# OPERATORS OF HARMONIC ANALYSIS IN VARIABLE EXPONENT LEBESGUE SPACES. TWO-WEIGHT ESTIMATES 

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#### Abstract

In the paper two-weighted norm estimates with general weights for Hardy-type transforms, maximal functions, potentials and Calderón-Zygmund singular integrals in variable exponent Lebesgue spaces defined on quasimetric measure spaces $(X, d, \mu)$ are established. In particular, we derive integral-type easily verifiable sufficient conditions governing two-weight inequalities for these operators. If exponents of Lebesgue spaces are constants, then most of the derived conditions are simultaneously necessary and sufficient for appropriate inequalities. Examples of weights governing the boundedness of maximal, potential and singular operators in weighted variable exponent Lebesgue spaces are given.


Key words: Variable exponent Lebesgue spaces, Hardy transforms, fractional and singular integrals, quasimetric measure spaces, spaces of homogeneous type, two-weight inequality.
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## Introduction

We study the two-weight problem for Hardy-type, maximal, potentials and singular operators in Lebesgue spaces with non-standard growth defined on quasimetric measure spaces. In particular, our aim is to derive easily verifiable sufficient conditions for the boundedness of these operators in weighted $L^{p(\cdot)}(X)$ spaces which enable us effectively construct examples of appropriate weights. The conditions are simultaneously necessary and sufficient for corresponding inequalities when the weights are of special type and the exponent $p$ of the space is constant. We assume that the exponent $p$ satisfies the local log-Hölder continuity condition and if the diameter of $X$ is infinite, then we suppose that $p$ is constant outside some ball. In the framework of variable exponent analysis such a condition first appeared in the paper [8] where the author established the boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. As far as we know, unfortunately, it is not known an analog of the log-Hölder decay condition (at infinity) for $p: X \rightarrow[1, \infty)$ even in the unweighted case, which is well-known and natural for the Euclidean spaces (see [5], [41, [3]). The local log-Hölder continuity condition for the exponent $p$ together with the log-Hölder decay condition guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ spaces (see e.g., 7]).

A considerable interest of researchers is attracted to the study of mapping properties of integral operators defined on (quasi-)metric measure spaces. Such spaces with doubling measure and all their generalities naturally arise when studying boundary value problems for partial differential equations with variable coefficients, for instance, when the quasimetric might be induced by a differential operator, or tailored to fit kernels of integral operators. The problem of the boundedness of integral operators naturally arises also in the Lebesgue spaces with non-standard growth.

Historically the boundedness of the Hardy-Littlewood maximal, potential and singular operators in $L^{p(\cdot)}$ spaces defined on (quasi)metric measure spaces was derived in [21], [22, [27, [29], 33]- 36], [1] (see also references cited therein).

Weighted inequalities for classical operators in $L_{w}^{p(\cdot)}$ spaces, where $w$ is a power-type weight, were established in the papers [31]-[33], 34]-36, [30, [19], [46, 42], [13] (see also the survey papers [45], [24), while the same problems with general weights for Hardy, maximal, potential and singular operators were studied in [16]-18, [28, [33], 34, [38, [10, [2, 40], 11]. Moreover, in the paper [11] a complete solution of the one-weight problem for maximal functions defined on Euclidean spaces are given in terms of Muckenhoupt-type conditions. Finally we notice that in the paper [18] modular-type sufficient conditions governing the two-weight inequality for maximal and singular operators were established.

It should be emphasized that in the classical Lebesgue spaces the two-weight problem for fractional integrals is already solved (see [26], [25]) but it is often useful to construct concrete examples of weights from transparent and easily verifiable conditions. This problem for singular integrals still remains open. However, some sufficient conditions governing two-weight estimates for the Calderón-Zygmund operators were given in the papers [14, [6] (see also the monographs [15], 49] and references cited therein).

To derive two-weight estimates for maximal, singular and potential operators we use the appropriate inequalities for Hardy-type transforms on $X$ (which are also derived in this paper).

The paper is organized as follows: In Section 1 we give some definitions and auxiliary results regarding quasimetric measure spaces and the variable exponent Lebesgue spaces. Section 2 is devoted to the sufficient conditions governing two-weight inequalities for Hardy-type defined on quasimetric measure spaces, while in Section 3 we study the two-weight problem for potentials defined on quasimetric measure spaces. In Section 4 we discuss weighted estimates for maximal and singular integrals.

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by $c$ or $C$. The symbol $f(x) \approx g(x)$ means that there are positive constants $c_{1}$ and $c_{2}$ independent of $x$ such that the inequality $f(x) \leq c_{1} g(x) \leq c_{2} f(x)$ holds. Throughout the paper by the symbol $p^{\prime}(x)$ is denoted the function $p(x) /(p(x)-1)$.

## 1. PRELIMINARIES

Let $X:=(X, d, \mu)$ be a topological space with a complete measure $\mu$ such that the space of compactly supported continuous functions is dense in $L^{1}(X, \mu)$ and there exists a non-negative real-valued function (quasi-metric) $d$ on $X \times X$ satisfying the conditions:
(i) $d(x, y)=0$ if and only if $x=y$;
(ii) there exists a constant $a_{1}>0$, such that $d(x, y) \leq a_{1}(d(x, z)+d(z, y))$ for all $x, y, z \in X$;
(iii) there exists a constant $a_{0}>0$, such that $d(x, y) \leq a_{0} d(y, x)$ for all $x, y, \in X$.

We assume that the balls $B(x, r):=\{y \in X: d(x, y)<r\}$ are measurable and $0 \leq \mu(B(x, r))<$ $\infty$ for all $x \in X$ and $r>0$; for every neighborhood $V$ of $x \in X$, there exists $r>0$, such that $B(x, r) \subset V$. Throughout the paper we also suppose that $\mu\{x\}=0$ and that

$$
\begin{equation*}
B(x, R) \backslash B(x, r) \neq \emptyset \tag{1}
\end{equation*}
$$

for all $x \in X$, positive $r$ and $R$ with $0<r<R<L$, where

$$
L:=\operatorname{diam}(X)=\sup \{d(x, y): x, y \in X\}
$$

We call the triple $(X, d, \mu)$ a quasimetric measure space. If $\mu$ satisfies the doubling condition $\mu(B(x, 2 r)) \leq c \mu(B(x, r))$, where the positive constant $c$ does not depend on $x \in X$ and $r>0$, then $(X, d, \mu)$ is called a space of homogeneous type (SHT). For the definition and some properties of an $S H T$ see, e.g., 4, 48, 20.

A quasimetric measure space, where the doubling condition is not assumed and may fail, is called a non-homogeneous space.

Notice that the condition $L<\infty$ implies that $\mu(X)<\infty$ because every ball in $X$ has a finite measure.

We say that the measure $\mu$ is upper Ahlfors $Q$ - regular if there is a positive constant $c_{1}$ such that $\mu B(x, r) \leq c_{1} r^{Q}$ for for all $x \in X$ and $r>0$. Further, $\mu$ is lower Ahlfors $q$ - regular if there is a positive constant $c_{2}$ such that $\mu B(x, r) \geq c_{2} r^{q}$ for all $x \in X$ and $r>0$. It is easy to check that if $L<\infty$, then $\mu$ is lower Ahlfors regular (see also, e.g., [22]).

For the boundedness of potential operators in weighted Lebesgue spaces with constant exponents on non-homogeneous spaces we refer, for example, to the monograph [17] (Ch. 6) and references cited therein.

Let $p$ be a non-negative $\mu$ - measurable function on $X$. Suppose that $E$ is a $\mu-$ measurable set in $X$ and $a$ is a constant satisfying the condition $1<a<\infty$. Throughout the paper we use the notation:

$$
\begin{gathered}
p_{-}(E):=\inf _{E} p ; \quad p_{+}(E):=\sup _{E} p ; \quad p_{-}:=p_{-}(X) ; \quad p_{+}:=p_{+}(X) \\
\bar{B}(x, r):=\{y \in X: d(x, y) \leq r\}, \quad k B(x, r):=B(x, k r) ; B_{x y}:=B(x, d(x, y)) ; \\
\bar{B}_{x y}:=\bar{B}(x, d(x, y)) ; g_{B}:=\frac{1}{\mu(B)} \int_{B}|g(x)| d \mu(x) .
\end{gathered}
$$

where $1<p_{-} \leq p_{+}<\infty$.
Assume now that $1 \leq p_{-} \leq p_{+}<\infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(X)$ (sometimes it is denoted by $L^{p(x)}(X)$ ) is the class of all $\mu$-measurable functions $f$ on $X$ for which $S_{p}(f):=\int_{X}|f(x)|^{p(x)} d \mu(x)<\infty$. The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$
\|f\|_{L^{p(\cdot)}(X)}=\inf \left\{\lambda>0: S_{p}(f / \lambda) \leq 1\right\}
$$

It is known (see e.g. 39, 43], [31, [22]) that $L^{p(\cdot)}$ space is a Banach space. For other properties of $L^{p(\cdot)}$ spaces we refer to [47], [39], 43], 45], [24], etc.

Now we introduce several definitions:
Definition 1.1. Let $(X, d, \mu)$ be a quasimetric measure space and let $N \geq 1$ be a constant. Suppose that $p$ satisfy the condition $0<p_{-} \leq p_{+}<\infty$. We say that $p \in \mathcal{P}(N, x)$, where $x \in X$, if there are positive constants $b$ and $c$ (which might be depended on $x$ ) such that

$$
\begin{equation*}
\mu(B(x, N r))^{p_{-}(B(x, r))-p_{+}(B(x, r))} \leq c \tag{2}
\end{equation*}
$$

holds for all $r, 0<r \leq b$. Further, $p \in \mathcal{P}(N)$ if there are a positive constants $b$ and $c$ such that (2) holds for all $x \in X$ and all $r$ satisfying the condition $0<r \leq b$.

Definition 1.2. Let $(X, d, \mu)$ be an $S H T$. Suppose that $0<p_{-} \leq p_{+}<\infty$. We say that $p \in L H(X, x)(p$ satisfies the log-Hölder- type condition at a point $x \in X)$ if there are positive constants $b$ and $c$ (which might be depended on $x$ ) such that

$$
\begin{equation*}
|p(x)-p(y)| \leq \frac{c}{-\ln \left(\mu\left(B_{x y}\right)\right)} \tag{3}
\end{equation*}
$$

holds for all $y$ satisfying the condition $d(x, y) \leq b$. Further, $p \in L H(X)(p$ satisfies the log-Hölder type condition on $X$ ) if there are positive constants $b$ and $c$ such that (3) holds for all $x, y$ with $d(x, y) \leq b$.
Definition 1.3. Let $(X, d, \mu)$ be a quasimetric measure space and let $0<p_{-} \leq p_{+}<\infty$. We say that $p \in \overline{L H}(X, x)$ if there are positive constants $b$ and $c$ (which might be depended on $x$ ) such that

$$
\begin{equation*}
|p(x)-p(y)| \leq \frac{c}{-\ln d(x, y)} \tag{4}
\end{equation*}
$$

for all $y$ with $d(x, y) \leq b$. Further, $p \in \overline{L H}(X)$ if (4) holds for all $x, y$ with $d(x, y) \leq b$.

It is easy to see that if the measure $\mu$ is upper Ahlfors $Q$-regular and $p \in L H(X)$ (resp. $p \in L H(X, x)$ ), then $p \in \overline{L H}(X)$ (resp. $p \in \overline{L H}(X, x)$. Further, if $\mu$ is lower Ahlfors $q$-regular and $p \in \overline{L H}(X)$ (resp. $p \in \overline{L H}(X, x)$ ), then $p \in L H(X)$ (resp. $p \in L H(X, x)$ ).

Remark 1.1. It can be checked easily that if $(X, d, \mu)$ is an SHT, then $\mu B_{x_{0} x} \approx \mu B_{x x_{0}}$.
Remark 1.2. Let $(X, d, \mu)$ be an SHT with $L<\infty$. It is known (see, e.g., [22, [27]) that if $p \in \overline{L H}(X)$ ), then $p \in \mathcal{P}(1)$. Further, if $\mu$ is upper Ahlfors $Q$-regular, then the condition $p \in \mathcal{P}(1)$ implies that $p \in \overline{L H}(X)$.
Proposition 1.4. If $0<p_{-}(X) \leq p_{+}(X)<\infty$ and $p \in L H(X)$ ( resp. $p \in \overline{L H}(X)$ ), then the functions cp $(\cdot)$ and $1 / p(\cdot)$ belong to the class $L H(X)($ resp. $\overline{L H}(X))$. Further if $p \in L H(X, x)$ (resp. $p \in \overline{L H}(X, x)$ ) then $c p(\cdot)$ and $1 / p(\cdot)$ belong to $L H(X, x)$ (resp. $p \in \overline{L H}(X, x)$ ), where $c$ is a positive constant. If $1<p_{-}(X) \leq p_{+}(X)<\infty$ and $p \in L H(X)($ resp. $p \in L H(X) \overline{L H}(X))$, then $p^{\prime} \in L H(X, x)\left(\right.$ resp. $\left.p^{\prime} \in \overline{L H}(X, x)\right)$.

Proof of this statement follows immediately from the definitions of the classes $L H(X, x), L H(X)$, $\overline{L H}(X, x), \overline{L H}(X)$; therefore we omit the details.

Proposition 1.5. Let $(X, d, \mu)$ be an $S H T$ and let $p \in \mathcal{P}(1)$. Then $\left(\mu B_{x y}\right)^{p(x)} \leq c\left(\mu B_{y x}\right)^{p(y)}$, for all $x, y \in X$ with $\mu(B(x, d(x, y))) \leq b$, where $b$ is a small constant, and the constant $c$ does not depend on $x, y \in X$.

Proof. Due to the doubling condition for $\mu$, Remark 1.1, the condition $p \in \mathcal{P}(1)$ and the fact $x \in$ $B\left(y, a_{1}\left(a_{0}+1\right) d(y, x)\right)$ we have the following estimates: $\mu\left(B_{x y}\right)^{p(x)} \leq \mu\left(B\left(y, a_{1}\left(a_{0}+1\right) d(x, y)\right)\right)^{p(x)} \leq$ $c \mu B\left(y, a_{1}\left(a_{0}+1\right) d(x, y)\right)^{p(y)} \leq c\left(\mu B_{y x}\right)^{p(y)}$, which proves the statement.

The proof of the next statement is trivial and follows directly from the definition of the classes $\mathcal{P}(N, x)$ and $\mathcal{P}(N)$. Details are omitted.

Proposition 1.6. Let $(X, d, \mu)$ be a quasimetric measure space and let $x_{0} \in X$. Suppose that $N \geq 1$ be a constant. Then the following statements hold:
(i) If $p \in \mathcal{P}\left(N, x_{0}\right)$ (resp. $p \in \mathcal{P}(N)$ ), then there are positive constants $r_{0}, c_{1}$ and $c_{2}$ such that for all $0<r \leq r_{0}$ and all $y \in B\left(x_{0}, r\right)$ (resp. for all $x_{0}, y$ with $\left.d\left(x_{0}, y\right)<r \leq r_{0}\right)$ we have that $\mu\left(B\left(x_{0}, N r\right)\right)^{p\left(x_{0}\right)} \leq c_{1} \mu\left(B\left(x_{0}, N r\right)\right)^{p(y)} \leq c_{2} \mu\left(B\left(x_{0}, N r\right)\right)^{p\left(x_{0}\right)}$.
(ii) Let $p \in \mathcal{P}\left(N, x_{0}\right)$. Then there are positive constants $r_{0}, c_{1}$ and $c_{2}$ (in general, depending on $x_{0}$ ) such that for all $r\left(r \leq r_{0}\right)$ and all $x, y \in B\left(x_{0}, r\right)$ we have $\mu\left(B\left(x_{0}, N r\right)\right)^{p(x)} \leq$ $c_{1} \mu\left(B\left(x_{0}, N r\right)\right)^{p(y)} \leq c_{2} \mu\left(B\left(x_{0}, N r\right)\right)^{p(x)}$.
(iii) Let $p \in \mathcal{P}(N)$. Then there are positive constants $r_{0}, c_{1}$ and $c_{2}$ such that for all balls $B$ with radius $r\left(r \leq r_{0}\right)$ and all $x, y \in B$, we have $\mu(N B)^{p(x)} \leq c_{1} \mu(N B)^{p(y)} \leq c_{2} \mu(N B)^{p(x)}$.

It is known that (see, e.g., [39, 43) if $f$ is a measurable function on $X$ and $E$ is a measurable subset of $X$, then the following inequalities hold:

$$
\begin{array}{ll}
\|f\|_{L^{p(\cdot)}(E)}^{p_{+}(E)} \leq S_{p}\left(f \chi_{E}\right) \leq\|f\|_{L^{p(\cdot)}(E)}^{p_{-}(E)}, \quad\|f\|_{L^{p(\cdot)}(E)} \leq 1 ; \\
\|f\|_{L^{p(\cdot)}(E)}^{p_{-}(E)} \leq S_{p}\left(f \chi_{E}\right) \leq\|f\|_{L^{p(\cdot)}(E)}^{p_{+}(E)}, \quad\|f\|_{L^{p(\cdot)}(E)}>1 .
\end{array}
$$

Hölder's inequality in variable exponent Lebesgue spaces has the following form:

$$
\int_{E} f g d \mu \leq\left(1 / p_{-}(E)+1 /\left(p^{\prime}\right)_{-}(E)\right)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p^{\prime}(\cdot)}(E)}
$$

Lemma 1.7. Let $(X, d, \mu)$ be an $S H T$.
(i) Let $\beta$ be a measurable function on $X$ such that $\beta_{+}<-1$ and let $r$ be a small positive number. Then there exists a positive constant $c$ independent of $r$ and $x$ such that

$$
\int_{X \backslash B\left(x_{0}, r\right)}\left(\mu B_{x_{0} y}\right)^{\beta(x)} d \mu(y) \leq c \frac{\beta(x)+1}{\beta(x)} \mu\left(B\left(x_{0}, r\right)\right)^{\beta(x)+1}
$$

(ii) Suppose that $p$ and $\alpha$ are measurable functions on $X$ satisfying the conditions $1<p_{-} \leq$ $p_{+}<\infty$ and $\alpha_{-}>1 / p_{-}$. Then there exists a positive constant $c$ such that for all $x \in X$ the inequality

$$
\int_{\bar{B}\left(x_{0}, 2 d\left(x_{0}, x\right)\right)}(\mu B(x, d(x, y)))^{(\alpha(x)-1) p^{\prime}(x)} d \mu(y) \leq c\left(\mu B\left(x_{0}, d\left(x_{0}, x\right)\right)\right)^{(\alpha(x)-1) p^{\prime}(x)+1}
$$

holds.
Proof. Part (i) was proved in [27] (see also [15], p.372, for constant $\beta$ ). The proof of Part (ii) was given in [15] (Lemma 6.5.2, p. 348) but repeating those arguments we can see that it is also true for variable $\alpha$ and $p$. Details are omitted.

Let $M$ be a maximal operator on $X$ given by

$$
M f(x):=\sup _{x \in X, r>0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)}|f(y)| d \mu(y)
$$

Definition 1.8. Let $(X, d, \mu)$ be a quasimetric measure space. We say that $p \in \mathcal{M}(X)$ if the operator $M$ is bounded in $L^{p(\cdot)}(X)$.
L. Diening [8] proved that if $\Omega$ is a bounded domain in $\mathbb{R}^{n}, 1<p_{-} \leq p_{+}<\infty$ and $p$ satisfies the local $\log$-Hölder continuity condition on $\Omega$ (i.e., $|p(x)-p(y)| \leq \frac{c}{-\ln (|x-y|)}$ for all $x, y \in \Omega$ with $|x-y| \leq 1 / 2)$, then the Hardy-Littlewood maximal operator defined on $\Omega$ is bounded in $L^{p(\cdot)}(\Omega)$.

Now we prove the following lemma:
Lemma 1.9. Let $(X, d, \mu)$ be an SHT. Suppose that $0<p_{-} \leq p_{+}<\infty$. Then $p$ satisfies the condition $p \in \mathcal{P}(1)$ (resp. $p \in \mathcal{P}(1, x)$ ) if and only if $p \in L H(X)$ (resp. $p \in L H(X, x)$ ).

Proof. Necessity. Let $p \in \mathcal{P}(1)$ and let $x, y \in X$ with $d(x, y)<c_{0}$ for some positive constant $c_{0}$. Observe that $x, y \in B$, where $B:=B(x, 2 d(x, y))$. By the doubling condition for $\mu$ we have that $\left(\mu B_{x y}\right)^{-|p(x)-p(y)|} \leq c(\mu B)^{-|p(x)-p(y)|} \leq c(\mu B)^{p_{-}(B)-p_{+}(B)} \leq C$, where $C$ is a positive constant which is greater than 1. Taking now the logarithm in the last inequality we have that $p \in L H(X)$. If $p \in \mathcal{P}(1, x)$, then by the same arguments we find that $p \in L H(X, x)$.

Sufficiency. Let $B:=B\left(x_{0}, r\right)$. First observe that If $x, y \in B$, then $\mu B_{x y} \leq c \mu B\left(x_{0}, r\right)$. Consequently, this inequality and the condition $p \in L H(X)$ yield $\left|p_{-}(B)-p_{+}(B)\right| \leq \frac{C}{-\ln \left(c_{0} \mu B\left(x_{0}, r\right)\right)}$. Further, there exists $r_{0}$ such that $0<r_{0}<1 / 2$ and $c_{1} \leq \frac{\ln (\mu(B))}{-\ln \left(c_{0} \ln \mu(B)\right)} \leq c_{2}, 0<r \leq$ $r_{0}$, where $c_{1}$ and $c_{2}$ are positive constants. Hence $(\mu(B))^{p_{-}(B)-p_{+}(B)} \leq(\mu(B))^{\frac{C}{\ln \left(c_{0} \mu(B)\right)}}=$ $\exp \left(\frac{C \ln (\mu(B))}{\ln \left(c_{0} \mu(B)\right)}\right) \leq C$.

Let now $p \in L H(X, x)$ and let $B_{x}:=B(x, r)$ where $r$ is a small number. We have that $p_{+}\left(B_{x}\right)-$ $p(x) \leq \frac{c}{-\ln \left(c_{0} \mu B(x, r)\right)}$ and $p(x)-p_{-}\left(B_{x}\right) \leq \frac{c}{-\ln \left(c_{0} \mu B(x, r)\right)}$ for some positive constant $c_{0}$. Consequently, $\left(\mu\left(B_{x}\right)\right)^{p_{-}\left(B_{x}\right)-p_{+}\left(B_{x}\right)}=\left(\mu\left(B_{x}\right)\right)^{p(x)-p_{+}\left(B_{x}\right)}\left(\mu\left(B_{x}\right)\right)^{p_{-}\left(B_{x}\right)-p(x)} \leq c\left(\mu\left(B_{x}\right)\right)^{\frac{-2 c}{\ln \left(c_{0} \mu B_{x}\right)}} \leq$ $C$.

Definition 1.10. A measure $\mu$ on $X$ is said to satisfy the reverse doubling condition $(\mu \in \operatorname{RDC}(X))$ if there exist constants $A>1$ and $B>1$ such that the inequality $\mu(B(a, A r)) \geq B \mu(B(a, r))$ holds.

Remark 1.3. It is known that if all annulus in $X$ are not empty, then $\mu \in \mathrm{DC}(X)$ implies that $\mu \in \operatorname{RDC}(X)$ (see, e.g., 48]).

In the sequel we will use the notation:

$$
\begin{gathered}
I_{1, k}:=\left\{\begin{array}{l}
B\left(x_{0}, A^{k-1} L / a_{1}\right) \text { if } L<\infty \\
B\left(x_{0}, A^{k-1} / a_{1}\right) \text { if } L=\infty
\end{array},\right. \\
I_{2, k}:=\left\{\begin{array}{l}
\bar{B}\left(x_{0}, A^{k+2} a_{1} L\right) \backslash B\left(x_{0}, A^{k-1} L / a_{1}\right), \text { if } L<\infty \\
\bar{B}\left(x_{0}, A^{k+2} a_{1}\right) \backslash B\left(x_{0}, A^{k-1} / a_{1}\right) \text { if } L=\infty,
\end{array}\right. \\
I_{3, k}:=\left\{\begin{array}{l}
X \backslash B\left(x_{0}, A^{k+2} L a_{1}\right) \text { if } L<\infty \\
X \backslash B\left(x_{0}, A^{k+2} a_{1}\right) \text { if } L=\infty
\end{array}\right. \\
E_{k}:=\left\{\begin{array}{l}
\bar{B}\left(x_{0}, A^{k+1} L\right) \backslash B\left(x_{0}, A^{k} L\right) \text { if } L<\infty \\
\bar{B}\left(x_{0}, A^{k+1}\right) \backslash B\left(x_{0}, A^{k}\right) \text { if } L=\infty
\end{array}\right.
\end{gathered}
$$

where the constant $A$ is defined in the reverse doubling condition and the constant $a_{1}$ is taken from the triangle inequality for the quasimetric $d$.

Lemma 1.11. Let $(X, d, \mu)$ be an SHT. Suppose that there is a point $x_{0} \in X$ such that $p \in$ $L H\left(X, x_{0}\right)$. Then there exist positive constants $r_{0}$ and $C$ ( which might be depended on $\left.x_{0}\right)$ such that for all $r, 0<r \leq r_{0}$, the inequity

$$
\left(\mu B_{A}\right)^{p_{-}\left(B_{A}\right)-p_{+}\left(B_{A}\right)} \leq C
$$

holds, where $B_{A}:=B\left(x_{0}, A r\right) \backslash B\left(x_{0}, r\right)$ and the constant $C$ is independent of $r$ and the constant $A$ is defined in Definition 1.10.

Proof. Let $B:=B\left(x_{0}, r\right)$. First observe that by the doubling and reverse doubling conditions we have that $\mu B_{A}=\mu B\left(x_{0}, A r\right)-\mu B\left(x_{0}, r\right) \geq(B-1) \mu B\left(x_{0}, r\right) \geq c \mu(A B)$. Suppose that $0<r<c_{0}$, where $c_{0}$ is a sufficiently small constant. Then by using Lemma 1.9 we find that $\left(\mu B_{A}\right)^{p_{-}\left(B_{A}\right)-p_{+}\left(B_{A}\right)} \leq c(\mu(A B))^{p_{-}\left(B_{A}\right)-p_{+}\left(B_{A}\right)} \leq c(\mu(A B))^{p_{-}(A B)-p_{+}(A B)} \leq c$.

Lemma 1.12. Let $(X, d, \mu)$ be an SHT and let $1<p_{-}(x) \leq p(x) \leq q(x) \leq q_{+}(X)<\infty$. Suppose that there is a point $x_{0} \in X$ such that $p, q \in L H\left(X, x_{0}\right)$. Assume that $p(x) \equiv p_{c} \equiv$ const, $q(x) \equiv q_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$ if $L=\infty$. Then there exist a positive constant $C$ such that

$$
\sum_{k}\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}\left\|g \chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)} \leq C\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{q^{\prime}(\cdot)}(X)}
$$

for all $f \in L^{p(\cdot)}(X)$ and $g \in L^{q^{\prime}(\cdot)}(X)$.

Proof. Suppose that $L=\infty$. To prove the lemma first observe that $\mu\left(E_{k}\right) \approx \mu B\left(x_{0}, A^{k}\right)$ and $\mu\left(I_{2, k}\right) \approx \mu B\left(x_{0}, A^{k-1}\right)$. This holds because $\mu$ satisfies the reverse doubling condition and, consequently,

$$
\begin{gathered}
\mu E_{k}=\mu\left(\bar{B}\left(x_{0}, A^{k+1}\right) \backslash B\left(x_{0}, A^{k}\right)\right)=\mu \bar{B}\left(x_{0}, A^{k+1}\right)-\mu B\left(x_{0}, A^{k}\right) \\
=\mu \bar{B}\left(x_{0}, A A^{k}\right)-\mu B\left(x_{0}, A^{k}\right) \geq B \mu B\left(x_{0}, A^{k}\right)-\mu B\left(x_{0}, A^{k}\right)=(B-1) \mu B\left(x_{0}, A^{k}\right)
\end{gathered}
$$

Moreover, using the doubling condition we have $\mu E_{k} \leq \mu B\left(x_{0}, A A^{k}\right) \leq c \mu B\left(x_{0}, A^{k}\right)$, where $c>1$. Hence, $\mu E_{k} \approx \mu B\left(x_{0}, A^{k}\right)$.

Further, since we can assume that $a_{1} \geq 1$, we find that

$$
\begin{gathered}
\mu I_{2, k}=\mu\left(\bar{B}\left(x_{0}, A^{k+2} a_{1}\right) \backslash B\left(x_{0}, A^{k-1} / a_{1}\right)\right)=\mu \bar{B}\left(x_{0}, A^{k+2} a_{1}\right)-\mu B\left(x_{0}, A^{k-1} / a_{1}\right) \\
=\mu \bar{B}\left(x_{0}, A A^{k+1} a_{1}\right)-\mu B\left(x_{0}, A^{k-1} / a_{1}\right) \geq B \mu B\left(x_{0}, A^{k+1} a_{1}\right)-\mu B\left(x_{0}, A^{k-1} / a_{1}\right) \\
\geq B^{2} \mu B\left(x_{0}, A^{k} / a_{1}\right)-\mu B\left(x_{0}, A^{k-1} / a_{1}\right) \geq B^{3} \mu B\left(x_{0}, A^{k-1} / a_{1}\right)-\mu B\left(x_{0}, A^{k-1} / a_{1}\right) \\
=\left(B^{3}-1\right) \mu B\left(x_{0}, A^{k-1} / a_{1}\right)
\end{gathered}
$$

Moreover, using the doubling condition we have $\mu I_{2, k} \leq \mu \bar{B}\left(x_{0}, A^{k+2} r\right) \leq c \mu B\left(x_{0}, A^{k+1} r\right) \leq$ $c^{2} \mu B\left(x_{0}, A^{k} / a_{1}\right) \leq c^{3} \mu B\left(x_{0}, A^{k-1} / a_{1}\right)$. This gives the estimates $\left(B^{3}-1\right) \mu B\left(x_{0}, A^{k-1} / a_{1}\right) \leq$ $\mu\left(I_{2, k}\right) \leq c^{3} \mu B\left(x_{0}, A^{k-1} / a_{1}\right)$.

For simplicity assume that $a=1$. Suppose that $m_{0}$ is an integer such that $\frac{A^{m_{0}-1}}{a_{1}}>1$. Let us split the sum as follows:

$$
\sum_{i}\left\|f \chi_{I_{2, i}}\right\|_{L^{p(\cdot)}(X)} \cdot\left\|g \chi_{I_{2, i}}\right\|_{L^{q^{\prime}(\cdot)}(X)}=\sum_{i \leq m_{0}}(\cdots)+\sum_{i>m_{0}}(\cdots)=: J_{1}+J_{2} .
$$

Since $p(x) \equiv p_{c}=$ const, $q(x)=q_{c}=$ const outside the ball $B\left(x_{0}, 1\right)$, by using Hölder's inequality and the fact that $p_{c} \leq q_{c}$, we have

$$
J_{2}=\sum_{i>m_{0}}\left\|f \chi_{I_{2, i}}\right\|_{L^{p_{c}}(X)} \cdot\left\|g \chi_{I_{2, i}}\right\|_{L^{\left(q_{c}\right)^{\prime}(X)}} \leq c\|f\|_{L^{p(\cdot)}(X)} \cdot\|g\|_{L^{q^{\prime}(\cdot)}(X)}
$$

Let us estimate $J_{1}$. Suppose that $\|f\|_{L^{p(\cdot)}(X)} \leq 1$ and $\|g\|_{L^{q^{\prime}(\cdot)}(X)} \leq 1$. Also, by Proposition 1.4 we have that $1 / q^{\prime} \in L H\left(X, x_{0}\right)$. Therefore by Lemma 1.11 and the fact that $1 / q^{\prime} \in$ $L H\left(X, x_{0}\right)$ we obtain that $\mu\left(I_{2, k}\right)^{\frac{1}{q_{+}\left(I_{2, k}\right)}} \approx\left\|\chi_{I_{2, k}}\right\|_{L^{q(\cdot)}(X)} \approx \mu\left(I_{2, k}\right)^{\frac{1}{q_{-}\left(I_{2, k}\right)}}$ and $\mu\left(I_{2, k}\right)^{\frac{1}{q_{+}^{\prime}\left(I_{2, k}\right)}} \approx$ $\left\|\chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)(X)}} \approx \mu\left(I_{2, k}\right)^{\frac{1}{q_{-}^{\prime}\left(I_{k}\right)}}$, where $k \leq m_{0}$. Further, observe that these estimates and Hölder's inequality yield the following chain of inequalities:

$$
\begin{aligned}
& J_{1} \leq c \sum_{k \leq m_{0}} \int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)} \cdot\left\|g \chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{q(\cdot)}(X)} \cdot\left\|\chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)}} \chi_{E_{k}}(x) d \mu(x) \\
& =c \int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)} \sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)} \cdot\left\|g \chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{q(\cdot)}(X)} \cdot\left\|\chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)(X)}}} \chi_{E_{k}}(x) d \mu(x) \\
& \quad \leq c\left\|\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{q(\cdot)}(X)}} \chi_{E_{k}}(x)\right\|_{L^{q \cdot \cdot}\left(\bar{B}\left(x_{0}, A^{m_{0}+1}\right)\right)} \\
& \times\left\|\sum_{k \leq m_{0}} \frac{\left\|g \chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{q^{\prime}(\cdot)}(X)}} \chi_{E_{k}}(x)\right\|_{L^{q^{\prime}(\cdot)}\left(\bar{B}\left(x_{0}, A^{m_{0}+1}\right)\right)}=: c S_{1}(f) \cdot S_{2}(g)
\end{aligned}
$$

Now we claim that $S_{1}(f) \leq c I(f)$, where

$$
I(f):=\left\|\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{I_{2 k}}\right\|_{L^{p(\cdot)}(X)}} \chi_{E_{k}(\cdot)}\right\|_{L^{p(\cdot)}\left(\bar{B}\left(x_{0}, A^{m_{0}+1}\right)\right)}
$$

and the positive constant $c$ does not depend on $f$. Indeed, suppose that $I(f) \leq 1$. Then taking into account Lemma 1.11 we have that

$$
\begin{aligned}
& \sum_{k \leq m_{0}} \frac{1}{\mu\left(I_{2, k}\right)} \int_{E_{k}}\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p(x)} d \mu(x) \\
& \leq c \int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)}\left(\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}} \chi_{E_{k}(x)}\right)^{p(x)} d \mu(x) \leq c .
\end{aligned}
$$

Consequently, since $p(x) \leq q(x), E_{k} \subseteq I_{2, k}$ and $\|f\|_{L^{p(\cdot)}(X)} \leq 1$, we find that

$$
\sum_{k \leq m_{0}} \frac{1}{\mu\left(I_{2, k}\right)} \int_{E_{k}}\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{q(x)} d \mu(x) \leq \sum_{k \leq m_{0}} \frac{1}{\mu\left(I_{2, k}\right)} \int_{E_{k}}\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p(x)} d \mu(x) \leq c .
$$

This implies that $S_{1}(f) \leq c$. Thus the desired inequality is proved. Further, let us introduce the following function:

$$
\mathbb{P}(y):=\sum_{k \leq 2} p_{+}\left(\chi_{I_{2, k}}\right) \chi_{E_{k}(y)}
$$

It is clear that $p(y) \leq \mathbb{P}(y)$ because $E_{k} \subset I_{2, k}$. Hence

$$
I(f) \leq c\left\|\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{I_{2 k}}\right\|_{L^{p(\cdot)}(X)}} \chi_{E_{k}(\cdot)}\right\|_{L^{\mathbb{P}(\cdot)}\left(\bar{B}\left(x_{0}, A^{m_{0}+1}\right)\right)}
$$

for some positive constant $c$. Then by using the this inequality, the definition of the function $\mathbb{P}$, the condition $p \in L H(X)$ and the obvious estimate $\left\|\chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p_{+}\left(I_{2, k}\right)} \geq c \mu\left(I_{2, k}\right)$, we find that

$$
\begin{gathered}
\int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)}\left(\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}} \chi_{E_{k}(x)}\right)^{\mathbb{P}(x)} d \mu(x) \\
=\int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)}\left(\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p_{+}\left(I_{2, k}\right)}}{\left\|\chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p_{+}\left(I_{2, k}\right)}} \chi_{E_{k}(x)}\right) d \mu(x) \\
\leq c \int_{\bar{B}\left(x_{0}, A^{m_{0}+1}\right)}\left(\sum_{k \leq m_{0}} \frac{\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)}(X)}^{p_{-}\left(I_{2, k}\right)}}{\mu\left(I_{2, k}\right)} \chi_{E_{k}(x)}\right) d \mu(x) \leq c \sum_{k \leq m_{0}}\left\|f \chi_{I_{2, k}}\right\|_{L^{p(\cdot)(X)}}^{p_{+}\left(I_{2, k}\right)} \\
\leq c \sum_{k \leq m_{0}} \int_{I_{2, k}}|f(x)|^{p(x)} d \mu(x) \leq c \int_{X}|f(x)|^{p(x)} d \mu(x) \leq c .
\end{gathered}
$$

Consequently, $I(f) \leq c\|f\|_{L^{p(\cdot)}(X)}$. Hence, $S_{1}(f) \leq c\|f\|_{L^{p(\cdot)}(X)}$. Analogously taking into account the fact that $q^{\prime} \in D L(X)$ and arguing as above we find that $S_{2}(g) \leq c\|g\|_{L^{q^{\prime}(\cdot)}(X)}$. Thus summarizing these estimates we conclude that

$$
\sum_{i \leq m_{0}}\left\|f \chi_{I_{i}}\right\|_{L^{p(\cdot)}(X)}\left\|g \chi_{I_{i}}\right\|_{L^{q^{\prime}(\cdot)}(X)} \leq c\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{q^{\prime}(\cdot)}(X)}
$$

The next statement for metric measure spaces was proved in [22] (see also [27], 29] for quasimetric measure spaces).

Theorem A. Let $(X, d, \mu)$ be an SHT and let $\mu(X)<\infty$. Suppose that $1<p_{-} \leq p_{+}<\infty$ and $p \in \mathcal{P}(1)$. Then $M$ is bounded in $L^{p(\cdot)}(X)$.

For the following statement we refer to [23]:
Theorem B. Let $(X, d, \mu)$ be an SHT and let $L=\infty$. Suppose that $1<p_{-} \leq p_{+}<\infty$ and $p \in \mathcal{P}(1)$. Suppose also that $p=p_{c}=$ const outside some ball $B:=B\left(x_{0}, R\right)$. Then $M$ is bounded in $L^{p(\cdot)}(X)$.

## 2. HARDY-TYPE TRANSFORMS

In this section we derive two-weight estimates for the operators:

$$
T_{v, w} f(x)=v(x) \int_{B_{x_{0} x}} f(y) w(y) d \mu(y) \text { and } T_{v, w}^{\prime} f(x)=v(x) \int_{X \backslash \bar{B}_{x_{0} x}} f(y) w(y) d \mu(y)
$$

Let $a$ is a positive constant and let $p$ be a measurable function defined on $X$. Let us introduce the notation:

$$
\begin{gathered}
p_{0}(x):=p_{-}\left(\bar{B}_{x_{0} x}\right) ; \quad \widetilde{p}_{0}(x):= \begin{cases}p_{0}(x) & \text { if } d\left(x_{0}, x\right) \leq a \\
p_{c}=\text { const } & \text { if } d\left(x_{0}, x\right)>a\end{cases} \\
p_{1}(x):=p_{-}\left(\bar{B}\left(x_{0}, a\right) \backslash B_{x_{0} x}\right) ; \widetilde{p}_{1}(x):= \begin{cases}p_{1}(x) & \text { if } d\left(x_{0}, x\right) \leq a \\
p_{c}=\text { const } & \text { if } d\left(x_{0}, x\right)>a .\end{cases}
\end{gathered}
$$

Remark 2.1. If we deal with a quasi-metric measure space with $L<\infty$, then we will assume that $a=L$. Obviously, $\widetilde{p}_{0} \equiv p_{0}$ and $\widetilde{p}_{1} \equiv p_{1}$ in this case.

Theorem 2.1. Let $(X, d, \mu)$ be a quasi-metric measure space. Assume that $p$ and $q$ are measurable functions on $X$ satisfying the condition $1<p_{-} \leq \widetilde{p}_{0}(x) \leq q(x) \leq q_{+}<\infty$. In the case when $L=\infty$ suppose that $p \equiv p_{c} \equiv$ const, $q \equiv q_{c} \equiv$ const, outside some ball $\bar{B}\left(x_{0}, a\right)$. If the condition

$$
A_{1}:=\sup _{0 \leq t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}(v(x))^{q(x)}\left(\int_{d\left(x_{0}, x\right) \leq t} w^{\left(\widetilde{p}_{0}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(\widetilde{p}_{0}\right)^{\prime}(x)}} d \mu(x)<\infty
$$

hold, then $T_{v, w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.
Proof. Here we use the arguments of the proofs of Theorem 1.1.4 in [15] (see p. 7) and of Theorem 2.1 in [17. First we notice that $p_{-} \leq p_{0}(x) \leq p(x)$ for all $x \in X$. Let $f \geq 0$ and let $S_{p}(f) \leq 1$. First assume that $L<\infty$. We denote

$$
I(s):=\int_{d\left(x_{0}, y\right)<s} f(y) w(y) d \mu(y) \text { for } s \in[0, L]
$$

Suppose that $I(L)<\infty$. Then $I(L) \in\left(2^{m}, 2^{m+1}\right]$ for some $m \in \mathbb{Z}$. Let us denote $s_{j}:=\sup \{s:$ $\left.I(s) \leq 2^{j}\right\}, j \leq m$, and $s_{m+1}:=L$. Then $\left\{s_{j}\right\}_{j=-\infty}^{m+1}$ is a non-decreasing sequence. It is easy to check that $I\left(s_{j}\right) \leq 2^{j}, I(s)>2^{j}$ for $s>s_{j}$, and $2^{j} \leq \int_{s_{j} \leq d\left(x_{0}, y\right) \leq s_{j+1}} f(y) w(y) d \mu(y)$. If $\beta:=\lim _{j \rightarrow-\infty} s_{j}$, then $d\left(x_{0}, x\right)<L$ if and only if $d\left(x_{0}, x\right) \in[0, \beta] \cup \bigcup_{j=-\infty}^{m}\left(s_{j}, s_{j+1}\right]$. If $I(L)=\infty$ then we take $m=\infty$. Since $0 \leq I(\beta) \leq I\left(s_{j}\right) \leq 2^{j}$ for every $j$, we have that $I(\beta)=0$. It is obvious that
$X=\bigcup_{j \leq m}\left\{x: s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}\right\}$. Further, we have that

$$
\begin{aligned}
& S_{q}\left(T_{v, w} f\right)= \int_{X}\left(T_{v, w} f(x)\right)^{q(x)} d \mu(x)=\int_{X}\left(v(x) \int_{B\left(x_{0}, d\left(x_{0}, x\right)\right)} f(y) w(y) d \mu(y)\right)^{q(x)} d \mu(x) \\
&=\int_{X}(v(x))^{q(x)}\left(\int_{B\left(x_{0}, d\left(x_{0}, x\right)\right)} f(y) w(y) d \mu(y)\right)^{q(x)} d \mu(x) \\
& \leq \sum_{j=-\infty}^{m} \int_{s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}}(v(x))^{q(x)}\left(\int_{d\left(x_{0}, y\right)<s_{j+1}} f(y) w(y) d \mu(y)\right)^{q(x)} d \mu(x) .
\end{aligned}
$$

Notice that $I\left(s_{j+1}\right) \leq 2^{j+1} \leq 4 \underset{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}}{ } w(y) f(y) d \mu(y)$. Consequently, by this estimate and Hölder's inequality with respect to the exponent $p_{0}(x)$ we find that

$$
\begin{aligned}
& S_{q}\left(T_{v, w} f\right) \leq c \sum_{j=-\infty}^{m} \int_{s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}}(v(x))^{q(x)}\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} f(y) w(y) d \mu(y)\right)^{q(x)} d \mu(x) \\
& \leq c \sum_{j=-\infty_{s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}}^{m}}(v(x))^{q(x)} J_{k}(x) d \mu(x),
\end{aligned}
$$

where

$$
J_{k}(x):=\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} f(y)^{p_{0}(x)} d \mu(y)\right)^{\frac{q(x)}{p_{0}(x)}}\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} w(y)^{\left(p_{0}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} .
$$

Observe now that $q(x) \geq p_{0}(x)$. Hence, this fact and the condition $S_{p}(f) \leq 1$ imply that

$$
\begin{gathered}
J_{k}(x) \leq c\left(\int_{\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\} \cap\{y: f(y) \leq 1\}} f(y)^{p_{0}(x)} d \mu(y)+\underset{\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\} \cap\{y: f(y)>1\}}{ } f(y)^{p(y)} d \mu(y)\right)^{\frac{q(x)}{p_{0}(x)}} \\
\times\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} w(y)^{\left(p_{0}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} \\
\leq c\left(\mu\left(\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\}\right)+\int_{\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\} \cap\{y: f(y)>1\}} f(y)^{p(y)} d \mu(y)\right) \\
\times\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} w(y)^{\left(p_{0}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} .
\end{gathered}
$$

It follows now that

$$
\begin{array}{r}
S_{q}\left(T_{v, w} f\right) \leq c\left(\sum_{j=-\infty}^{m} \mu\left(\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\}\right) \int_{s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}} v(x)^{q(x)}\right. \\
\times\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} w(y)^{\left(p_{0}^{\prime}\right)(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x) \\
\times \sum_{j=-\infty}^{m}\left(\int_{y:\left\{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\} \cap\{y: f(y)>1\}} f(y)^{p(y)} d \mu(y)\right) \\
\left.\times \int_{s_{j}<d\left(x_{0}, x\right) \leq s_{j+1}} v(x)^{q(x)}\left(\int_{s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}} w(y)^{\left(p_{0}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x)\right):=c\left(N_{1}+N_{2}\right) .
\end{array}
$$

It is obvious that

$$
N_{1} \leq A_{1} \sum_{j=-\infty}^{m+1} \mu\left(\left\{y: s_{j-1} \leq d\left(x_{0}, y\right) \leq s_{j}\right\}\right) \leq C A_{1}
$$

and

$$
N_{2} \leq A_{1} \sum_{j=-\infty^{\left\{y: s_{j-1}\right.}}^{m+1} \int_{\left.d\left(x_{0}, y\right) \leq s_{j}\right\}} f(y)^{p(y)} d \mu(y)=C \int_{X}(f(y))^{p(y)} d \mu(y)=A_{1} S_{p}(f) \leq A_{1}
$$

Finally $S_{q}\left(T_{v, w} f\right) \leq c\left(c A_{1}+A_{1}\right)<\infty$. Thus $T_{v, w}$ is bounded if $A_{1}<\infty$.
Let us now suppose that $L=\infty$. We have

$$
\begin{array}{r}
T_{v, w} f(x)=\chi_{B\left(x_{0}, a\right)}(x) v(x) \int_{B_{x_{0} x}} f(y) w(y) d \mu(y) \\
+\chi_{X \backslash B\left(x_{0}, a\right)}(x) v(x) \int_{B_{x_{0} x}} f(y) w(y) d \mu(y)=: T_{v, w}^{(1)} f(x)+T_{v, w}^{(2)} f(x)
\end{array}
$$

By using already proved result for $L<\infty$ and the fact that $\operatorname{diam}\left(B\left(x_{0}, a\right)\right)<\infty$ we find that $\left\|T_{v, w}^{(1)} f\right\|_{L^{q(\cdot)}\left(B\left(x_{0}, a\right)\right)} \leq c\|f\|_{L^{p(\cdot)}\left(B\left(x_{0}, a\right)\right)} \leq c$ because

$$
A_{1}^{(a)}:=\sup _{0 \leq t \leq a} \int_{t<d\left(x_{0}, x\right) \leq a}(v(x))^{q(x)}\left(\int_{d\left(x_{0}, x\right) \leq t} w^{\left(p_{0}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x) \leq A_{1}<\infty
$$

Further, observe that

$$
\begin{gathered}
T_{v, w}^{(2)} f(x)=\chi_{X \backslash B\left(x_{0}, a\right)}(x) v(x) \int_{B_{x_{0} x}} f(y) w(y) d \mu(y)=\chi_{X \backslash B\left(x_{0}, a\right)}(x) v(x) \int_{d\left(x_{0}, y\right) \leq a} f(y) w(y) d \mu(y) \\
+\chi_{X \backslash B\left(x_{0}, a\right)}(x) v(x) \int_{a \leq d\left(x_{0}, y\right) \leq d\left(x_{0}, x\right)} f(y) w(y) d \mu(y)=: T_{v, w}^{(2,1)} f(x)+T_{v, w}^{(2,2)} f(x) .
\end{gathered}
$$

It is easy to see that (see also Theorem 1.1.3 or 1.1.4 of [15]) the condition

$$
\bar{A}_{1}^{(a)}:=\sup _{t \geq a}\left(\int_{d\left(x_{0}, x\right) \geq t}(v(x))^{q_{c}} d \mu(x)\right)^{\frac{1}{q_{c}}}\left(\int_{a \leq d\left(x_{0}, y\right) \leq t} w(y)^{\left(p_{c}\right)^{\prime}} d \mu(y)\right)^{\frac{1}{\left(p_{c}\right)^{\prime}}}<\infty
$$

guarantees the boundedness of the operator

$$
T_{v, w} f(x)=v(x) \int_{\substack{a \leq d\left(x_{0}, y\right)<d\left(x_{0}, x\right)}} f(y) w(y) d \mu(y)
$$

from $L^{p_{c}}\left(X \backslash B\left(x_{0}, a\right)\right)$ to $L^{q_{c}}\left(X \backslash B\left(x_{0}, a\right)\right)$. Thus $T_{v, w}^{(2,2)}$ is bounded. It remains to prove that $T_{v, w}^{(2,1)}$ is bounded. We have

$$
\begin{aligned}
& \left\|T_{v, w}^{(2,1)} f\right\|_{L^{p(\cdot)}(X)}=\left(\int_{\left(B\left(x_{0}, a\right)\right)^{c}} v(x)^{q_{c}} d \mu(x)\right)^{\frac{1}{q_{c}}}\left(\int_{\bar{B}\left(x_{0}, a\right)} f(y) w(y) d \mu(y)\right) \\
& \quad \leq\left(\int_{\left(B\left(x_{0}, a\right)\right)^{c}} v(x)^{q_{c}} d \mu(x)\right)^{\frac{1}{q_{c}}}\|f\|_{L^{p(\cdot)}\left(\bar{B}\left(x_{0}, a\right)\right)}\|w\|_{L^{p^{\prime}(\cdot)}\left(\bar{B}\left(x_{0}, a\right)\right)} .
\end{aligned}
$$

Observe now that the condition $A_{1}<\infty$ guarantees that the integral

$$
\int_{\left(B\left(x_{0}, a\right)\right)^{c}} v(x)^{q_{c}} d \mu(x)
$$

is finite. Moreover, $N:=\|w\|_{L^{p^{\prime}(\cdot)}\left(\bar{B}\left(x_{0}, a\right)\right)}<\infty$. Indeed, we have that

$$
N \leq \begin{cases}\left(\int_{\bar{B}\left(x_{0}, a\right)} w(y)^{p^{\prime}(y)} d \mu(y)\right)^{\frac{1}{\left(p_{-}\left(\bar{B}\left(x_{0}, a\right)\right)\right)^{\prime}}} & \text { if }\|w\|_{L^{p^{\prime}(\cdot)}\left(\bar{B}\left(x_{0}, a\right)\right)} \leq 1 \\ \left(\int_{\bar{B}\left(x_{0}, a\right)} w(y)^{p^{\prime}(y)} d \mu(y)\right)^{\frac{1}{\left(p_{+}\left(\bar{B}\left(x_{0}, a\right)\right)\right)^{\prime}}} & \text { if }\|w\|_{L^{p^{\prime}(\cdot)}\left(\bar{B}\left(x_{0}, a\right)\right.}>1\end{cases}
$$

Further,

$$
\int_{\bar{B}\left(x_{0}, a\right)} w(y)^{p^{\prime}(y)} d \mu(y)=\int_{\bar{B}\left(x_{0}, a\right) \cap\{w \leq 1\}} w(y)^{p^{\prime}(y)} d \mu(y)+\int_{\bar{B}\left(x_{0}, a\right) \cap\{w>1\}} w(y)^{p^{\prime}(y)} d \mu(y):=I_{1}+I_{2} .
$$

For $I_{1}$, we have that $I_{1} \leq \mu\left(\bar{B}\left(x_{0}, a\right)\right)<\infty$. Since $L=\infty$ and condition (1) holds, there exists a point $y_{0} \in X$ such that $a<d\left(x_{0}, y_{0}\right)<2 a$. Consequently, $\bar{B}\left(x_{0}, a\right) \subset \bar{B}\left(x_{0}, d\left(x_{0}, y_{0}\right)\right)$ and $p(y) \geq p_{-}\left(\bar{B}\left(x_{0}, d\left(x_{0}, y_{0}\right)\right)\right)=p_{0}\left(y_{0}\right)$, where $y \in \bar{B}\left(x_{0}, a\right)$. Consequently, the condition $A_{1}<\infty$ yields $I_{2} \leq \int_{\bar{B}\left(x_{0}, a\right)} w(y)^{\left(p_{0}\right)^{\prime}\left(y_{0}\right)} d y<\infty$. Finally we have that $\left\|T_{v, w}^{(2,1)} f\right\|_{L^{p(\cdot)}(X)} \leq C$. Hence, $T_{v, w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

The proof of the following statement is similar to that of Theorem 2.1; therefore we omit it (see also the proofs of Theorem 1.1.3 in [15] and Theorems 2.6 and 2.7 in [17] for similar arguments).

Theorem 2.2. Let $(X, d, \mu)$ be a quasi-metric measure space. Assume that $p$ and $q$ are measurable functions on $X$ satisfying the condition $1<p_{-} \leq \widetilde{p}_{1}(x) \leq q(x) \leq q_{+}<\infty$. If $L=\infty$, then we assume that $p \equiv p_{c} \equiv$ const, $q \equiv q_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$. If

$$
B_{1}=\sup _{0 \leq t \leq L} \int_{d\left(x_{0}, x\right) \leq t}(v(x))^{q(x)}\left(\int_{t \leq d\left(x_{0}, x\right) \leq L} w^{\left(\widetilde{p}_{1}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(\widetilde{p}_{1}\right)^{\prime}(x)}} d \mu(x)<\infty
$$

then $T_{v, w}^{\prime}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Remark 2.2. If $p \equiv$ const, then the condition $A_{1}<\infty$ in Theorem 2.1 (resp. $B_{1}<\infty$ in Theorem 2.2) is also necessary for the boundedness of $T_{v, w}\left(\right.$ resp. $\left.T_{v, w}^{\prime}\right)$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. See [15], pp.4-5, for the details.

## 3. Potentials

In this section we discuss two-weight estimates for the potential operators $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ on quasi-metric measure spaces, where $0<\alpha_{-} \leq \alpha_{+}<1$. If $\alpha \equiv$ const, then we denote $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ by $T_{\alpha}$ and $I_{\alpha}$ respectively.

The boundedness of Riesz potential operators in $L^{p(\cdot)}(\Omega)$ spaces, where $\Omega$ is a domain in $\mathbb{R}^{n}$ was established in [9, [44, [7, [3].

For the following statement we refer to [34]:
Theorem C. Let $(X, d, \mu)$ be an SHT. Suppose that $1<p_{-} \leq p_{+}<\infty$ and $p \in \mathcal{P}(1)$. Assume that if $L=\infty$, then $p \equiv$ const outside some ball. Let $\alpha$ be a constant satisfying the condition $0<\alpha<1 / p_{+}$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. Then $T_{\alpha}$ is bounded in $L^{p(\cdot)}(X)$.

Theorem D 29]. Let $(X, d, \mu)$ be a non-homogeneous space with $L<\infty$ and let $N$ be a constant defined by $N=a_{1}\left(1+2 a_{0}\right)$, where the constants $a_{0}$ and $a_{1}$ are taken from the definition of the quasi-metric $d$. Suppose that $1<p_{-}<p_{+}<\infty, p, \alpha \in \mathcal{P}(N)$ and that $\mu$ is upper Ahlfors 1 -regular. We define $q(x)=\frac{p(x)}{1-\alpha(x) p(x)}$, where $0<\alpha_{-} \leq \alpha_{+}<1 / p_{-}$. Then $I_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

For the statements and their proofs of this section we keep the notation of the previous sections and, in addition, introduce the new notation:

$$
\begin{array}{r}
v_{\alpha}^{(1)}(x):=v(x)\left(\mu B_{x_{0} x}\right)^{\alpha-1}, w_{\alpha}^{(1)}(x):=w^{-1}(x) ; v_{\alpha}^{(2)}(x):=v(x) ; \\
w_{\alpha}^{(2)}(x):=w^{-1}(x)\left(\mu B_{x_{0} x}\right)^{\alpha-1} ; \\
F_{x}:=\begin{array}{l}
\left\{y \in X: \frac{d\left(x_{0}, y\right) L}{A^{2} a_{1}} \leq d\left(x_{0}, y\right) \leq A^{2} L a_{1} d\left(x_{0}, x\right)\right\}, \text { if } L<\infty, \\
\left\{y \in X: \frac{d\left(x_{0}, y\right)}{A^{2} a_{1}} \leq d\left(x_{0}, y\right) \leq A^{2} a_{1} d\left(x_{0}, x\right)\right\}, \text { if } L=\infty,
\end{array},
\end{array}
$$

where $A$ and $a_{1}$ are constants defined in Definition 1.10 and the triangle inequality for $d$ respectively. We begin this section with the following general-type statement:

Theorem 3.1. Let $(X, d, \mu)$ be an SHT without atoms. Suppose that $1<p_{-} \leq p_{+}<\infty$ and $\alpha$ is a constant satisfying the condition $0<\alpha<1 / p_{+}$. Let $p \in \mathcal{P}(1)$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. Further, if $L=\infty$, then we assume that $p \equiv p_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$. Then the inequality

$$
\begin{equation*}
\left\|v\left(T_{\alpha} f\right)\right\|_{L^{q(\cdot)}(X)} \leq c\|w f\|_{L^{p(\cdot)}(X)} \tag{5}
\end{equation*}
$$

holds if the following three conditions are satisfied:
(a) $T_{v_{\alpha}^{(1)}, w_{\alpha}^{(1)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;
(b) $T_{v_{\alpha}^{(2)}, w_{\alpha}^{(2)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;
(c) there is a positive constant b such that one of the following inequality holds: 1) $v_{+}\left(F_{x}\right) \leq$ $b w(x)$ for $\mu-$ a.e. $x \in X \quad$; 2) $v(x) \leq b w_{-}\left(F_{x}\right)$ for $\mu-$ a.e. $x \in X$.

Proof. For simplicity suppose that $L<\infty$. The proof for the case $L=\infty$ is similar to that of the previous case. Recall that the sets $I_{i, k}, i=1,2,3$ and $E_{k}$ are defined in Section 1 . Let $f \geq 0$ and let $\|g\|_{L^{q^{\prime}(\cdot)}(X)} \leq 1$. We have

$$
\begin{array}{r}
\int_{X}\left(T_{\alpha} f\right)(x) g(x) v(x) d \mu(x)=\sum_{k=-\infty}^{0} \int_{E_{k}}\left(T_{\alpha} f\right)(x) g(x) v(x) d \mu(x) \\
\leq \sum_{k=-\infty}^{0} \int_{E_{k}}\left(T_{\alpha} f_{1, k}\right)(x) g(x) v(x) d \mu(x)+\sum_{k=-\infty_{E_{k}}}^{0} \int_{\alpha}\left(T_{\alpha} f_{2, k}\right)(x) g(x) v(x) d \mu(x) \\
+\sum_{k=-\infty}^{0} \int_{E_{k}}\left(T_{\alpha} f_{3, k}\right)(x) g(x) v(x) d \mu(x):=S_{1}+S_{2}+S_{3},
\end{array}
$$

where $f_{1, k}=f \cdot \chi_{I_{1, k}}, f_{2, k}=f \cdot \chi_{I_{2, k}}, f_{3, k}=f \cdot \chi_{I_{3, k}}$.
Observe that if $x \in E_{k}$ and $y \in I_{1, k}$, then $d\left(x_{0}, y\right) \leq d\left(x_{0}, x\right) / A a_{1}$. Consequently, the triangle inequality for $d$ yields $d\left(x_{0}, x\right) \leq A^{\prime} a_{1} a_{0} d(x, y)$, where $A^{\prime}=A /(A-1)$. Hence, by using Remark 1.1 we find that $\mu\left(B_{x_{0} x}\right) \leq c \mu\left(B_{x y}\right)$. Applying now condition (a) we have that

$$
S_{1} \leq c\left\|\left(\mu B_{x_{0} x}\right)^{\alpha-1} v(x) \int_{B_{x_{0} x}} f(y) d \mu(y)\right\|_{L^{q(x)}(X)}\|g\|_{L^{q^{\prime}(\cdot)}(X)} \leq c\|f\|_{L^{p(\cdot)}(X)}
$$

Further, observe that if $x \in E_{k}$ and $y \in I_{3, k}$, then $\mu\left(B_{x_{0} y}\right) \leq c \mu\left(B_{x y}\right)$. By condition (b) we find that $S_{3} \leq c\|f\|_{L^{p(\cdot)}(X)}$.

Now we estimate $S_{2}$. Suppose that $v_{+}\left(F_{x}\right) \leq b w(x)$. Theorem C and Lemma 1.12 yield

$$
\begin{array}{r}
S_{2} \leq \sum_{k}\left\|\left(T_{\alpha} f_{2, k}\right)(\cdot) \chi_{E_{k}}(\cdot) v(\cdot)\right\|_{L^{q(\cdot)}(X)}\left\|g \chi_{E_{k}}(\cdot)\right\|_{L^{q^{\prime}(\cdot)}(X)} \\
\leq \sum_{k}\left(v_{+}\left(E_{k}\right)\right)\left\|\left(T_{\alpha} f_{2, k}\right)(\cdot)\right\|_{L^{q(\cdot)}(X)}\left\|g(\cdot) \chi_{E_{k}}(\cdot)\right\|_{L^{q^{\prime}(\cdot)}(X)} \\
\quad \leq c \sum_{k}\left(v_{+}\left(E_{k}\right)\right)\left\|f_{2, k}\right\|_{L^{p(\cdot)}(X)}\left\|g(\cdot) \chi_{E_{k}}(\cdot)\right\|_{L^{q^{\prime}(\cdot)}(X)} \\
\leq c \sum_{k}\left\|f_{2, k}(\cdot) w(\cdot) \chi_{I_{2, k}}(\cdot)\right\|_{L^{p(\cdot)}(X)}\left\|g(\cdot) \chi_{E_{k}}(\cdot)\right\|_{L^{q^{\prime}(\cdot)}(X)} \\
\leq c\|f(\cdot) w(\cdot)\|_{L^{p(\cdot)}(X)}\|g(\cdot)\|_{L^{q^{\prime}(\cdot)}(X)} \leq c\|f(\cdot) w(\cdot)\|_{L^{p(\cdot)}(X)} .
\end{array}
$$

The estimate of $S_{2}$ for the case when $v(x) \leq b w_{-}\left(F_{x}\right)$ is similar to that of the previous one. Details are omitted.

Theorems 3.1, 2.1 and 2.2 imply the following statement:
Theorem 3.2. Let $(X, d, \mu)$ be an SHT. Suppose that $1<p_{-} \leq p_{+}<\infty$ and $\alpha$ is a constant satisfying the condition $0<\alpha<1 / p_{+}$. Let $p \in \mathcal{P}(1)$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. If $L=\infty$, then we suppose that $p \equiv p_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$. Then inequality (5) holds if the following three conditions are satisfied:
(i) $\quad P_{1}:=\sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v(x)}{\left(\mu\left(B_{x_{0} x}\right)\right)^{1-\alpha}}\right)^{q(x)}\left(\int_{d\left(x_{0}, y\right) \leq t} w^{-\left(\widetilde{p}_{0}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(\widetilde{P}_{0}\right)^{\prime}(x)}} d \mu(x)<\infty$;
(ii) $\quad P_{2}:=\sup _{0<t \leq L} \int_{d\left(x_{0}, x\right) \leq t}(v(x))^{q(x)}\left(\int_{t<d\left(x_{0}, y\right) \leq L}\left(w(y)\left(\mu B_{x_{0} y}\right)^{1-\alpha}\right)^{-\left(\widetilde{p}_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(\tilde{p}_{1}\right)^{\prime}(x)}} d \mu(x)<\infty$,
(iii) condition (c) of Theorem 3.1 holds.

Remark 3.1. If $p=p_{c} \equiv$ const on $X$, then the conditions $P_{i}<\infty, i=1,2$, are necessary for (5). Necessity of the condition $P_{1}<\infty$ follows by taking the test function $f=$ $w^{-\left(p_{c}\right)^{\prime}} \chi_{B\left(x_{0}, t\right)}$ in (5) and observing that $\mu B_{x y} \leq c \mu B_{x_{0} x}$ for those $x$ and $y$ which satisfy the conditions $d\left(x_{0}, x\right) \geq t$ and $d\left(x_{0}, y\right) \leq t$ (see also [15], Theorem 6.6.1, p. 418 for the similar arguments), while necessity of the condition $P_{2}<\infty$ can be derived by choosing the test function $f(x)=w^{-\left(p_{c}\right)^{\prime}}(x) \chi_{X \backslash B\left(x_{0}, t\right)}(x)\left(\mu B_{x_{0} x}\right)^{(\alpha-1)\left(\left(p_{c}\right)^{\prime}-1\right)}$ and taking into account the estimate $\mu B_{x y} \leq \mu B_{x_{0} y}$ for $d\left(x_{0}, x\right) \leq t$ and $d\left(x_{0}, y\right) \geq t$.

The next statement follows in the same manner as the previous one. In this case Theorem D is used instead of Theorem C. The proof is omitted.

Theorem 3.3. Let $(X, d, \mu)$ be a non-homogeneous space with $L<\infty$. Let $N$ be a constant defined by $N=a_{1}\left(1+2 a_{0}\right)$. Suppose that $1<p_{-} \leq p_{+}<\infty, p, \alpha \in \mathcal{P}(N)$ and that $\mu$ is upper Ahlfors 1-regular. We define $q(x)=\frac{p(x)}{1-\alpha(x) p(x)}$, where $0<\alpha_{-} \leq \alpha_{+}<1 / p_{+}$. Then the inequality

$$
\begin{equation*}
\left\|v(\cdot)\left(I_{\alpha(\cdot)} f\right)(\cdot)\right\|_{L^{q(\cdot)}(X)} \leq c\|w(\cdot) f(\cdot)\|_{L^{p(\cdot)}(X)} \tag{6}
\end{equation*}
$$

holds if

$$
\begin{aligned}
& \text { (i) } \sup _{0 \leq t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v(x)}{\left(d\left(x_{0}, x\right)\right)^{1-\alpha(x)}}\right)^{q(x)}\left(\int_{\bar{B}\left(x_{0}, t\right)} w^{-\left(p_{0}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x)<\infty ; \\
& \text { (ii) } \sup _{0 \leq t \leq \frac{L}{B}} \int_{\left(x_{0}, t\right)}(v(x))^{q(x)}\left(\int_{t<d\left(x_{0}, y\right) \leq L}\left(w(y) d\left(x_{0}, y\right)^{1-\alpha(y)}\right)^{-\left(p_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{1}\right)^{\prime}(x)}} d \mu(x)<\infty,
\end{aligned}
$$

(iii) condition (c) of Theorem 3.1 is satisfied.

Remark 3.2. It is easy to check that if $p$ and $\alpha$ are constants, then conditions (i) and (ii) in Theorem 3.3 are also necessary for (6). This follows easily by choosing appropriate test functions in (6) (see also Remark 3.1)

Theorem 3.4. Let $(X, d, \mu)$ be an SHT without atoms. Let $1<p_{-} \leq p_{+}<\infty$ and let $\alpha$ be a constant with the condition $0<\alpha<1 / p_{+}$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. Assume that $p$ has a minimum at $x_{0}$ and that $p \in L H(X)$. Suppose also that if $L=\infty$, then $p$ is constant outside some ball $B\left(x_{0}, a\right)$. Let $v$ and $w$ be positive increasing functions on $(0,2 L)$. Then the inequality

$$
\begin{equation*}
\left\|v\left(d\left(x_{0}, \cdot\right)\right)\left(T_{\alpha} f\right)(\cdot)\right\|_{L^{q(\cdot)}(X)} \leq c\left\|w\left(d\left(x_{0}, \cdot\right)\right) f(\cdot)\right\|_{L^{p(\cdot)}(X)} \tag{7}
\end{equation*}
$$

holds if

$$
\begin{aligned}
I_{1} & :=\sup _{0<t \leq L} I_{1}(t):=\sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v\left(d\left(x_{0}, x\right)\right)}{\left(\mu\left(B_{x_{0} x}\right)\right)^{1-\alpha}}\right)^{q(x)} \\
& \times\left(\int w^{-\left(\widetilde{p}_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{\left(\tilde{p}_{0}\right)^{\prime}(x)}} d \mu(x)<\infty \\
& d\left(x_{0}, y\right) \leq t
\end{aligned}
$$

for $L=\infty ;$

$$
J_{1}:=\sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v\left(d\left(x_{0}, x\right)\right)}{\left(\mu\left(B_{x_{0} x}\right)\right)^{1-\alpha}}\right)^{q(x)}\left(\int_{d\left(x_{0}, y\right) \leq t} w^{-p^{\prime}\left(x_{0}\right)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{p^{\prime}\left(x_{0}\right)}} d \mu(x)<\infty
$$

for $L<\infty$.

Proof. Let $L=\infty$. Observe that by Lemma 1.9 the condition $p \in \operatorname{LH}(X)$ implies $p \in \mathcal{P}(1)$. We will show that the condition $I_{1}<\infty$ implies the inequality $\frac{v\left(A^{2} a_{1} t\right)}{w(t)} \leq C$ for all $t>0$, where $A$ and $a_{1}$ are constants defined in Definition 1.10 and the triangle inequality for $d$ respectively. Indeed, let us assume that $t \leq b_{1}$, where $b_{1}$ is a small positive constant. Then, taking into account the monotonicity of $v$ and $w$, and the facts that $\widetilde{p}_{0}(x)=p_{0}(x)$ (for small $\left.d\left(x_{0}, x\right)\right)$ and $\mu \in \operatorname{RDC}(X)$, we have

$$
\begin{gathered}
I_{1}(t) \geq \int_{A^{2} a_{1} t \leq d\left(x_{0}, x\right)<A^{3} a_{1} t}\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q(x)}\left(\mu B\left(x_{0}, t\right)\right)^{\left(\alpha-1 / p_{0}(x)\right) q(x)} d \mu(x) \\
\geq\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q_{-}} \int_{A^{2} a_{1} t \leq d\left(x_{0}, x\right)<A^{3} a_{1} t}\left(\mu B\left(x_{0}, t\right)\right)^{\left(\alpha-1 / p_{0}(x)\right) q(x)} d \mu(x) \geq c\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q_{-}} .
\end{gathered}
$$

Hence, $\bar{c}:=\varlimsup_{t \rightarrow 0} \frac{v\left(A^{2} a_{1} t\right)}{w(t)}<\infty$. Further, if $t>b_{2}$, where $b_{2}$ is a large number, then since $p$ and $q$ are constants, for $d\left(x_{0}, x\right)>t$, we have that

$$
\begin{array}{r}
I_{1}(t) \geq\left(\int_{A^{2} a_{1} t \leq d\left(x_{0}, x\right)<A^{3} a_{1} t} v\left(d\left(x_{0}, x\right)\right)^{q_{c}}\left(\mu B\left(x_{0}, t\right)\right)^{(\alpha-1) q_{c}} d \mu(x)\right) \\
\times\left(\int_{B\left(x_{0}, t\right)} w^{-\left(p_{c}\right)^{\prime}}(x) d \mu(x)\right)^{q_{c} /\left(p_{c}\right)^{\prime}} d \mu(x) \\
\geq C\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q_{c}} \int_{A^{2} a_{1} t \leq d\left(x_{0}, x\right)<A^{3} a_{1} t}\left(\mu B\left(x_{0}, t\right)\right)^{\left(\alpha-1 / p_{c}\right) q_{c}} d \mu(x) \geq c\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q_{c}} .
\end{array}
$$

In the last inequality we used the fact that $\mu$ satisfies the reverse doubling condition.
Now we show that the condition $I_{1}<\infty$ implies

$$
\begin{aligned}
& \sup _{t>0} I_{2}(t):=\sup _{t>0} \int_{d\left(x_{0}, x\right) \leq t}\left(v\left(d\left(x_{0}, x\right)\right)\right)^{q(x)}\left(\int_{d\left(x_{0}, y\right)>t} w^{-\left(\widetilde{p}_{1}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right)\right. \\
&\left.\times\left(\mu\left(B_{x_{0} y}\right)\right)^{(\alpha-1)\left(\widetilde{p}_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(\widetilde{p}_{1}\right)^{\prime}(x)}} d \mu(x)<\infty
\end{aligned}
$$

Due to monotonicity of functions $v$ and $w$, the condition $p \in L H(X)$, Proposition 1.4, Lemma 1.7, Lemma 1.9 and the assumption that $p$ has a minimum at $x_{0}$, we find that for all $t>0$,

$$
\begin{array}{r}
I_{2}(t) \leq \int_{d\left(x_{0}, x\right) \leq t}\left(\frac{v(t)}{w(t)}\right)^{q(x)}\left(\mu\left(B\left(x_{0}, t\right)\right)\right)^{\left(\alpha-1 / p\left(x_{0}\right)\right) q(x)} d \mu(x) \\
\leq c \int_{d\left(x_{0}, x\right) \leq t}\left(\frac{v(t)}{w(t)}\right)^{q(x)}\left(\mu\left(B\left(x_{0}, t\right)\right)\right)^{\left(\alpha-1 / p\left(x_{0}\right)\right) q\left(x_{0}\right)} d \mu(x) \\
\leq c\left(\int_{d\left(x_{0}, x\right) \leq t}\left(\frac{v\left(A^{2} a_{1} t\right)}{w(t)}\right)^{q(x)} d \mu(x)\right)\left(\mu\left(B\left(x_{0}, t\right)\right)\right)^{-1} \leq C .
\end{array}
$$

Now Theorem 3.2 completes the proof.
Theorem 3.5. Let $(X, d, \mu)$ be an SHT with $L<\infty$. Suppose that $p, q$ and $\alpha$ are measurable functions on $X$ satisfying the conditions: $1<p_{-} \leq p(x) \leq q(x) \leq q_{+}<\infty$ and $1 / p_{-}<\alpha_{-} \leq$
$\alpha_{+}<1$. Assume that there is a point $x_{0}$ such that $\mu\left\{x_{0}\right\}=0$ and $p, q, \alpha \in L H\left(X, x_{0}\right)$. Suppose also that $w$ is a positive increasing function on $(0,2 L)$. Then the inequality

$$
\left\|\left(T_{\alpha(\cdot)} f\right) v\right\|_{L^{q(\cdot)}(X)} \leq c\left\|w\left(d\left(x_{0}, \cdot\right)\right) f(\cdot)\right\|_{L^{p(\cdot)}(X)}
$$

holds if the following two conditions are satisfied:

$$
\begin{array}{r}
\widetilde{I}_{1}:=\sup _{0<t \leq L} \int_{t \leq d\left(x_{0}, x\right) \leq L}\left(\frac{v(x)}{\left(\mu B_{x_{0} x}\right)^{1-\alpha(x)}}\right)^{q(x)} \\
\times\left(\int_{d\left(x_{0}, x\right) \leq t} w^{-\left(p_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x)<\infty ; \\
\widetilde{I}_{2}:=\sup _{0<t \leq L} \int_{d\left(x_{0}, x\right) \leq t}(v(x))^{q(x)}\left(\int _ { t \leq d ( x _ { 0 } , x ) \leq L } \left(w\left(d\left(x_{0}, y\right)\right)\right.\right. \\
\left.\left.\times\left(\mu B_{x_{0} y}\right)^{1-\alpha(x)}\right)^{-\left(p_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{1}\right)^{\prime}(x)}} d \mu(x)<\infty .
\end{array}
$$

Proof. For simplicity assume that $L=1$. First observe that by Lemma 1.9 we have $p, q, \alpha \in \mathcal{P}(1)$. Suppose that $f \geq 0$ and $S_{p}\left(w\left(d\left(x_{0}, \cdot\right)\right) f(\cdot)\right) \leq 1$. We will show that $S_{q}\left(v\left(T_{\alpha(\cdot)} f\right)\right) \leq C$.

We have

$$
\begin{aligned}
& S_{q}\left(v T_{\alpha(\cdot)} f\right) \leq C_{q}\left[\int_{X}\left(v(x) \int_{d\left(x_{0}, y\right) \leq d\left(x_{0}, x\right) /\left(2 a_{1}\right)} f(y)\left(\mu B_{x y}\right)^{\alpha(x)-1} d \mu(y)\right)^{q(x)} d \mu(x)\right. \\
& \quad+\int_{X}\left(v(x) \int_{d\left(x_{0}, x\right) /\left(2 a_{1}\right) \leq d\left(x_{0}, y\right) \leq 2 a_{1} d\left(x_{0}, x\right)} f(y)\left(\mu B_{x y}\right)^{\alpha(x)-1} d \mu(y)\right)^{q(x)} d \mu(x) \\
& \left.+\int_{X}\left(v(x) \int_{d\left(x_{0}, y\right) \geq 2 a_{1} d\left(x_{0}, x\right)} f(y)\left(\mu B_{x y}\right)^{\alpha(x)-1} d \mu(y)\right)^{q(x)} d \mu(x)\right]:=C_{q}\left[I_{1}+I_{2}+I_{3}\right] .
\end{aligned}
$$

First observe that by virtue of the doubling condition for $\mu$, Remark 1.1 and simple calculation we find that $\mu\left(B_{x_{0} x}\right) \leq c \mu\left(B_{x y}\right)$. Taking into account this estimate and Theorem 2.1 we have that

$$
I_{1} \leq c \int_{X}\left(\frac{v(x)}{\left(\mu B_{x_{0} x}\right)^{1-\alpha(x)}} \int_{d\left(x_{0}, y\right)<d\left(x_{0}, x\right)} f(y) d \mu(y)\right)^{q(x)} d \mu(x) \leq C
$$

Further, it is easy to see that if $d\left(x_{0}, y\right) \geq 2 a_{1} d\left(x_{0}, x\right)$, then the triangle inequality for $d$ and the doubling condition for $\mu$ yield that $\mu B_{x_{0} y} \leq c \mu B_{x y}$. Hence due to Proposition 1.5 we see that $\left(\mu B_{x_{0} y}\right)^{\alpha(x)-1} \geq c\left(\mu B_{x y}\right)^{\alpha(y)-1}$ for such $x$ and $y$. Therefore, Theorem 2.2 implies that $I_{3} \leq C$.

It remains to estimate $I_{2}$. Let us denote:

$$
E^{(1)}(x):=\bar{B}_{x_{0} x} \backslash B\left(x_{0}, d\left(x_{0}, x\right) /\left(2 a_{1}\right)\right) ; \quad E^{(2)}(x):=\bar{B}\left(x_{0}, 2 a_{1} d\left(x_{0}, x\right)\right) \backslash B_{x_{0} x}
$$

Then we have that

$$
\begin{gathered}
I_{2} \leq C\left[\int_{X}\left[v(x) \int_{E^{(1)}(x)} f(y)\left(\mu B_{x y}\right)^{\alpha(x)-1} d \mu(y)\right]^{q(x)} d \mu(x)\right. \\
\left.+\int_{X}\left[v(x) \int_{E^{(2)}(x)} f(y)\left(\mu B_{x y}\right)^{\alpha(x)-1} d \mu(y)\right]^{q(x)} d \mu(x)\right]:=c\left[I_{21}+I_{22}\right] .
\end{gathered}
$$

Using Hölder's inequality for the classical Lebesgue spaces we find that

$$
\begin{gathered}
I_{21} \leq \int_{X} v^{q(x)}(x)\left(\int_{E^{(1)}(x)} w^{p_{0}(x)}\left(d\left(x_{0}, y\right)\right)(f(y))^{p_{0}(x)} d \mu(y)\right)^{q(x) / p_{0}(x)} \\
\times\left(\int_{E^{(1)}(x)} w^{-\left(p_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right)\left(\mu B_{x y}\right)^{(\alpha(x)-1)\left(p_{0}\right)^{\prime}(x)} d \mu(y)\right)^{q(x) /\left(p_{0}\right)^{\prime}(x)} d \mu(x) .
\end{gathered}
$$

Denote the first inner integral by $J^{(1)}$ and the second one by $J^{(2)}$.
By using the fact that $p_{0}(x) \leq p(y)$, where $y \in E^{(1)}(x)$, we see that $J^{(1)} \leq \mu\left(B_{x_{0} x}\right)+$ $\int_{E^{(1)}(x)}(f(y))^{p(y)}\left(w\left(d\left(x_{0}, y\right)\right)\right)^{p(y)} d \mu(y)$, while by applying Lemma 1.7 , for $J^{(2)}$, we have that

$$
\begin{gathered}
J^{(2)} \leq c w^{-\left(p_{0}\right)^{\prime}(x)}\left(\frac{d\left(x_{0}, x\right)}{2 a_{1}}\right) \int_{E^{(1)}(x)}\left(\mu B_{x y}\right)^{(\alpha(x)-1)\left(p_{0}\right)^{\prime}(x)} d \mu(y) \\
\leq c w^{-\left(p_{0}\right)^{\prime}(x)}\left(\frac{d\left(x_{0}, x\right)}{2 a_{1}}\right)\left(\mu B_{x_{0} x}\right)^{(\alpha(x)-1)\left(p_{0}\right)^{\prime}(x)+1}
\end{gathered}
$$

Summarizing these estimates for $J^{(1)}$ and $J^{(2)}$ we conclude that

$$
\begin{array}{r}
I_{21} \leq \int_{X} v^{q(x)}(x)\left(\mu B_{x_{0} x}\right)^{q(x) \alpha(x)} w^{-q(x)}\left(\frac{d\left(x_{0}, x\right)}{2 a_{1}}\right) d \mu(x)+\int_{X} v^{q(x)}(x) \\
\times\left(\int_{E^{(1)}(x)} w^{p(y)}\left(d\left(x_{0}, y\right)\right)(f(y))^{p(y)} d \mu(y)\right)^{q(x) / p_{0}(x)}\left(\mu B_{x_{0} x}\right)^{q(x)\left(\alpha(x)-1 / p_{0}(x)\right)} \\
\times w^{-q(x)}\left(\frac{d\left(x_{0}, x\right)}{2 a_{1}}\right) d \mu(x)=: I_{21}^{(1)}+I_{21}^{(2)} .
\end{array}
$$

By applying monotonicity of $w$, the reverse doubling property for $\mu$ with the constants $A$ and $B$ (see Remark 1.3), and the condition $\widetilde{I}_{1}<\infty$ we have that

$$
\begin{aligned}
& \left.I_{21}^{(1)} \leq c \sum_{k=-\infty_{\bar{B}}\left(x_{0}, A^{k}\right) \backslash B\left(x_{0}, A^{k-1}\right)}^{0} \int_{B(x)^{q(x)}\left(x_{0}, \frac{A^{k-1}}{2 a_{1}}\right)} w^{-\left(p_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime \prime}(x)}} \\
& \quad \times\left(\mu B_{x_{0}, x}\right)^{\frac{q(x)}{p_{0}(x)}+(\alpha(x)-1) q(x)} d \mu(x) \leq c \sum_{k=-\infty}^{0}\left(\mu \bar{B}\left(x_{0}, A^{k}\right)\right)^{q_{-} / p_{+}} \\
& \times \int_{\bar{B}\left(x_{0}, A^{k}\right) \backslash B\left(x_{0}, A^{k-1}\right)}^{v(x)^{q(x)}\left(\int_{B\left(x_{0}, A^{k}\right)} w^{-\left(p_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}}} \\
& \times\left(\mu B_{x_{0}, x}\right)^{q(x)(\alpha(x)-1)} d \mu(x) \leq c \sum_{k=-\infty}^{0}\left(\mu \bar{B}\left(x_{0}, A^{k}\right) \backslash B\left(x_{0}, A^{k-1}\right)\right)^{q_{-} / p_{+}} \\
& \leq c \sum_{k=-\infty}^{0} \int_{\mu \bar{B}\left(x_{0}, A^{k}\right) \backslash B\left(x_{0}, A^{k-1}\right)}\left(\mu B_{x_{0}, x}\right)^{q_{-} / p_{+}-1} d \mu(y) \\
& \leq c \int_{X}\left(\mu B_{x_{0}, x}\right)^{q_{-} / p_{+}-1} d \mu(y)<\infty .
\end{aligned}
$$

Due to the facts that $q(x) \geq p_{0}(x), S_{p}\left(w\left(d\left(x_{0}, \cdot\right) f(\cdot)\right)\right) \leq 1, \widetilde{I}_{1}<\infty$ and $w$ is increasing, for $I_{21}^{(2)}$, we find that

$$
\begin{gathered}
I_{21}^{(2)} \leq c \sum_{k=-\infty}^{0}\left(\int_{\mu \bar{B}\left(x_{0}, A^{k+1} a_{1}\right) \backslash B\left(x_{0}, A^{k-2}\right)} w^{p(y)}\left(d\left(x_{0}, y\right)\right)(f(y))^{p(y)} d \mu(y)\right) \\
\times\left(\int_{\mu \bar{B}\left(x_{0}, A^{k}\right) \backslash B\left(x_{0}, A^{k-1}\right)}^{v^{q(x)}(x)\left(\int_{B\left(x_{0}, A^{k-1}\right)} w^{-\left(p_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}}}\right. \\
\left.\times\left(\mu B_{x_{0}, x}\right)^{(\alpha(x)-1) q(x)} d \mu(x)\right) \leq c S_{p}\left(f(\cdot) w\left(d\left(x_{0}, \cdot\right)\right) \leq c .\right.
\end{gathered}
$$

Analogously, it follows the estimate for $I_{22}$. In this case we use the condition $\widetilde{I}_{2}<\infty$ and the fact that $p_{1}(x) \leq p(y)$ when $d\left(x_{0}, y\right) \leq d\left(x_{0}, y\right)<2 a_{1} d\left(x_{0}, x\right)$. The details are omitted. The theorem is proved.

Taking into account the proof of Theorem 3.5 we can easily derive the following statement proof of which is omitted:

Theorem 3.6. Let $(X, d, \mu)$ be an $S H T$ with $L<\infty$. Suppose that $p, q$ and $\alpha$ are measurable functions on $X$ satisfying the conditions $1<p_{-} \leq p(x) \leq q(x) \leq q_{+}<\infty$ and $1 / p_{-}<\alpha_{-} \leq \alpha_{+}<$ 1. Assume that there is a point $x_{0}$ such that $p, q, \alpha \in L H\left(X, x_{0}\right)$ and $p$ has a minimum at $x_{0}$. Let $v$ and $w$ be positive increasing function on $(0,2 L)$ satisfying the condition $J_{1}<\infty$ ( see Theorem 3.4 ). Then inequality (7) is fulfilled.

Theorem 3.7. Let $(X, d, \mu)$ be an SHT with $L<\infty$ and let $\mu$ be upper Ahlfors 1-regular. Suppose that $1<p_{-} \leq p_{+}<\infty$ and that $p \in \overline{L H}(X)$. Let $p$ have a minimum at $x_{0}$. Assume that $\alpha$ is constant satisfying the condition $\alpha<1 / p_{+}$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. If $v$ and $w$ are positive increasing functions on $(0,2 L)$ satisfying the condition

$$
E:=\sup _{0 \leq t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v\left(d\left(x_{0}, x\right)\right)}{\left(d\left(x_{0}, x\right)\right)^{1-\alpha}}\right)^{q(x)}\left(\int_{d\left(x_{0}, x\right) \leq t} w^{-\left(p_{0}\right)^{\prime}\left(d\left(x_{0}, x\right)\right)}(y) d \mu(y)\right)^{\frac{q(x)}{\left(p_{0}\right)^{\prime}(x)}} d \mu(x)<\infty
$$

then the inequality

$$
\left\|v\left(d\left(x_{0}, \cdot\right)\right)\left(I_{\alpha} f\right)(\cdot)\right\|_{L^{q(\cdot)}(X)} \leq c\left\|w\left(d\left(x_{0}, \cdot\right)\right) f(\cdot)\right\|_{L^{p(\cdot)}(X)}
$$

holds.
Proof is similar to that of Theorem 3.4. We only discuss some details. First observe that due to Remark 1.2 we have that $p \in \mathcal{P}(N)$, where $N=a_{1}\left(1+2 a_{0}\right)$. It is easy to check that the condition $E<\infty$ implies that $\frac{v\left(A^{2} a_{1} t\right)}{w(t)} \leq C$ for all t , where the constant $A$ is defined in Definition 1.10 and $a_{1}$ is from the triangle inequality for $d$. Further, Lemmas $1.7,1.9$, the fact that $p$ has a minimum at $x_{0}$ and the inequality

$$
\int_{d\left(x_{0}, y\right)>t}\left(d\left(x_{0}, y\right)\right)^{(\alpha-1)\left(p_{1}\right)^{\prime}(x)} d \mu(y) \leq c t^{(\alpha-1)\left(p_{1}\right)^{\prime}(x)+1}
$$

where the constant $c$ does not depend on $t$ and $x$, yield that

$$
\sup _{0 \leq t \leq L} \int_{d\left(x_{0}, x\right) \leq t}\left(v\left(d\left(x_{0}, x\right)\right)\right)^{q(x)}\left(\int_{d\left(x_{0}, y\right)>t}\left(\frac{w\left(d\left(x_{0}, y\right)\right)}{\left(d\left(x_{0}, y\right)\right)^{1-\alpha}}\right)^{-\left(p_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{q(x)}{\left(p_{1}\right)^{\prime}(x)}} d \mu(x)<\infty
$$

Theorem 3.3 completes the proof. $\square$
Example 3.8. Let $v(t)=t^{\gamma}$ and $w(t)=t^{\beta}$, where $\gamma$ and $\beta$ are constants satisfying the condition $0 \leq \beta<1 /\left(p_{-}\right)^{\prime}, \gamma \geq \max \left\{0,1-\alpha-\frac{1}{q_{+}}-\frac{q_{-}}{q_{+}}\left(-\beta+\frac{1}{\left(p_{-}\right)^{\prime}}\right)\right\}$. Then $(v, w)$ satisfies the conditions of Theorem 3.4.

## 4. Maximal and Singular Operators

Let

$$
K f(x)=p \cdot v \cdot \int_{X} k(x, y) f(y) d \mu(y)
$$

where $k: X \times X \backslash\{(x, x): x \in X\} \rightarrow \mathbb{R}$ be a measurable function satisfying the conditions:

$$
\begin{gathered}
|k(x, y)| \leq \frac{c}{\mu B(x, d(x, y))}, \quad x, y \in X, x \neq y \\
\left|k\left(x_{1}, y\right)-k\left(x_{2}, y\right)\right|+\left|k\left(y, x_{1}\right)-k\left(y, x_{2}\right)\right| \leq c \omega\left(\frac{d\left(x_{2}, x_{1}\right)}{d\left(x_{2}, y\right)}\right) \frac{1}{\mu B\left(x_{2}, d\left(x_{2}, y\right)\right)}
\end{gathered}
$$

for all $x_{1}, x_{2}$ and $y$ with $d\left(x_{2}, y\right) \geq c d\left(x, x_{2}\right)$, where $\omega$ is a positive non-decreasing function on $(0, \infty)$ which satisfies the $\Delta_{2}$ condition: $\omega(2 t) \leq c \omega(t)(t>0)$; and the Dini condition: $\int_{0}^{1}(\omega(t) / t) d t<\infty$.

We also assume that for some constant $s, 1<s<\infty$, and all $f \in L^{s}(X)$ the limit $K f(x)$ exists almost everywhere on $X$ and that $K$ is bounded in $L^{s}(X)$.

It is known (see, e.g., [15, Ch. 7) that if $r$ is constant such that $1<r<\infty,(X, d, \mu)$ is an SHT and the weight function $w \in A_{r}(X)$, i.e.

$$
\sup _{B}\left(\frac{1}{\mu(B)} \int_{B} w(x) d \mu(x)\right)\left(\frac{1}{\mu(B)} \int_{B} w^{1-r^{\prime}}(x) d \mu(x)\right)^{r-1}<\infty
$$

where the supremum is taken over all balls $B$ in $X$, then the one-weight inequality $\left\|w^{1 / r} K f\right\|_{L^{r}(X)} \leq$ $c\left\|w^{1 / r} f\right\|_{L^{r}(X)}$ holds.

The boundedness of Calderón-Zygmund operators in $L^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ was establish in [12].
Theorem E 37]. Let $(X, d, \mu)$ be an SHT. Suppose that $p \in \mathcal{P}(1)$. Then the singular operator $K$ is bounded in $L^{p(\cdot)}(X)$.

Before formulating the main results of this section we introduce the notation:

$$
\bar{v}(x):=\frac{v(x)}{\mu\left(B_{x_{0} x}\right)}, \quad \widetilde{w}(x):=\frac{1}{w(x)}, \quad \widetilde{w}_{1}(x):=\frac{1}{w(x) \mu\left(B_{x_{0} x}\right)}
$$

The following statements follows in the same way as Theorem 3.1 was proved. In this case Theorem 1.2 (for the maximal operator) and Theorem E (for singular integrals) are used instead of Theorem C. Details are omitted.

Theorem 4.1. Let $(X, d, \mu)$ be an SHT and let $1<p_{-} \leq p_{+}<\infty$. Further suppose that $p \in \mathcal{P}(1)$. If $L=\infty$, then we assume that $p$ is constant outside some ball $B\left(x_{0}, a\right)$. Then the inequality

$$
\begin{equation*}
\|v(N f)\|_{L^{p(\cdot)}(X)} \leq C\|w f\|_{L^{p(\cdot)}(X)} \tag{8}
\end{equation*}
$$

where $N$ is $M$ or $K$, holds if the following three conditions are satisfied:
(a) $T_{\bar{v}, \widetilde{w}}$ is bounded in $L^{p(\cdot)}(X)$;
(b) $T_{v, \widetilde{w}_{1}}^{\prime}$ is bounded in $L^{p(\cdot)}(X)$;
(c) there is a positive constant $b$ such that one of the following two conditions hold: 1) $v_{+}\left(F_{x}\right) \leq$ $b w(x) \mu-a . e . \quad x \in X ; 2) v(x) \leq b w_{-}\left(F_{x}\right) \mu-a . e . \quad x \in X$, where $F_{x}$ is the set depended on $x$ which is defined in Section 3.

The next two statements are direct consequences of Theorems 4.1, 2.1 and 2.2.
Theorem 4.2. Let $(X, d, \mu)$ be an SHT and let $1<p_{-} \leq p_{+}<\infty$. Further suppose that $p \in \mathcal{P}(1)$. If $L=\infty$, then we assume that $p \equiv p_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$. Let $N$ be $M$ or $K$. Then inequality (8) holds if:

$$
\begin{aligned}
& \text { (i) } \sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v(x)}{\mu B_{x_{0} x}}\right)^{p(x)}\left(\int_{\bar{B}\left(x_{0}, t\right)} w^{-\left(\widetilde{p}_{0}\right)^{\prime}(x)}(y) d \mu(y)\right)^{\frac{p(x)}{\left(\widetilde{p}_{0}\right)^{\prime}(x)}} d \mu(x)<\infty ; \\
& \text { (ii) } \sup _{0<t \leq L} \int_{\bar{B}\left(x_{0}, t\right)}(v(x))^{p(x)}\left(\int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{w(y)}{\mu B_{x_{0} y}}\right)^{-\left(\widetilde{p}_{1}\right)^{\prime}(x)} d \mu(y)\right)^{\frac{p(x)}{\left(\widetilde{p}_{1}\right)^{\prime}(x)}} d \mu(x)<\infty ;
\end{aligned}
$$

(iii) condition (c) of the previous theorem is satisfied.

Remark 4.1. It is known (see [14]) that if $p \equiv$ const, then conditions (i) and (ii) (written for $X=\mathbb{R}$, the Euclidean distance and the Lebesgue measure) of Theorem 4.2 are also necessary for the two-weight inequality

$$
\|v(H f)\|_{L^{p(\cdot)}(\mathbb{R})} \leq C\|w f\|_{L^{p(\cdot)}(\mathbb{R})}
$$

where $H$ is the Hilbert transform on $\mathbb{R}:(H f)(x)=$ p.v. $\int_{\mathbb{R}} \frac{f(t)}{x-t} d t$.
Remark 4.2. If $p \equiv$ const and $N=M$, then condition (i) of Theorem 4.2 is necessary for (8). This follows from the obvious estimate $M f(x) \geq \frac{c}{\mu\left(B_{x_{0} x}\right)} \int_{B_{x_{0} x}} f(y) d \mu(y)(f \geq 0)$ and Remark 2.2.
Theorem 4.3. Let $(X, d, \mu)$ be an SHT without atoms. Let $1<p_{-} \leq p_{+}<\infty$. Assume that $p$ has a minimum at $x_{0}$ and that $p \in L H(X)$. If $L=\infty$ we also assume that $p \equiv p_{c} \equiv$ const outside some ball $B\left(x_{0}, a\right)$. Let $v$ and $w$ be positive increasing functions on $(0,2 L)$. Then the inequality

$$
\begin{equation*}
\left\|v\left(d\left(x_{0}, \cdot\right)\right)(N f)(\cdot)\right\|_{L^{p(\cdot)}(X)} \leq c\left\|w\left(d\left(x_{0}, \cdot\right)\right) f(\cdot)\right\|_{L^{p(\cdot)}(X)} \tag{9}
\end{equation*}
$$

where $N$ is $M$ or $K$, holds if the following condition is satisfied:

$$
\sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v\left(d\left(x_{0}, x\right)\right)}{\mu\left(B_{x_{0} x}\right)}\right)^{p(x)}\left(\int_{\bar{B}\left(x_{0}, t\right)} w^{-\left(\widetilde{p}_{0}\right)^{\prime}(x)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{p(x)}{\left(\widetilde{p}_{0}\right)^{\prime}(x)}} d \mu(x)<\infty .
$$

Proof of this statement is similar to that of Theorem 3.4; therefore we omit it. Notice that Lemma 1.9 yields that $p \in L H(X) \Rightarrow p \in \mathcal{P}(1)$.

Example 4.4. Let $(X, d, \mu)$ be a quasimetric measure space with $L<\infty$. Suppose that $1<p_{-} \leq$ $p_{+}<\infty$ and $p \in L H(X)$. Assume that the measure $\mu$ is upper and lower Ahlfors $1-$ regular. Let there exist $x_{0} \in X$ such that $p$ has a minimum at $x_{0}$. Then the condition

$$
S:=\sup _{0<t \leq L} \int_{t<d\left(x_{0}, x\right) \leq L}\left(\frac{v\left(d\left(x_{0}, x\right)\right)}{\mu\left(B_{x_{0} x}\right)}\right)^{p(x)}\left(\int_{\bar{B}\left(x_{0}, t\right)} w^{-p^{\prime}\left(x_{0}\right)}\left(d\left(x_{0}, y\right)\right) d \mu(y)\right)^{\frac{p(x)}{p^{\prime}\left(x_{0}\right)}} d \mu(x)<\infty
$$

is satisfied for the weight functions $v(t)=t^{1 / p^{\prime}\left(x_{0}\right)}, w(t)=t^{1 / p^{\prime}\left(x_{0}\right)} \ln \frac{2 L}{t}$ and, consequently, by Theorem 4.3 inequality (9) holds, where $N$ is $M$ or $K$.

Indeed, first observe that $v$ and $w$ are both increasing on $[0, L]$. Further, it is easy to check that $S \leq c \sup _{0<t \leq L} V(t)(W(t))^{\frac{p\left(x_{0}\right)}{p^{\prime}\left(x_{0}\right)}}<\infty$ because $W(L)<\infty$.

By using the representation formula of a general integral by improper integral and the fact that $\mu$ is Ahlfors 1 - regular, it follows that $W(t) \leq C_{1} \ln ^{-1} \frac{2 L}{t}$ and $V(t) \leq C_{2} \ln \frac{2 L}{t}$ for $0<t \leq L$, where the positive constants does not depend on $t$. Hence the result follows.

Observe that for the constant $p$ both weights $v$ and $w$ are outside the Muckenhoupt class $A_{p}(X)$ (see e.g. [15], Ch. 8).

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