \tau) = \frac{m_2(z, t)}{|t - z|^\alpha}, \tag{4}$$

where the functions $m_1(z, t)$ and $m_2(z, t)$ are continuous and bounded.

In [4] for the case of L^p the following statement was proved.

Proposition A. *Let Γ be a Carleson curve. The operator $\widetilde{S}_\Gamma$ is bounded in $L^p(\Gamma)$ if*

$$p > \frac{r}{r - 2}. \tag{5}$$

In [5] it was established more general result for variable exponent Lebesgue space $L^{p(t)}$ which even in the case of constant p is stronger than the existing result of Proposition A because it was admitted the whole range $1 < p < \infty$ avoiding restriction (5). In the same paper [5] it was also proved the boundedness of the operator $\widetilde{S}_\Gamma$ in weighted variable exponent Lebesgue spaces with a certain class of weights including power type weights.

This paper deals with the boundedness of $\widetilde{S}_\Gamma$ in weighted grand Lebesgue spaces. All possible cases of weighted grand Lebesgue spaces are discussed, namely the case when in the definition of the norm a weight generates absolutely continuous measure and the other case, when a weight plays a role of multiplier. It is known (see, e.g., [6]) that these two spaces are not reducible to each other unlike the classical weighted Lebesgue spaces.

Let us define the aforementioned two weighted grand Lebesgue spaces.

Let $1 < p < \infty$, $\theta > 0$. Let w be a weight function defined on the rectifiable curve Γ of the complex plane, i.e. w be a.e. positive and integrable function on Γ. We define the weighted grand Lebesgue space $L_w^{p),\theta}(\Gamma)$ as follows:

$$L_w^{p),\theta}(\Gamma) = \left\{ f : \|f\|_{L_w^{p),\theta}(\Gamma)} < \infty \right\},$$

where

$$\|f\|_{L_w^{p),\theta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Gamma |f(t)|^{p-\varepsilon} w(t)\, |dt| \right)^{\frac{1}{p-\varepsilon}}.$$

Now, we define another type weighted grand Lebesgue space $\mathcal{L}_w^{p),\theta}(\Gamma)$ by the norm

$$\|f\|_{\mathcal{L}_w^{p),\theta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Gamma |f(t)w(t)|^{p-\varepsilon}\, |dt| \right)^{\frac{1}{p-\varepsilon}}.$$

Both these spaces are non-reflexive, non-separable Banach spaces. It should be noted that in unweighted case when $\theta = 1$ the spaces $L^{p)}$ were introduced by T. Iwaniec and C. Sbordone [7]. More general space $L^{p),\theta}$, $\theta > 0$, was introduced by L. Greco, T. Iwaniec and C. Sbordone [8].

Let Γ be a simple finite rectifiable curve $\Gamma = \{t \in \mathbb{C} : \ t = t(s),\ 0 \le s \le t\}$ with arc-length measure $\nu(t) = s$. In the sequel we use the notation

$$D(t, r) = \Gamma \cap B(t, r), \quad B(t, r) = \left\{ \tau \in \mathbb{C} : \ |\tau - t| < r \right\}, \quad t \in \Gamma, \quad r > 0.$$

Γ is called Carleson if

$$\nu D(t, r) \le c_0 r$$

with $c_0 > 0$ not depending on t and r.

We need also the definition of the Muckenhoupt $\mathcal{A}_p(\Gamma)$ class of weights suited to the curves.

By the definition $w \in \mathcal{A}_p(\Gamma)$ if

$$\sup_{\substack{r>0\\ t\in\Gamma}} \left(\frac{1}{r}\int_{D(t,r)} w(t)\, d|t|\right)\left(\frac{1}{r}\int_{D(t,r)} (w(t))^{1-p'}\, d|t|\right)^{p-1} < \infty.$$

The main results of this paper read as follows:

Theorem 1. *Let* $1 < p < \infty$, $\theta > 0$. *Assume that* $w \in \mathcal{A}_p(\Gamma)$. *Then the operator* $\widetilde{S}_\Gamma$ *is bounded in* $L_w^{p),\theta}(\Gamma)$.

Theorem 2. *Let* $1 < p < \infty$, $\theta > 0$. *If* $w^p \in \mathcal{A}_p(\Gamma)$, *then the operator* $\widetilde{S}_\Gamma$ *is bounded in* $\mathcal{L}_w^{p),\theta}(\Gamma)$.

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