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INTEGRAL OPERATORS IN GRAND VARIABLE LEBESGUE SPACES

1. Introduction

In this note new Banach function spaces are introduced. These spaces unify two non-standard function spaces: variable exponent Lebesgue spaces and grand Lebesgue space. Comprehensive study and some aspects of applications of one these spaces were delivered in the recently published books [1], [6], [23]. The variable exponent Lebesgue space represents the special case of that introduced by W. Orlicz in the 30-th of the last century and then generalized by I. Musielak and W. Orlicz. H. Nakano [28] then specified it.

The grand Lebesgue spaces were introduced in the 90-th of the last century by T. Iwaniec and C. Sbordone [12]. Lately number of problems of Harmonic analysis and the theory of non-linear differential equations were studied in these spaces (see e.g. the papers [9], [16], [17], [18], [15], [29], [20], [21], etc.).

The spaces introduced in this paper are non-reflexive, non-separable and non-rearrangement invariant. The boundedness results of the Hardy-Littlewood maximal and Calderón-Zygmund operators defined on spaces of homogeneous type are given. From the above-mentioned solutions quite a number of interesting results are obtained.

2. Preliminaries

Throughout the paper we assume that (X, d, μ) is a space of homogeneous type (SHT) with finite measure, i.e. X is a set, d is a quasi-metric on X and μ is a finite measure on X satisfying the well–known doubling condition. We will assume that X does not contain any atoms. Let p be a measurable function on X satisfying the condition

$$1 < p_- \le p_+ < \infty, \tag{1}$$

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where

$$p_- := \inf_X p; \quad p_+ := \sup_X p.$$

We denote the class of all exponent satisfying condition (1) by $\mathcal{P}(X)$.

Let us denote by $\mathcal{D}(X)$ the class of bounded functions on X with compact support, d_X be the diameter of X.

Let $p(\cdot) \in \mathcal{P}(X)$. By the symbol $L^{p(\cdot)}$ we denote the variable exponent Lebesgue spaces (see e.g. [26], [6] for the definition). Further, let $\theta > 0$. We denote by $L^{p(\cdot),\theta}(X)$ the class of all measurable functions $f : X \mapsto \mathbb{R}$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}(X)} := \sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{L^{p(x)-\varepsilon}(X)}$$

is finite.

Together with the space $L^{p(\cdot),\theta}$ it is interesting to consider the space $\mathcal{L}^{p(\cdot),\theta}$ which is defined with respect to the norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\theta}} := \sup_{0<\varepsilon<p_- -1} \left\|\varepsilon^{\frac{\theta}{p(x)-\varepsilon}} f\right\|_{L^{p(x)-\varepsilon}(X)}.$$

It is obvious that

$$\mathcal{L}^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot),\theta}(X).$$

Further, there exists a function f such that $f \in L^{p(\cdot),\theta}(X)$ but $f \notin \mathcal{L}^{p(\cdot),\theta}(X)$.

It can be checked that $L^{p(\cdot),\theta}(X)$ and $\mathcal{L}^{p(\cdot),\theta}(X)$ are Banach spaces.

Remark. Let X be a bounded domain in $\mathbb{R}^n$, d be an Euclidean metric, and let μ be the Lebesgue measure. If $p = p_c =$ const, then $L^{p(\cdot),\theta} = \mathcal{L}^{p(\cdot),\theta}$ is the grand Lebesgue space $L^{p_c),\theta}$ introduced in [10]. In the case $p = p_c =$ const and $\theta = 1$, then we have Iwaniec-Sbordone [12] space $L^{p_c)}$. The space $L^{p_c)}$ naturally arises, for example, to study integrability problems of the Jacobian under minimal hypothesis (see [12]), while $L^{p_c),\theta}$ is related to the investigation of the nonhomogeneous n- harmonic equation div $A(x, \nabla u) = \mu$ (see [3]).

Proposition A. *The spaces $L^{p(\cdot),\theta}(X)$ and $\mathcal{L}^{p(\cdot),\theta}(X)$ are complete. The closure of $L^{p(\cdot)}(X)$ in $L^{p(\cdot),\theta}(X)$ (resp. in $\mathcal{L}^{p(\cdot),\theta}(X)$) consists of those $f \in L^{p(\cdot),\theta}(X)$ (resp. $f \in \mathcal{L}^{p(\cdot),\theta}(X)$) for which $\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(X)} = 0$ (resp. $\lim_{\varepsilon\to 0} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(X)} = 0$).*

Proposition B. *Let $p \in \mathcal{P}(X)$. Then the following embeddings hold:*

$$L^{p(\cdot)}(X) \hookrightarrow L^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot)-\varepsilon}(X), \ \ 0 < \varepsilon < p_- - 1;$$

$$L^{p(\cdot)}(X) \hookrightarrow \mathcal{L}^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot)-\varepsilon}(X), \ \ 0 < \varepsilon < p_- - 1.$$

We define the Hardy–Littlewood maximal operator on X by

$$(M_X f)(x) = \sup_{0<r<d_X} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)| d\mu(y), \quad x \in X,$$

where $B(x,r)$ is the ball in X with center x and radius r.

Definition 1. Suppose that $\mathcal{P}^{\log}_{loc}(X)$ is the class of those exponents p satisfying the local log-Hölder continuity condition: there is a positive constant c_0 such that for all $x, y \in X$ with $d(x,y) < 1/2$,

$$|p(x) - p(y)| \leq \frac{c_0}{-\ln\big(d(x,y)\big)}.$$

Further, let $\widetilde{\mathcal{P}}^{\log}_{loc}(X)$ be the class of those exponents satisfying the condition: there exists a positive constants a and b such that if $d(x,y) < b$, then

$$|p(x) - p(y)| \leq \frac{a}{-\ln\big(\mu B(x, d(x,y))\big)}.$$

It is easy to check that $\mathcal{P}^{\log}_{loc}(X) \subset \widetilde{\mathcal{P}}^{\log}_{loc}(X)$.

The boundedness of M_X in $L^{p(\cdot)}(X)$ spaces was established by L. Diening [5] for Euclidean spaces and by M. Khabazi [13] for an SHT.

3. The Main Results

Now we formulate the main results of this paper:

Theorem 1 (General-type theorem)**.** *Let $p \in \mathcal{P}(X)$ and let $\theta > 0$.*

(a) *Suppose that $\mathcal{F}$ be a family of pairs (f,g) such that*

$$\|f\|_{L^{p(\cdot)-\varepsilon}} \leq c_{p,\varepsilon} \|g\|_{L^{p(\cdot)-\varepsilon}}.$$

If

$$\sup_{0<\varepsilon<\sigma} c_{p,\varepsilon} < \infty$$

for some positive constant σ, then for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot),\theta}(X)} \leq c \|g\|_{L^{p(\cdot),\theta}(X)};$$

(b) *Suppose that $\mathcal{F}$ be a family of pairs (f,g) such that*

$$\|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f\|_{L^{p(\cdot)-\varepsilon}(X)} \leq b_{p,\varepsilon} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} g\|_{L^{p(\cdot)-\varepsilon}(X)}$$

for some positive constant $b_{p,\varepsilon}$. If

$$\sup_{0<\varepsilon<\sigma} b_{p,\varepsilon} < \infty$$

for some positive constant σ, then there exists a positive constant c such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{\mathcal{L}^{p(\cdot),\theta}(X)} \leq c \|g\|_{\mathcal{L}^{p(\cdot),\theta}(X)}.$$

Theorem 2. *Let $p \in \mathcal{P}(X) \cap \widetilde{\mathcal{P}}^{\log}_{loc}(X)$ and let $\theta > 0$. Then the Hardy–Littlewood maximal operator M_X is bounded in $L^{p(\cdot),\theta}(X)$.*

Let $k : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \leq \frac{c}{\mu B(x, d(x,y))}, \quad x, y \in X, \quad x \neq y;$$

$$|k(x_1,y)-k(x_2,y)|+|k(y,x_1)-k(y,x_2)| \leq c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big)\frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) > d(x,x_2)$, where ω is a positive, non-decreasing function on $(0,\infty)$ satisfying Δ_2 condition ($\omega(2t) \leq c\omega(t)$, $t > 0$) and the Dini condition $\int_0^1 \omega(t)/t dt < \infty$.

We also assume that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}(X)$ the limit

$$(Kf)(x) = p.v. \int\limits_X k(x,y)f(y)d\mu(y)$$

exists almost everywhere on X and that K is bounded in $L^{p_0}(X)$.

The following statement is known (see [24], [25]) (for Euclidean spaces see [7], [3]).

Theorem A. *Let $p \in \mathcal{P}(X) \cap \mathcal{P}^{\log}_{loc}(X)$. Then K is bounded in $L^{p(\cdot)}(X)$.*

Theorem 3. *Let $p \in \mathcal{P}(X) \cap \widetilde{\mathcal{P}}^{\log}_{loc}(X)$ and let $\theta > 0$. Then there is a positive constant c depending only on p such that the following inequality*

$$\|Kf\|_{L^{p(\cdot),\theta}(X)} \leq c\|f\|_{L^{p(\cdot),\theta}(X)}, \quad f \in \mathcal{D}(X),$$

holds, where the positive constant c does not depend on f.

Regarding the space $\mathcal{L}^{p(\cdot),\theta}(X)$ we have the following statement:

Theorem 4. *Let p satisfy the conditions of Theorem 2. Then the operator M_X is bounded in $\mathcal{L}^{p(\cdot),\theta}(X)$.*

4. Some Applications

Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be arc-length measure on Γ. By definition, Γ is regular if there is a positive constant c such that

$$\nu(D(z,r) \cap \Gamma) \leq cr$$

for every $z \in \Gamma$ and all $r > 0$, where $D(z,r)$ is a disc in $\mathbb{C}$ with center z and radius r. The reverse inequality

$$\nu(D(z,r) \cap \Gamma) \geq r$$

holds for all $z \in \Gamma$ and $r < L/2$, where L is a diameter of Γ. If we equip Γ with the measure ν and the Euclidean metric, the regular curve becomes an SHT.

The associate kernel in which we are interested is

$$k(z,w) = \frac{1}{z-w}.$$

The Cauchy integral

$$S_\Gamma f(t) = \int\limits_\Gamma \frac{f(\tau)}{t-\tau} d\nu(\tau)$$

is the corresponding singular operator.

The above-mentioned kernel in the case of regular curves is a Calderón-Zygmund kernel. As was proved by G. David [4], a necessary and sufficient condition for continuity of the operator S_Γ in $L^r(\Gamma)$, where r is a constant ($1 < r < \infty$), is that Γ is regular.

We denote by M_Γ the Hardy–Littlewood maximal operator defined on Γ.

The above-formulated results yield the next statement:

Proposition 1. *Let Γ be a regular curve. Suppose that $p \in \mathcal{P}(\Gamma) \cap \mathcal{P}^{\log}_{loc}(\Gamma)$. Assume that $L < \infty$. Then*

(i) M_Γ is bounded in $L^{p(\cdot),\theta}(\Gamma)$;

(ii) M_Γ is bounded in $\mathcal{L}^{p(\cdot),\theta}(\Gamma)$;

(iii) the operator S_Γ is a bounded operator in $L^{p(\cdot),\theta}(\Gamma)$.

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