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On Fourier Multipliers in Weighted Triebel–Lizorkin Spaces

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The paper is devoted to two-weighted estimates for multipliers of Fourier transforms. Apparently this problem has not been studied before. Easy to verify conditions for the pairs of weights ensuring two-weight estimates for several classes of multipliers in Triebel–Lizorkin spaces are derived. Two-weighted boundedness criteria for the Riemann–Liouville and Weyl transforms on the line are established. The authors' recent results on trace inequalities for one-sided potentials in the "diagonal" case ($p = q$) are applied.

The method of proof consists of the representation of the operators under consideration in the form of compositions of certain "elementary" transformations.

Various examples of pairs of weights governing two-weighted estimates for Fourier multipliers are presented.

Keywords: Fourier multipliers; Triebel–Lizorkin spaces; Riemann–Liouville and Weyl operators; Riesz potentials; Singular integrals

Classification: 2000 *Mathematics Subject Classification*: 42A45, 42B15, 42B20, 47B34

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INTRODUCTION

The main goal of this paper is to give two-weighted estimates for multipliers of Fourier transforms in Triebel–Lizorkin spaces. One-weighted multiplier theorems of Mikhlin and Hörmander type in Lebesgue spaces with Muckenhoupt's A_p weights are given in [1, 2]. Multiplier theorems with exponential weights are proved in [3, 4].

General (L^p, L^q) $(1 < p \leq q < \infty)$ Fourier multipliers in unweighted cases have been studied in [5–9]. For extensions of these results we refer to [10, 11].

An improvement of Hörmander's multiplier theorem in terms of spaces of fractional smoothness is obtained in [12]. On the basis of integral representations of functions certain spaces of differentiable functions were studied in [13], where sufficient conditions are established for Fourier integral multipliers in $L^p(\mathbb{R}^n)$ when $|1/2 - 1/p| < 1/q$ for some $q > 2$.

The setting of the problem in the framework of two-weight theory enables us to determine new classes of multipliers even in the unweighted case. At the same time we obtain easy-to-verify conditions for pairs of weights ensuring the validity of two-weight inequalities for multipliers. It should be noted that these conditions are not only sufficient but also necessary for the whole class of multipliers under consideration. The main results in this direction are based on criteria for boundedness from L^p_w to L^q_v $(1 < p \leq q < \infty)$ for fractional and singular integrals.

In general the paper is organized as follows: Section 1 is devoted to the establishment of two-weighted criteria for fractional integrals. Here some known results for related operators are given as well. In Section 2 we define weighted Triebel–Lizorkin spaces and prove some auxiliary results. Sections 3 and 4 contain multiplier theorems for Fourier transforms in the one-dimensional case. Section 5 is devoted to the multidimensional case.

In part of the multiplier statements there are connections with the papers of L. Hörmander [14], J. Schwartz [15] and P. Lizorkin [6]. The initial method consists of the representation of the operators under consideration in the form of compositions of certain elementary transformations.

1 TWO-WEIGHTED BOUNDEDNESS CRITERIA FOR FRACTIONAL INTEGRALS

Let ν be a locally finite measure defined on the Borel sets from $\mathbb{R}^n$. In the sequel by $L^p_\nu(\mathbb{R}^n)$ $(1 < p < \infty)$ we denote the set of ν-measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which

$$\|f\|_{L^p_\nu(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}\nu\right)^{1/p} < \infty.$$

For absolutely continuous measure $\mathrm{d}\nu(x) = w(x)\,\mathrm{d}x$, where w is a locally integrable, a.e. positive function (*i.e.* a weight) we use the symbol $L^p_w(\mathbb{R}^n)$. For an arbitrary Borel set E in $\mathbb{R}^n$ we define

$$w(E) = \int_E w(x)\,\mathrm{d}x;$$

if $n = 1$ and $E = (a, b)$ we shall also denote this by $w(a, b)$.

By $L^{p\infty}_\nu(\mathbb{R}^n)$ we denote the set of measurable functions for which

$$\|f\|_{L^{p\infty}_\nu} = \sup_{\lambda>0} \lambda(\nu\{x \in \mathbb{R}^n\colon |f(x)| > \lambda\})^{1/p} < \infty.$$

As usual, the number p' is defined by $1/p' + 1/p = 1$.

For a measurable $f\colon \mathbb{R} \to \mathbb{R}$ put

$$\mathcal{R}_\alpha f(x) = \int_{-\infty}^{x} (x - y)^{\alpha-1} f(y)\,\mathrm{d}y, \tag{1.1}$$

$$\mathcal{W}_\alpha f(x) = \int_{x}^{+\infty} (y - x)^{\alpha-1} f(y)\,\mathrm{d}y. \tag{1.2}$$

where $x \in \mathbb{R}$ and $\alpha > 0$.

Both of these transformations are particular cases of the general integral transform

$$Kf(x) = \int_{-\infty}^{+\infty} k(x, y) f(y)\,\mathrm{d}y,$$

where $k\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive measurable kernel.

We shall need the following

THEOREM A *Let $1 < p < q < \infty$ and let v, w be weights. Then the operator K is bounded from $L^p_w(\mathbb{R})$ into $L^{q\infty}_v(\mathbb{R})$ if*

$$C_0 \equiv \sup_{\substack{x,h\\ x\in R, h>0}} \left(\int_{x-h}^{x+h} v(x)\,\mathrm{d}x\right)^{1/q} \left(\int_{\mathbb{R}\setminus(x-h,x+h)} k^{p'}(x,y) w^{1-p'}(y)\,\mathrm{d}y\right)^{1/p'} < \infty. \tag{1.3}$$

The proof of this theorem in a more general setting is given in [16] (see Theorem 3.1.1).

THEOREM 1.1 *Let $1 < p < q < \infty$, $0 < \alpha < 1$ and let v, w be weights. Then the inequality*

$$\|\mathcal{R}_\alpha f\|_{L^{q\infty}_v(\mathbb{R})} \le c\|f\|_{L^p_w(\mathbb{R})}, \quad f \in L^p_w(\mathbb{R}), \tag{1.4}$$

with a positive constant c independent of f, holds if and only if

$$C_1 \equiv \sup_{\substack{x,h\\ x\in R, h>0}} (v(x-h, x+h))^{1/q} \left(\int_{-\infty}^{x-h} w^{1-p'}(y)(x-y)^{(\alpha-1)p'}\,\mathrm{d}y\right)^{1/p'} < \infty. \tag{1.5}$$

Proof From Theorem A we see that if $C_1 < \infty$, then the two-weight weak type inequality (1.4) holds. Now we show that the condition $C_1 < \infty$ is also necessary.

First we show that

$$I(x,h) \equiv \int_{-\infty}^{x-h} w^{1-p'}(y)(x-y)^{(\alpha-1)p'}\,\mathrm{d}y < \infty$$

for every x and h. Indeed, if we assume on the contrary that for some $x \in \mathbb{R}$ and $h > 0$, $I(x,h) = \infty$, then there exists a non-negative $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{x-h} g^p(y) w(y)\,\mathrm{d}y \le 1$$

and

$$\int_{-\infty}^{x-h} g(y)(x-y)^{\alpha-1}\,\mathrm{d}y = \infty.$$

On the other hand, if $z \in (x-h, x+h)$, then

$$\mathcal{R}_\alpha g(z) \geq \int_{-\infty}^{x-h} g(y)(z-y)^{\alpha-1}\,\mathrm{d}y \geq c_1 \int_{-\infty}^{x-h} g(y)(x-y)^{\alpha-1}\,\mathrm{d}y = \infty.$$

Consequently

$$(x-h, x+h) \subset \{z\colon \mathcal{R}_\alpha g(z) > \lambda\}$$

for every $\lambda > 0$. From (1.4) we have

$$\int_{x-h}^{x+h} v(y)\,\mathrm{d}y \leq c\lambda^{-q}.$$

As λ is an arbitrary positive number, we conclude that $\int_{x-h}^{x+h} v(x)\,\mathrm{d}x = 0$, which is absurd. Hence $I(x,h) < \infty$.

Now let $f \geq 0$, $x \in \mathbb{R}$, $h > 0$ and $z \in (x-h, x+h)$. We have

$$\mathcal{R}_\alpha f(z) = \int_{-\infty}^{z} (z-y)^{\alpha-1} f(y)\,\mathrm{d}y \geq c_2 \int_{-\infty}^{x-h} (x-y)^{\alpha-1} f(y)\,\mathrm{d}y.$$

From the two-weight weak type inequality (1.4) we obtain that

$$\begin{aligned}\int_{x-h}^{x+h} v(y)\,\mathrm{d}y &\leq \int_{\{z:\mathcal{R}_\alpha f(z) \geq c_2 \int_{-\infty}^{x-h}(x-y)^{\alpha-1} f(y)\,\mathrm{d}y\}} v(z)\,\mathrm{d}z \\ &\leq c c_2^{-q}\left(\int_{-\infty}^{x-h}(x-y)^{\alpha-1}f(y)\,\mathrm{d}y\right)^{-q}\left(\int_{-\infty}^{+\infty} f^p(y)w(y)\,\mathrm{d}y\right)^{q/p},\end{aligned}$$

where the constants c and c_2 are independent of $x \in (-\infty, +\infty)$, $h > 0$ and $f \geq 0$. If we put here

$$f(y) = \chi_{(-\infty, x-h)}(y) w^{1-p'}(y)(x-y)^{(p'-1)(\alpha-1)},$$

then we obtain

$$\int_{x-h}^{x+h} v(y)\,\mathrm{d}y \le c_3\left(\int_{-\infty}^{x-h}(x-y)^{(\alpha-1)p'}w^{1-p'}(y)\,\mathrm{d}y\right)^{-q/p'}$$

and finally we see that $C_1 < \infty$. ■

THEOREM 1.2 *Let* $1 < p < q < \infty$, $\alpha \in (0,1)$ *and let* v, w *be weights. Then* $\mathcal{W}_\alpha$ *is bounded from* $L^p_w(\mathbb{R})$ *into* $L^{q\infty}_v(\mathbb{R})$ *if and only if*

$$C_2 \equiv \sup_{\substack{a,h\\ a\in\mathbb{R},h>0}}\left(\int_{a-h}^{a+h} v(y)\,\mathrm{d}y\right)^{1/q}\left(\int_{a+h}^{+\infty} w^{1-p'}(y)(y-a)^{(\alpha-1)p'}\,\mathrm{d}y\right)^{1/p'} < \infty.$$

The proof is similar to that of the previous theorem.

From the last statement obviously we have that $\mathcal{W}_\alpha$ acts boundedly from $L^{q'}_{v^{1-q'}}(\mathbb{R})$ into $L^{p'\infty}_{w^{1-p'}}(\mathbb{R})$ if and only if

$$C_3 \equiv \sup_{\substack{a,h\\ a\in\mathbb{R},h>0}}\left(\int_{a-h}^{a+h} w^{1-p'}\,\mathrm{d}y\right)^{1/p'}\left(\int_{a+h}^{+\infty} v(y)(y-a)^{(\alpha-1)q}\,\mathrm{d}y\right)^{1/q} < \infty. \tag{1.6}$$

In the papers [17, 18], criteria of Sawyer type were derived for the operators $\mathcal{R}_\alpha$ and $\mathcal{W}_\alpha$. In fact, the above-mentioned results lead to

THEOREM B *Let* $1 < p \le q < \infty$ *and let* $0 < \alpha < 1$; *let* v, w *be weights. Then* $\mathcal{R}_\alpha$ *is bounded from* $L^p_w(\mathbb{R})$ *to* $L^q_v(\mathbb{R})$ *if and only if* $\mathcal{R}_\alpha$ *is bounded from* $L^p_w(\mathbb{R})$ *to* $L^{q\infty}_v(\mathbb{R})$ *and simultaneously* $\mathcal{W}_\alpha$ *acts boundedly from* $L^{q'}_{v^{1-q'}}(\mathbb{R})$ *into* $L^{p'\infty}_{w^{1-p'}}(\mathbb{R})$.

Now combining the results stated above we have

THEOREM 1.3 *Let* $1 < p < q < \infty$ *and* $0 < \alpha < 1$. *For the boundedness of* $\mathcal{R}_\alpha$ *from* $L^p_w(\mathbb{R})$ *into* $L^q_v(\mathbb{R})$ *it is necessary and sufficient that the conditions* (1.5) *and* (1.6) *are satisfied simultaneously.*

From duality arguments we have the following result for $\mathcal{W}_\alpha$:

THEOREM 1.4 *Let* $1 < p < q < \infty$ *and let* $\alpha \in (0, 1)$. *Then* $\mathcal{W}_\alpha$ *is bounded from* $L^p_w(\mathbb{R})$ *into* $L^q_v(\mathbb{R})$ *if and only if*

$$\sup_{\substack{a,h \\ a\in\mathbb{R},h>0}} \left(\int_{a-h}^{a+h} w^{1-p'}(y)\,\mathrm{d}y\right)^{1/p'} \left(\int_{-\infty}^{a-h} v(y)(a-y)^{(\alpha-1)q}\,\mathrm{d}y\right)^{1/q} < \infty \tag{1.7}$$

and

$$\sup_{\substack{a,h \\ a\in\mathbb{R},h>0}} \left(\int_{a-h}^{a+h} v(y)\,\mathrm{d}y\right)^{1/q} \left(\int_{a+h}^{+\infty} w^{1-p'}(y)(y-a)^{(\alpha-1)p'}\,\mathrm{d}y\right)^{1/p'} < \infty. \tag{1.8}$$

The solution of the two-weighted problem for the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}\,\mathrm{d}y, \quad 0 < \alpha < n,$$

when $1 < p < q < \infty$ gives the following statement:

THEOREM C ([19, 20]) *Let* $1 < p < q < \infty$ *and let* $\alpha \in (0, n)$. *Then* I_α *is bounded from* $L^p_w(\mathbb{R}^n)$ *into* $L^q_v(\mathbb{R}^n)$ *if and only if*

$$\sup_{\substack{x\in\mathbb{R}^n \\ r>0}} (v(B(x,2r)))^{1/q} \left(\int_{|x-y|>r} w^{1-p'}|x-y|^{(\alpha-n)p'}\,\mathrm{d}y\right)^{1/p'} < \infty \tag{1.9}$$

and

$$\sup_{\substack{x\in\mathbb{R}^n \\ r>0}} (w^{1-p'}(B(x,2r)))^{1/p'} \left(\int_{|x-y|>r} v(y)|x-y|^{(\alpha-n)q}\,\mathrm{d}y\right)^{1/q} < \infty. \tag{1.10}$$

Note also the following statement (see [16]):

THEOREM D *Let $1 < p < q < \infty$ and let $\alpha \in (0, n)$. It is assumed that v and $w^{1-p'}$ satisfy the reverse doubling condition. Then I_α is bounded from $L^p_w(\mathbb{R}^n)$ to $L^q_v(\mathbb{R}^n)$ if and only if the following condition holds:*

$$\sup_{\substack{x\in\mathbb{R}^n\\ r>0}} r^{\alpha-n}\left(\int_{|x-y|<r} v(y)\,\mathrm{d}y\right)^{1/q}\left(\int_{|x-y|<r} w^{1-p'}(y)\,\mathrm{d}y\right)^{1/p'} < \infty.$$

For $p = q$ the following result is known:

THEOREM E ([21]) *Let $1 < p < \infty$ and $0 < \alpha < n$. The operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if $I_\alpha \in L^{p'}_{loc}(\mathbb{R}^n)$ and*

$$I_\alpha(I_\alpha v)^{p'}(x) \leq cI_\alpha v(x) \tag{1.11}$$

almost everywhere on $\mathbb{R}^n$.

We also mention the following statement:

THEOREM F ([22]) *Let $1 \leq q < p < \infty$ and $0 < \alpha < n$. The operator I_α acts boundedly from $L^p(\mathbb{R}^n)$ into $L^q_v(\mathbb{R}^n)$ if and only if*

$$\int_0^\infty \left[r^{\alpha p-1}\int_{B(x,r)} v(y)\,\mathrm{d}y\right]^{p'-1} r^{-1}\,\mathrm{d}r \in L^{q(p-1)/(p-q)}(\mathbb{R}^n). \tag{1.12}$$

where $B(x, r) = \{y\colon |x-y| < r\}$.

In the sequel we shall need the following results for the operators $\mathcal{R}_\alpha$ and $\mathcal{W}_\alpha$:

THEOREM G ([23]) *Let $1 < p < \infty$ and let $\alpha \in (0, 1)$. Suppose that v is a locally integrable a.e. positive function on $\mathbb{R}$. Then the following statements hold:*

(i) *The operator $\mathcal{R}_\alpha$ is bounded from $L^p(\mathbb{R})$ to $L^p_v(\mathbb{R})$* if and only if *$\mathcal{W}_\alpha \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$ and*

$$\mathcal{W}_\alpha[\mathcal{W}_\alpha v]^{p'}(x) \leq c(\mathcal{W}_\alpha v)(x) < \infty \tag{1.13}$$

for a.a. $x \in \mathbb{R}$.

(ii) *The operator $\mathcal{W}_\alpha$ acts boundedly from $L^p(\mathbb{R})$ into $L^p_v(\mathbb{R})$ if and only if $\mathcal{R}_\alpha \in L^{p'}_{loc}(\mathbb{R})$ and*

$$\mathcal{R}_\alpha[\mathcal{R}_\alpha v]^{p'}(x) \leq c(\mathcal{R}_\alpha v)(x) < \infty \tag{1.14}$$

for a.a. $x \in \mathbb{R}$.

2 WEIGHTED TRIEBEL–LIZORKIN SPACES

Let $S(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions (see [24]). For $\varphi \in S(\mathbb{R}^n)$ the Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\lambda) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) \exp\{-i\lambda x\}\, dx;$$

let $\hat{\varphi}$ denote the inverse Fourier transform of φ. For the Fourier transform and its inverse the notation $F(\varphi)$ and $F^{-1}(\varphi)$ respectively will be also used.

The Fourier transform determines a topological isomorphism of the space S onto itself.

Let S' be the space of tempered distributions, *i.e.* the space of linear bounded functionals on $S(\mathbb{R}^n)$. In the sequel the Fourier transforms in the framework of the theory of S'-distributions will be considered.

Now we give a definition of a weighted Triebel–Lizorkin space in a general setting.

Let $\{m_j\}_{j=-\infty}^{\infty}$ be a two-sided increasing sequence of positive numbers such that $\lim_{j\to-\infty} m_j = 0$ and $\lim_{j\to+\infty} m_j = +\infty$. Let $\mathbb{I}$ be the collection of all intervals $(m_j, m_{j+1}]$ and $[-m_{j+1}, -m_j)$, $j \in \mathbb{Z}$. Any interval of this type we shall denote by I. It is clear that $\cup_{\mathbb{I}} I = \mathbb{R}\backslash\{0\}$. Now considering n similar decompositions of $R\backslash\{0\}$ by the sets

$$\Gamma_j^i = [m_{j,i}, m_{j+1,i}) \cup (-m_{j+1,i}, -m_{j,i}), \quad j \in Z, \quad i = 1, \ldots, n,$$

we denote by J the collection of all intervals of the form

$$J = I_1^{(1)} \times \cdots \times I_n^{(n)},$$

where $I_j^{(i)}$ is an arbitrary one-dimensional interval of the above-mentioned type. This gives a decomposition of $\mathbb{R}\backslash\{0\} \times \cdots \times \mathbb{R}\backslash\{0\}$.

Let $\{\beta_{j,i}\}_{j=-\infty}^{\infty}$, $i = 1, \ldots, n$ be sequences of positive numbers which for arbitrary i, $i = 1, \ldots, n$, satisfy the following conditions:

(i)

$$\sum_{j=-\infty}^{0} \beta_{j,i}(m_{j+1,i} - m_{j,i}) < \infty; \tag{2.1}$$

(ii) there exists some ε, $0 < \varepsilon < 1$, such that

$$\sum_{j=-\infty}^{0} \beta_{j,i} m_{j,i}^{\varepsilon} < \infty; \tag{2.2}$$

(iii) there exists some natural number k such that

$$\sum_{j=1}^{\infty} \beta_{j,i} m_{j,i}^{-k} < \infty. \tag{2.3}$$

Now let

$$A = \Gamma_{j_1}^{i_1} \times \cdots \times \Gamma_{j_m}^{i_m}, \quad j_m \in \mathbb{Z}, \quad i_l \in \{1, \ldots, n\}.$$

Put

$$\beta_A = \prod_{l=1}^{n} \beta_{j_l, i_l}.$$

For $\varphi \in S(\mathbb{R}^n)$ let

$$\varphi_A = F^{-1}(\chi_J \hat{\varphi}).$$

Suppose now that $1 < p$, $\theta < \infty$. If for some locally finite regular measure ν and for any $\varphi \in S(\mathbb{R}^n)$ the quantity

$$|\varphi, F_{\beta,\nu}^{p,\theta}| = \left(\int_{\mathbb{R}^n} \left(\sum_A \beta_A^{\theta} |\varphi_A(x)|^{\theta} \right)^{p/\theta} \mathrm{d}\nu \right)^{1/p}. \tag{2.4}$$

is finite, then the completion of $S(\mathbb{R}^n)$ with respect to the norm will be called a weighted Triebel–Lizorkin space and denoted by $F_{\beta,\nu}^{p,\theta}(\mathbb{R}^n)$. For unweighted Triebel–Lizorkin spaces we refer to [8, 25].

PROPOSITION 2.1 *Let* $1 < p,\ \theta < \infty$ *and suppose that*

$$(1+|x|)^{-p(1-\varepsilon)} \in L^1_\nu(\mathbb{R}^n) \tag{2.5}$$

for the constant ε *from* (2.2). *Then* (2.4) *is finite for arbitrary* $\varphi \in S(\mathbb{R}^n)$.

Proof To avoid awkward computations we give a proof for $n = 1$; the case $n > 1$ may be handled in a similar manner. Without loss of generality we can consider a function $\varphi \in S(\mathbb{R})$ whose Fourier transform $\hat{\varphi}$ vanishes for $\lambda < 0$. Let $I_j = [m_j, m_{j+1}], j \in \mathbb{Z}$, and let $\varphi_j = \varphi_{I_j}$. We have

$$\begin{aligned}
|\varphi, F^{p,\theta}_{\beta,\nu}| &= \left(\int_{\mathbb{R}} \left(\sum_{j=-\infty}^{\infty} |\varphi_j(x)|^\theta \beta_j^\theta\right)^{p/\theta} \mathrm{d}\nu\right)^{1/p} \\
&\le \left(\int_{\mathbb{R}} \left(\sum_{j=-\infty}^{\infty} |\varphi_j(x)|\beta_j\right)^{p} \mathrm{d}\nu\right)^{1/p} \\
&\le \left(\int_{|x|<1} \left(\sum_{j=-\infty}^{0} \beta_j|\varphi_j(x)|\right)^{p} \mathrm{d}\nu\right)^{1/p} \\
&\quad + \left(\int_{|x|>1} \left(\sum_{j=-\infty}^{0} \beta_j|\varphi_j(x)|\right)^{p} \mathrm{d}\nu\right)^{1/p} \\
&\quad + \left(\int_{|x|<1} \left(\sum_{j=1}^{\infty} \beta_j|\varphi_j(x)|\right)^{p} \mathrm{d}\nu\right)^{1/p} \\
&\quad + \left(\int_{|x|>1} \left(\sum_{j=1}^{\infty} \beta_j|\varphi_j(x)|\right)^{p} \mathrm{d}\nu\right)^{1/p} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is obvious that

$$I_1 \le c \sum_{j=-\infty}^{0} \beta_j(m_{j+1} - m_j)\left(\int_{-1}^{1} \mathrm{d}\nu\right)^{1/p} < \infty$$

by condition (2.1).

Integration by parts leads to the estimate

$$(2\pi)^{1/2}|\varphi_j(x)| \leq |\hat{\varphi}(m_{j+1}) - \hat{\varphi}(m_j)|\frac{1}{|x|} + \hat{\varphi}(m_j)|\exp\{im_jx\} - 1||x|^{-1}$$
$$+ \frac{1}{|x|}\int_{m_j}^{m_{j+1}} |\hat{\varphi}(\lambda)|\,\mathrm{d}\lambda.$$

Hence

$$\begin{aligned} I_2 &\leq c\Bigg(\left(\int_{|x|>1}\left(\sum_{j=-\infty}^{0}\beta_j(m_{j+1}-m_j)\right)^p |x|^{-p}\,\mathrm{d}v\right)^{1/p} \\ &\quad + \left(\int_{|x|>1}\left(\sum_{j=-\infty}^{0}\beta_j|\exp\{im_jx\}-1|\right)^p |x|^{-p}\,\mathrm{d}v\right)^{1/p} \\ &\quad + \left(\int_{|x|>1}\left(\sum_{j=-\infty}^{0}\beta_j\int_{m_j}^{m_{j+1}}|\hat{\varphi}(\lambda)|\,\mathrm{d}\lambda\right)^p |x|^{-p}\,\mathrm{d}v\right)^{1/p}\Bigg) \\ &= I_2^{(1)} + I_2^{(2)} + I_2^{(3)}. \end{aligned}$$

Since $(1+|x|)^{-p} \in L_v^1(\mathbb{R})$ and the condition (2.1) is satisfied, we conclude that $I_2^{(1)}$ is finite. For $I_2^{(2)}$ we derive

$$\begin{aligned} I_2^{(2)} &\leq c\sum_{j=-\infty}^{0}\beta_j\left(\int_{|x|>1}|\exp\{im_jx\}-1||x|^{-p}\,\mathrm{d}v\right)^{1/p} \\ &\leq c\sum_{j=-\infty}^{\infty}\beta_j\left(\int_{|x|>1}|\sin m_jx/2|^{p\varepsilon}|x|^{-p}\,\mathrm{d}v\right)^{1/p} \\ &\leq c\sum_{j=-\infty}^{0}\beta_j m_j^{\varepsilon}\left(\int_{|x|>1}|x|^{-p(1-\varepsilon)}\,\mathrm{d}v\right)^{1/p} < \infty. \end{aligned}$$

The boundedness of $\hat{\varphi}$ and the condition (2.2) imply that $I_2^{(3)}$ is finite. Further since $\hat{\varphi} \in S(\mathbb{R})$ and $|\hat{\varphi}(\lambda)| \leq c\lambda^{-(k+1)}$ (where k is as in (2.3)) we have by (2.3),

$$I_3 \leq c \sum_{j=1}^{\infty} \beta_j m_j^{-k} \left(\int_{-1}^{1} \mathrm{d}\nu \right)^{1/p} < \infty.$$

Integrating by parts and using the estimates $|\hat{\varphi}(\lambda)| \leq c|\lambda|^{-k}$ and $|\hat{\varphi}'(\lambda)| \leq c\lambda^{-(k+1)}$, we get that

$$\begin{aligned} I_4 &\leq c \left(\int_{|x|>1} \left(\sum_{j=1}^{\infty} \beta_j (|\hat{\varphi}(m_{j+1})| + |\hat{\varphi}(m_j)|) \right)^p |x|^{-p} \,\mathrm{d}\nu \right)^{1/p} \\ &\quad + c \left(\int_{|x|>1} \left(\sum_{j=1}^{\infty} \beta_j \int_{m_j}^{m_{j+1}} |\hat{\varphi}'(\lambda)| \,\mathrm{d}\lambda \right)^p |x|^{-p} \,\mathrm{d}\nu \right)^{1/p} \\ &\leq c \sum_{j=1}^{\infty} \beta_j m_j^{-k} \left(\int_{|x|>1} |x|^{-p} \,\mathrm{d}\nu \right)^{1/p} < \infty \end{aligned}$$

thanks to (2.1), (2.3) and (2.5).

Summarizing all these estimates we conclude that (2.4) is finite for arbitrary $\varphi \in S(\mathbb{R})$. ■

For an absolutely continuous measure $\mathrm{d}\nu = w(x)\,\mathrm{d}x$, where w is a locally integrable a.e. positive function, we write $F_{\beta,w}^{p,\theta}$ instead of $F_{\beta,\nu}^{p,\theta}$. The function w, as usual, will be called a weight function. If $w \equiv 1$, then we use the notation $L_{\beta,w}^{p,\theta} \equiv L_{\beta}^{p,\theta}$.

It is easy to see that the space $F_{\beta,\nu}^{p,\theta_1}$ is continuously imbedded into $F_{\beta,\nu}^{p,\theta_2}$ when $\theta_1 \leq \theta_2$ thanks to the inequality

$$\left(\sum_{j=1}^{\infty} a_j^{\theta_2} \right)^{1/\theta_2} \leq \left(\sum_{j=1}^{\infty} a_j^{\theta_1} \right)^{1/\theta_1}.$$

The spaces $F_{\beta,\nu}^{p,\theta}$ $(1 < p, \theta < \infty)$ are Banach spaces and each $f \in F_{\beta,\nu}^{p,\theta}$ can be regarded as an element of S'.

Remark When $\theta = 2$, $\beta_A = 1$, the decomposition is lacunary and ν is absolutely continuous with weight function w, the norm (2.4) is equivalent to the $L^p_w(\mathbb{R}^n)$ norm, thanks to the weighted version of the Littlewood–Paley theorem (see [1]) when the weight function w satisfies the condition A_p:

$$\sup \frac{1}{|J|}\int_J w(x)\,\mathrm{d}x\left(\frac{1}{|J|}\int_J w^{1-p'}(x)\,\mathrm{d}x\right)^{p'-1} < \infty,$$

where the supremum is taken over all n-dimensional boxes with sides parallel to the coordinate axes.

PROPOSITION 2.2 *Let ν be an arbitrary locally finite, regular measure. Then $S(\mathbb{R}^n)$ is dense in $L^p_\nu(\mathbb{R}^n)$.*

Proof It is sufficient to prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p_w(\mathbb{R}^n)$. Given $\varphi \in L^p_\nu(\mathbb{R}^n)$ and $\varepsilon > 0$, choose a continuous function g with compact support such that

$$\|f - g\|_{L^p_\nu(\mathbb{R}^n)} < \frac{\varepsilon}{2}.$$

(see [26], Lemma IV.8.19)

Let ψ be a non-negative, infinitely differentiable function supported in the unit ball of $\mathbb{R}^n$ with total integral equal to 1.

Define

$$\psi_t(x) = t^{-n}\psi\left(\frac{x}{t}\right), \quad t > 0.$$

It is easy to see that $\psi_t * g \in C_0^\infty(\mathbb{R}^n)$ for all $t > 0$ and $\psi_t * g \to g$ as $t \to 0$ uniformly on compact subsets of $\mathbb{R}^n$. If B is a large ball containing the support of g in its interior, choose t so small that

$$\|g - \psi_t * g\|_\infty < \frac{\varepsilon}{2}(\nu B)^{-1/p}.$$

Then

$$\|f - \psi_t * g\|_{L^p_\nu} \le \|f - g\|_{L^p_\nu} + \|g - \psi_t * g\|_{L^p_\nu} < \varepsilon.$$

Hence $C_0^\infty(\mathbb{R}^n)$ and therefore $S(\mathbb{R}^n)$ is dense in $L^p_\nu(\mathbb{R}^n)$. ∎

3 TWO-WEIGHT MULTIPLIERS IN TRIEBEL–LIZORKIN SPACES (ONE-DIMENSIONAL CASE)

Let X and Y be two function spaces on $\mathbb{R}^n$ with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Assume that $S(\mathbb{R}^n)$ is dense in both X and Y.

DEFINITION 3.1 *A distribution $m \in S'$ is called an (X, Y) multiplier if for the operator $\mathcal{K}$ defined by the Fourier transform equation*

$$\widehat{\mathcal{K}f} = m\hat{f}, \quad f \in S, \tag{3.1}$$

there exists a constant c such that

$$\|\mathcal{K}f\|_Y \le c\|f\|_X$$

for all $f \in S(\mathbb{R}^n)$.

In this case we write $m \in \mathcal{M}(X, Y)$. The number $\sup_{\|\varphi\|_X=1} \|F^{-1}(m\hat{\varphi})\|_Y$ is the norm of the (X, Y)-multiplier m.

In the sequel we shall need the following definitions of weight classes, the weights being defined on $\mathbb{R}$.

DEFINITION 3.2 *Let $\alpha \in (0, 1)$. We say that the weight pair (v, w) belongs to the class $\cup_\alpha^{q,p}$ if $1 < p < q < \infty$ and for v and w conditions (1.5) and (1.6) are satisfied. Further, the weight pair (v, w) belongs to $\tilde{\cup}_\alpha^{q,p}$ if $1 < p < q < \infty$ and conditions (1.7) and (1.8) hold.*

DEFINITION 3.3 *Let $0 < \alpha < 1$.*

(i) *The pair of weight functions (v, w) belongs to the class $W_\alpha^{q,p}$ if $1 < p < q < \infty$, $n = 1$ and (v, w) satisfies conditions (1.9) and (1.10).*
(ii) *Let $1 < p < \infty$, $n = 1$ and let I_α be the Riesz potential on $\mathbb{R}$. We say that the weight function v belongs to the class V_α^p if $I_\alpha v \in L_{loc}^{p'}$ and the condition (1.11) holds.*
(iii) *The weight $v \in \Gamma_\alpha^{q,p}$ if $n = 1$ and the condition (1.12) is fulfilled.*

Note that

$$W_\alpha^{q,p} = \cup_\alpha^{q,p} \cap \tilde{\cup}_\alpha^{q,p}.$$

On the other hand, for $w \equiv 1$ we have

$$W_\alpha^{q,p} = \mathsf{U}_\alpha^{q,p} = \tilde{\mathsf{U}}_\alpha^{q,p}.$$

DEFINITION 3.4 *Let $0 < \alpha < 1$. The weight function v on $\mathbb{R}$ is said to be of the class B_α^p (resp. $\tilde{B}_\alpha^p$) if $\mathcal{W}_\alpha(v) \in L_{\mathrm{loc}}^{p'}$ and* (1.13) *holds (resp. $\mathcal{R}_\alpha(v) \in L_{\mathrm{loc}}^{p'}$ and* (1.14) *is fulfilled).*

Now as in the previous section, let $\mathbb{I}$ be a decomposition of $\mathbb{R}$ with corresponding numbers β_j satisfying conditions (2.1), (2.2) and (2.3).

We have the following statement:

THEOREM 3.1 *Let $1 < p < q < \infty$, $1 < \theta < \infty$ and $(v, w) \in W_\alpha^{q,p}$. Let $\mathcal{K}$ be defined by* (3.1), *where the function m is represented in an arbitrary interval $I \in \mathbb{I}$ as*

$$m(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha}\, \mathrm{d}\mu_I, \quad 0 < \alpha < 1, \tag{3.2}$$

and the μ_I are finite measures for which

$$\sup_{I \in \mathbb{I}} \operatorname{var} \mu_I = M < \infty. \tag{3.3}$$

Then $m \in \mathcal{M}(F_{\beta,w}^{q,\theta}, F_{\beta,v}^{q,\theta})$ and, moreover,

$$|\mathcal{K}f, F_{\beta,v}^{q,\theta}| \le cM|f, F_{\beta,w}^{p,\theta}|, \tag{3.4}$$

where c does not depend on f and m.

The statement of Theorem 3.1 remains valid if the condition $(v, w) \in W_\alpha^{q,p}$ is replaced by the condition of Theorem D.

In the sequel by $B(x, r)$ we understand the interval $[x - r, x + r]$.

THEOREM 3.1′ *Let $1 < p < q < \infty$, $1 < \theta < \infty$. Assume that the measure v on $\mathbb{R}$ satisfies the condition*

$$vB(x, r) \le cr^{q(1/p-\alpha)}, \quad 0 < \alpha < 1,$$

with the constant c independent of x and r. Then for any measurable function m satisfying the conditions of the previous theorem we have that $m \in \mathcal{M}(F_\beta^{q,\theta}, F_{\beta,v}^{p,\theta})$.

In this section we shall *a priori* assume that

$$(1+|x|)^{-q+\varepsilon} \in L^1_v(\mathbb{R})$$

and

$$(1+|x|)^{-p+\varepsilon} \in L^1_w(\mathbb{R})$$

for some $\varepsilon > 0$.

Note that if $w \in A_p(\mathbb{R})$, then the last condition is satisfied.

The following statements also hold:

THEOREM 3.2 *Let $1 < p < q < \infty$, $1 < \theta < \infty$ and $(v, w) \in \mathbb{U}^{q,p}_\alpha$. Suppose that a function m in each I, $I \in \mathbb{I}$, is defined by the formula*

$$m(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha}\, \mathrm{d}\mu_I(t) + \exp\{i\alpha\pi\} \int_{\lambda}^{\infty} (\lambda - t)^{-\alpha}\, \mathrm{d}\mu_I(t), \quad 0 < \alpha < 1, \tag{3.5}$$

where the finite measures μ_I satisfy (3.3). *Then $m \in \mathcal{M}(F^{p,\theta}_{\beta,w}, F^{q,\theta}_{\beta,v})$ and* (3.4) *holds.*

THEOREM 3.3 *Let $1 < p < q < \infty$, $1 < \theta < \infty$ and let $(v, w) \in \tilde{\mathbb{U}}^{q,p}_\alpha$. Let a measurable function m be represented in each $I \in \mathbb{I}$ by*

$$m(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha}\, \mathrm{d}\mu_I(t) + \exp\{-i\alpha\pi\} \int_{\lambda}^{\infty} (t - \lambda)^{-\alpha}\, \mathrm{d}\mu_I(t), \quad 0 < \alpha < 1, \tag{3.6}$$

where μ_I satisfies (3.3). *Then $m \in \mathcal{M}(F^{p,\theta}_{\beta,w}, F^{q,\theta}_{\beta,v})$ and* (3.4) *holds.*

THEOREM 3.4 *Let $1 < p$, $\theta < \infty$ and $0 < \alpha < 1/p$. Assume that $v \in B^p_\alpha$ (resp. $v \in \tilde{B}^p_\alpha$). Then for the function m represented in each $I \in \mathbb{I}$ by* (3.5) (*resp. by* (3.6)) *we have $m \in \mathcal{M}(F^{p,\theta}_{\beta}, F^{p,\theta}_{\beta,v})$.*

THEOREM 3.5 *Let $1 \le q < p < \infty$, $1 < \theta < \infty$ and $v \in \Gamma^{q,p}_\alpha$, $0 < \alpha < 1$. Then for the functions m defined by* (3.2) *and satisfying the condition* (3.3) *we have $m \in \mathcal{M}(F^{p,\theta}_{\beta}, F^{q,\theta}_{\beta,v})$.*

THEOREM 3.6 *Let* $1 < p$, $\theta < \infty$ *and let* $v \in V_\alpha^p$. *Then the function* m *from Theorem* 3.1 *is a* $(F_\beta^{p,\theta}, F_{\beta,v}^{p,\theta})$ *multiplier.*

Remark 3.1 Let $\theta = 2$ and suppose that in addition to the above-mentioned conditions for a pair of weights (v,w), we have $v \in A_q(\mathbb{R})$ and $w \in A_p(\mathbb{R})$. Then the foregoing theorems give multiplier statements for (L_w^p, L_v^q).

PROPOSITION 3.1 *If for some pair of weights* (v,w) *all functions* m *of type* (3.2) *with condition* (3.3) *belong to* $\mathcal{M}(L_w^p, L_v^q)$ *then* $(v,w) \in W_\alpha^{q,p}$.

The same is true for other multipliers and appropriate classes of pairs (v, w).

The proofs of all the theorems formulated above are carried out essentially by the same method, that is, by the representation of the operator under consideration as a composition of certain elementary transformations.

Let

$$x_+^\alpha = \begin{cases} x^\alpha, & x > 0 \\ 0, & x < 0 \end{cases}$$

and

$$x_-^\alpha = \begin{cases} 0, & x > 0 \\ |x|^\alpha, & x < 0. \end{cases}$$

We consider the following distributions:

$$l(\lambda) = \lambda_+^{-\alpha},$$

$$h(\lambda) = \lambda_+^{-\alpha} + \exp\{i\alpha\pi\}\lambda_-^{-\alpha}$$

and

$$\gamma(\lambda) = \lambda_+^{-\alpha} + \exp\{-i\alpha\pi\}\lambda_-^{-\alpha},$$

where $0 < \alpha < 1$.

It is known that their Fourier preimages are given by

$$\check{l}(x) = A(\alpha)x_+^{\alpha-1} + B(\alpha)x_-^{\alpha-1},$$

$$\check{h} = C(\alpha)x_+^{\alpha-1}$$

and

$$\check{\gamma}(x) = D(\alpha)x_-^{\alpha-1},$$

where

$$A(\alpha) = (2\pi)^{-1/2}\exp\left\{\frac{-i\alpha\pi}{2}\right\}\Gamma(1-\alpha),$$
$$B(\alpha) = (2\pi)^{-1/2}\exp\left\{\frac{i\alpha\pi}{2}\right\}\Gamma(1-\alpha),$$
$$C(\alpha) = (2\pi)^{-1/2}\Gamma(\alpha)\exp\left\{\frac{i\alpha\pi}{2}\right\},$$
$$D(\alpha) = (2\pi)^{-1/2}\Gamma(\alpha)\exp\left\{\frac{-i\alpha\pi}{2}\right\}.$$

(see [27], p. 172).

LEMMA 3.1 *After completion with respect to the norm of $L_w^p(\mathbb{R})$, the mapping $\varphi \to \psi$ defined on $S(\mathbb{R})$ by the Fourier transform equation*

$$\hat{\psi}(\lambda) = h(\lambda)\hat{\varphi}(\lambda)$$

generates a bounded operator from $L_w^p(\mathbb{R})$ to $L_v^q(\mathbb{R})$ $(1 < p < q < \infty)$ if $(v, w) \in U_\alpha^{q,p}$.

Proof The convolution of φ with the preimage of h, *i.e.* with $C(\alpha)x_+^{\alpha-1}$, gives the Riemann–Liouville operator $\mathcal{R}^\alpha$ on $\mathbb{R}$ (see [28], Theorem 7.1). By the assumptions $(v,w) \in U_\alpha^{q,p}$ it acts boundedly from L_w^p into L_v^q (see Theorem 1.3).

Similar propositions hold for the other "elementary multipliers" l and γ and for appropriate classes of pairs of weights. Henceforth the proofs will be given only for h.

Let us consider the family of operators $\mathcal{R}_\alpha^t$ defined by the Fourier transform equation

$$\hat{\psi}(\lambda) = h(\lambda - t)\hat{\varphi}(\lambda), \quad \varphi \in S(\mathbb{R}), \ t \in \mathbb{R}.$$

Since the shift of t in the Fourier image corresponds to the multiplication by $\exp\{itx\}$ of the Fourier image, the norms of $\mathcal{R}_\alpha^t$ coincide with that of $\mathcal{R}_\alpha$. ■

THEOREM 3.7 *Let $1 < p < q < \infty$ and let $(v, w) \in U_\alpha^{q,p}$. Suppose that a function m is defined by the formula*

$$m(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha} \, \mathrm{d}\mu(t) + \exp\{i\alpha\pi\} \int_{\lambda}^{\infty} (t - \lambda)^{-\alpha} \, \mathrm{d}\mu(t), \quad 0 < \alpha < 1 \tag{3.7}$$

where μ is a finite measure on $\mathbb{R}$. Then the operator $\mathcal{K}$ acts boundedly from $L_w^p(\mathbb{R})$ into $L_v^q(\mathbb{R})$.

Proof It is easy to see that m is a regular tempered distribution. Indeed, since the images of $\varphi \in S$ by the Riemann–Liouville and Weyl operators are bounded functions we have

$$\left| \int_{\mathbb{R}} m(\lambda)\varphi(\lambda) \, \mathrm{d}\lambda \right| = \left| \int_{\mathbb{R}} \left(\int_t^{\infty} \varphi(\lambda)(\lambda - t)^{-\alpha} \, \mathrm{d}\lambda + \exp\{i\alpha\pi\} \right. \right.$$
$$\left. \left. \times \int_{-\infty}^{t} \varphi(\lambda)(t - \lambda)^{-\alpha} \, \mathrm{d}\lambda \right) \mathrm{d}\mu(t) \right| \le c \operatorname{var} \mu < \infty.$$

Therefore $m\hat{\varphi} \in L^1(\mathbb{R})$. By definition of the operator $\mathcal{K}$ and the function m we have

$$\mathcal{K}\varphi(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} m(\lambda)\hat{\varphi}(\lambda) \exp\{i\lambda x\} \, \mathrm{d}\lambda$$
$$= (2\pi)^{-1/2} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(\lambda - t) \, \mathrm{d}\mu(t) \right) \hat{\varphi}(\lambda) \exp\{i\lambda x\} \, \mathrm{d}\lambda \right).$$

Changing the order of integration we get

$$\mathcal{K}_\varphi(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(\lambda - t)\hat{\varphi}(\lambda) \exp\{i\lambda x\} \, d\lambda \right) d\mu(t)$$
$$= (2\pi)^{-1/2} \int_{\mathbb{R}} \mathcal{R}_\alpha^t \, d\mu(t).$$

From Lemma 3.1 and the previous remark with respect to $\mathcal{R}_\alpha^t$ it follows that

$$\|\mathcal{K}f\|_{L_v^q} \le (2\pi)^{-1/2} \int_{\mathbb{R}} \|\mathcal{R}_\alpha^t \varphi\|_{L_v^q} \, d\mu(t) \le cM\|\varphi\|_{L_w^p},$$

where M is the total variation of μ. ■

Now we shall deal with Fourier multipliers in weighted Lebesgue spaces of vector-valued functions with values in $l^\theta(1 < \theta < \infty)$.

DEFINITION 3.5 *Let* $1 < p < \infty$, $1 \le \theta < \infty$. *By* $L_v^p(l^\theta)$ *is denoted the set of vector-valued functions* $f(x) = \{f_j(x)\}_{j=1}^\infty$, $x \in \mathbb{R}$, *with measurable components and with finite norm*

$$|f, L_v^p(l^\theta)| = \left(\int_{\mathbb{R}} \left(\sum_{j=1}^\infty |f_j(x)|^\theta \right)^{p/\theta} dv(x) \right)^{1/p}.$$

It is well-known that $L_v^p(l^\theta)$ is a Banach space (see [26], p. 162). Further, it is evident that if $f \in L_v^p(l^\theta)$ then $f_j \in L_v^p$ for all $j \in \mathbb{N}$.

Let $\vec{S}$ be the set of all vector-valued functions $\varphi = (\varphi_1, \varphi_2, \ldots)$ where $\varphi_j \in S$, $j \in \mathbb{N}$.

Note that the set $\vec{S} \cap L_v^p(l^\theta)$ is dense in $L_v^p(l^\theta)$ for $1 < p$, $\theta < \infty$ and an arbitrary measure v.

Indeed, let $f \in L_v^p(l^\theta)$ and a positive ε be given. By Proposition 2.2 for any j we can choose φ_j such that

$$\|f_j - \varphi_j\|_{L_v^p} < \varepsilon 2^{-j/p} \quad \text{when } 1 < p \le \theta$$

and

$$\|f_j - \varphi_j\|_{L^p_v} < \varepsilon 2^{-j/\theta} \quad \text{when } p > \theta.$$

When $1 < p \leq \theta$ for $\varphi = (\varphi_1, \varphi_2, \ldots)$ we have

$$\begin{aligned} \|f - \varphi, L^p_v(l^\theta)\| &= \left(\int_{\mathbb{R}} \left(\sum_{j=1}^{\infty} |f_j - \varphi_j|^\theta \right)^{p/\theta} \mathrm{d}v \right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}} \sum_{j=1}^{\infty} |f_j - \varphi_j|^p \, \mathrm{d}v \right)^{1/p} = \left(\sum_{k=1}^{\infty} \|f_j - \varphi_j\|_{L^p_v} \right)^{1/p} < \varepsilon. \end{aligned}$$

If $p > \theta$ then it is sufficient to use Jessen's inequality ([29], p. 182):

$$\begin{aligned} \|f - \varphi\|_{L^p_v(l^\theta)} &= \left(\int_{\mathbb{R}} \left(\sum_{j=1}^{\infty} |f_j - \varphi_j|^\theta \right)^{p/\theta} \mathrm{d}v \right)^{1/p} \\ &\leq \left(\sum_{j=1}^{\infty} \left(\int_{\mathbb{R}} |f_j - \varphi_j|^p \, \mathrm{d}v \right)^{\theta/p} \right)^{1/p} \\ &= \left(\sum_{j=1}^{\infty} \|f_j - \varphi_j\|^\theta_{L^p_v} \right)^{1/p} < \varepsilon. \end{aligned}$$

Thus we see that $\varphi \in \vec{S} \cap L^p_v(l^\theta)$ and $\|f - \varphi\|_{L^p_v(l^\theta)} < \varepsilon$. The Fourier transform of the vector-valued function $f \in L^p_w(l^\theta)$ is defined by $\hat{f} = Ff = \{\hat{f}_j\}_{j=1}^{\infty}$.

Recall that Ff_j is defined by means of the Fourier transform of distributions.

The convolution of the vector-valued function $f \in L^p_v(l^\theta)$ with a tempered distribution $h \in S'$ is considered coordinate-wise:

$$h * f = \{h * f_j\}_{j=1}^{\infty}.$$

The following equality for the Fourier transform of a convolution holds:

$$F(h * f) = F(h) \cdot F(f).$$

LEMMA 3.2 *The transform* $\varphi \longmapsto \psi$ *defined by the equality*

$$\psi(\lambda) = h(\lambda)\hat{\varphi}(\lambda), \quad \varphi \in \vec{S},$$

generates a bounded operator from $L^p_w(l^\theta)$ *to* $L^q_v(l^\theta)$ *when* $1 < p < q < \infty$, $1 < \theta < \infty$ *and the pair of weights* (v, w) *belongs to* $\cup^{q,p}_\alpha$.

The proof is analogous to that of Lemma 3.1. It should only be noted that

$$\left\| \int_{-\infty}^{x} f_j(y)(x-y)^{\alpha-1}\,\mathrm{d}y \right\|_{l^\theta} \le \int_{-\infty}^{x} \|f_j(y)\|_{l^\theta}(x-y)^{\alpha-1}\,\mathrm{d}y.$$

THEOREM 3.8 *Let* $1 < p < q < \infty$, $1 < \theta < \infty$ *and let the pair of weights* (v, w) *belong to* $\cup^{q,p}_\alpha$. *Then for the transform* $\mathcal{K}$ *defined by*

$$\hat{\mathcal{K}} f(\lambda) = m(\lambda)\hat{\varphi}(\lambda), \quad \varphi \in \vec{S},$$

where $m(\lambda)$ *is represented by* (3.7), *we have*

$$|\mathcal{K}_\varphi, L^q_w(l^\theta)| \le cM|\varphi, L^p_w(l^\theta)|,$$

with a constant c independent of f and m. The operator $\mathcal{K}$ *is extendable to a bounded operator from* $L^p_w(l^\theta)$ *to* $L^q_v(l^\theta)$.

Proof If we consider the superposition in $L^p_w(l^\theta)$ defined by

$$\hat{\psi}(\lambda) = h(\lambda - t)\hat{\varphi}(\lambda), \quad \varphi \in \vec{S}, \quad t \in \mathbb{R},$$

the norm of this operator coincides with its value when $t = 0$. The rest of the proof is the same as in Theorem 3.7. ∎

THEOREM 3.9 *Let $1 < p < q < \infty$, $1 < \theta < \infty$ and let $(v, w) \in U_{\alpha}^{q,p}$. Assume that the measurable functions m_j are defined by*

$$m_j(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha} \, d\mu_j(t) + \exp\{i\alpha\pi\} \int_{\lambda}^{\infty} (t - \lambda)^{-\alpha} \, d\mu_j(t) \qquad (3.8)$$

where μ_j are finite measures for which

$$\sup_j \operatorname{var} \mu_j < \infty. \qquad (3.9)$$

Then the operator $\mathcal{K}$ defined on $\vec{S}$ by the Fourier-transform equation

$$\widehat{\mathcal{K}_{\varphi}} = \{\mu_j \varphi_j\}_{j=1}^{\infty}, \quad \varphi \in \vec{S}, \qquad (3.10)$$

is extendable to a bounded operator from $L_w^p(l^\theta)$ to $L_v^q(l^\theta)$.

The proof of Theorem 3.9 will be divided into several steps.

Let us consider the transform $T(\lambda)$, $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$, $\lambda_j \in \mathbb{R}$, defined on $\vec{S}$:

$$T(\lambda): f_j \to g_j, \qquad \hat{g}_j(\lambda) = h(\lambda - \lambda_j)\hat{f}_j(\lambda), \quad j = 1, 2, \ldots, \quad \lambda \in \mathbb{R},$$

where as above

$$h(\lambda) = \lambda_+^{-\alpha} + \exp\{i\alpha\pi\}\lambda_-^{\alpha\pi} \quad \text{and} \quad \check{h}(x) = C(\alpha)x_+^{\alpha-1}.$$

LEMMA 3.3 *The transform $T(\lambda)$ is extendable to a bounded operator from $L_w^p(l^\theta)$ to $L_v^q(l^\theta)$ under the conditions that $1 < p < q < \infty$, $1 < \theta < \infty$ and $(v, w) \in U_{\alpha}^{q,p}$.*

Proof The operator $T(\lambda)$ can be represented as

$$T(\lambda) = L(\lambda)T(0)L(-\lambda)$$

where the operator L is defined by

$$\{f_j(x)\}_{j=1}^{\infty} \to \{\exp\{i\lambda x\} f_j(x)\}_{j=1}^{\infty}.$$

It is clear that $T(0)$ is the Riemann–Liouville operator and since the operator L is an isometry of $L^p_w(l^\theta)$ the desired result follows from Theorem 3.8.

Further we truncate the operator T and for any given n consider the operator

$$T(t_1, \dots, t_n): f \to g, \qquad \hat{g}_j(\lambda) = h(\lambda - t_j)\hat{f}_j(\lambda),$$

when $j \leq n$ and $\hat{g}_j(\lambda) = \hat{f}_j(\lambda)$ when $j > n$. ■

LEMMA 3.4 *Let* $1 < p < q < \infty$, $1 < \theta < \infty$, *and let* $(v, w) \in \cup^{q,p}_{\alpha}$. *Let* m_j *be defined by* (3.8). *Assume that the transform* $\mathcal{K}_n$ *is determined in* $L^p_w(l^\theta)$ *by*

$$\mathcal{K}_n f = G = \{G_j\},$$

$$\hat{G}_j = \begin{cases} m\hat{f}_j, & j \leq n, \\ \hat{f}_j, & j > n. \end{cases}$$

Then there exists a positive constant c independent of f and n such that

$$|\mathcal{K}_n f, L^q_v(l^\theta)| \leq cM|f, L^p_w(l^\theta)|.$$

Proof Without loss of generality we assume that the measures μ_j are positive and normalized to 1.

For $j \leq n$ we have

$$\begin{aligned} G_j(x) &= (2\pi)^{-1/2} \int_{\mathbb{R}} m(\lambda)\hat{f}_j(\lambda) \exp\{i\lambda x\}\, \mathrm{d}\lambda \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(\lambda - t)\, \mathrm{d}\mu_j \right) \hat{f}_j(\lambda) \exp\{i\lambda x\}\, \mathrm{d}\lambda \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(\lambda - t)\hat{f}_j(\lambda) \exp\{i\lambda x\}\, \mathrm{d}\lambda \right) \mathrm{d}\mu_j(t_j) \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \\ &\quad \times \left(\int_{\mathbb{R}} h(\lambda - t_j)\hat{f}_j(\lambda) \exp\{i\lambda x\}\, \mathrm{d}\lambda \right) \mathrm{d}\mu_1(t_1) \cdots \mathrm{d}\mu_n(t_n). \end{aligned}$$

Hence $\mathcal{K}_n$ can be represented in the form

$$\mathcal{K}_n f(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} T(t_1, \ldots, t_n) \, \mathrm{d}\mu_1(t_1) \cdots \mathrm{d}\mu_n(t_n).$$

Therefore

$$|\mathcal{K}_n f, L_v^q(l^\theta)| \le c|T(t_1, \ldots, t_n)f, L_v^q(l^\theta)|.$$

Applying Lemma 3.3 we can see that $\mathcal{K}_n$ is bounded from $L_w^p(l^\theta)$ to $L_v^q(l^\theta)$ with an upper estimate of the norm independent of n. ■

Proof of Theorem 3.9 First of all we show that $\lim_{n\to\infty} \mathcal{K}_n f = \mathcal{K} f$ exists in the sense of convergence in the norm of $L_v^q(l^\theta)$ for arbitrary $f \in \vec{S} \cap L_w^p(l^\theta)$.

Let

$$\mathcal{K}_{n_1} f - \mathcal{K}_n f = \{\psi_j\}_{j=1}^{\infty}, \quad n > n_1.$$

By definition of $\mathcal{K}_n$ and Lemma 3.4:

$$|\mathcal{K}_{n_1} f - \mathcal{K}_{n_2} f, L_v^q(l^\theta)| \le cM \left\| \left(\sum_{j=n_1}^{n} |f_j(x)|^\theta \right)^{1/\theta} \right\|_{L_w^p}.$$

The right-hand side tends to zero. The proof of the theorem now follows from the uniform boundedness principle (see [26], p. 73).

The following statements can be proved analogously:

THEOREM 3.10 *Let* $1 < p < q < \infty$, $1 < \theta < \infty$ *and let* $(v, w) \in \tilde{\mho}_\alpha^{q,p}$. *Let the functions* m_j *be defined by*

$$m_j(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha} \, \mathrm{d}\mu_j(t) + \exp\{-i\alpha\pi\} \int_{\lambda}^{\infty} (t - \lambda)^{-\alpha} \, \mathrm{d}\mu_j(t), \qquad (3.11)$$

where the m_j *are finite measures satisfying condition* (3.9). *Then the operator* $\mathcal{K}$ *defined by* (3.10) *is extendable to a bounded operator from* $L^p_w(l^\theta)$ *to* $L^q_v(l^\theta)$ *and*

$$\|\mathcal{K}\|_{L^p_w(l^\theta)\to L^q_v(l^\theta)} \leq cM.$$

THEOREM 3.11 *Suppose that the functions* m_j *are representable in the form*

$$m_j(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha}\,\mathrm{d}\mu_j(t)$$

with the condition (3.9). *The following statements hold*:

(i) *If* $1 < p < q < \infty$, $1 < \theta < \infty$ *and the pair* (v, w) *belongs to the class* $W^{q,p}_\alpha$ $(v \in V^p_\alpha)$, *then the operator* $\mathcal{K}$ *is bounded from* $L^p_w(l^\theta)$ *to* $L^q_v(l^\theta)$ *(acts boundedly from* $L^p(l^\theta)$ *to* $L^p_v(l^\theta)$);

(ii) *If* $1 \leq q < p < \infty$, $1 < \theta < \infty$ *and* $v \in \Gamma^{q,p}_\alpha$ *then the operator* $\mathcal{K}$ *acts boundedly from* $L^p(l^\theta)$ *into* $L^q_v(l^\theta)$.

Below the intervals of decomposition forming $\mathbb{I}$ we regard as enumerated by $\{I_j\}_{j=1}^\infty$.

PROPOSITION 3.1 *Let* m_j *be the functions for which the operator*

$$\widehat{Tf_j} = m_j\hat{f}_j$$

is bounded from $L^p_w(l^\theta)$ *to* $L^q_v(l^\theta)$ *for the weight pair* $(v, w) \in \cup^{q,p}_\alpha$ *and for* p *and* q *with* $1 < p < q < \infty$. *Let the function* m *be defined by* $m(\lambda) = m_j(\lambda)$, $\lambda \in I_j$. *Then the operator* $\mathcal{K}$ *defined by* (3.1) *is bounded from* $F^{p,\theta}_w$ *to* $F^{q,\theta}_v$.

The proof is evident in view of the equality

$$|\mathcal{K}f, F^{q,\theta}_{\beta,v}| = |Tf_\beta, L^q_v(l^\theta)|,$$

where $f_\beta = \{\beta_j f_j\}$.

Now Theorem 3.2 immediately follows from Proposition 3.1 and Theorem 3.9.

4 $\left(F^{q,\theta}_{\beta,w}, F^{p,\theta}_{\beta,v}\right)$ MULTIPLIERS. THE CASE $1 < q \le p < \infty$

Let us consider the function

$$\theta(\lambda) = \begin{cases} 1, & \text{when } \lambda > 0 \\ 0, & \text{when } \lambda < 0. \end{cases}$$

It is known (see [27]) that the tempered distribution $\check{\theta}$ is given by the equality

$$\check{\theta}(x) = (2\pi)^{1/2}\left(\frac{\delta(x)}{2} + (2\pi i)^{-1}\frac{1}{x}\right) \tag{4.1}$$

where δ is the Dirac function.

Let the transform $\varphi \longmapsto \psi$ be defined on $S(\mathbb{R})$ by the Fourier equation

$$\hat{\psi} = \theta\hat{\varphi}.$$

This equality corresponds to the convolution

$$\psi = \check{\theta} * \varphi.$$

according to (4.1) the latter leads to the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)(x-y)^{-1}\,dy.$$

The considerations of previous sections together with two-weighted estimates for singular integrals proved in [30, 31] (see also [32]) enable us to prove assertions about $(F^{p,0}_{\beta,w}, F^{p,0}_{\beta,v})$ $(1 < p < \infty)$ and $(F^{p}_{\beta,w}, F^{q}_{\beta,v})$ $(1 < q < p < \infty)$ multipliers.

DEFINITION 4.1 *A pair of weights* (v, w) *belongs to* a_p $(1 < p < \infty)$, *if* $v(x) = \sigma(|x|)\rho(x)$, $w(x) = u(|x|)\rho(x)$, $\rho \in A_p(R)$, σ *and* u *are increasing functions on* $(0, \infty)$ *and*

$$\sup_{t>0}\int_{|x|>t} v(x)|x|^{-p}\,dx\left(\int_{|x|<t} w^{1-p'}(x)\,dx\right)^{p-1} < \infty.$$

DEFINITION 4.2 *A pair* (v, w) *is said to be of class* b_p $(1 < p < \infty)$, *if* σ *and* u *decrease on* $(0, \infty)$, $\rho \in A_p(R)$ *and*

$$\sup_{t>0} \int_{|x|<t} v(x)\,\mathrm{d}x \left(\int_{|x|>t} w^{1-p'}(x)|x|^{-p'}\,\mathrm{d}x \right)^{p-1} < \infty.$$

In [30] it is proved that if $(v, w) \in a_p \cup b_p$, $1 < p < \infty$, then H is bounded from L^p_w into L^q_v.

THEOREM 4.1 *Let* $1 < p$, $\theta < \infty$ *and let* $(v, w) \in a_p \cup b_p$. *Suppose that the function* m *is expressed in any interval* I *of the decomposition* $\mathbb{I}$ *by the following form*

$$m(\lambda) = \int_{-\infty}^{\lambda} \mathrm{d}\mu\,(\lambda \in I),$$

where the positive measures μ_I *satisfy the condition*

$$\sup_I \operatorname{var} \mu_I < \infty.$$

Then $m \in \mathcal{M}(F^{p,\theta}_{\beta,w}, F^{p,\theta}_{\beta,v})$.

THEOREM 4.2 *Let* $1 < q < p < \infty$, $1 < \theta < \infty$. *If the pair of even, increasing on* $(0, \infty)$ *weight functions satisfies the condition*

$$\int_0^\infty \left[\left(\int_t^\infty v(x)x^{-nq}\,\mathrm{d}x \right) \left(\int_0^{t/2} w^{1-p'}(x)\,\mathrm{d}x \right)^{q-1} \right]^{p/(p-q)} \cdot w^{1-p'}\left(\frac{t}{2}\right) \mathrm{d}t < \infty,$$

then the function m *from the previous theorem is an* $(F^{p,\theta}_{\beta,w}, F^{q,\theta}_{\beta,v})$ *multiplier.*

These statements follow in the same manner as the above-formulated corresponding theorems.

5 MULTIPLIERS IN WEIGHTED SPACES WITH MIXED NORMS

Let $\bar{p} = (p_1, \dots, p_n)$, $1 < p_i < \infty$, $i = 1, \dots, n$. Put $\bar{w} \equiv (w_1, \dots, w_n)$, where $w_i = w_i(x_i)$ $(i = 1, \dots, n)$ are weight functions defined on $\mathbb{R}$.

By definition $L^{\bar{p}}_{\bar{w}}(\mathbb{R}^n)$ is the space of functions f: $\mathbb{R}^n \to \mathbb{R}$ with the condition

$$\|f\|_{L^{\bar{p}}_{\bar{w}}} = \left(\int_{-\infty}^{\infty} \mathrm{d}v_1 \left(\int_{-\infty}^{\infty} \mathrm{d}v_2 \cdots \left(\int_{-\infty}^{\infty} |f(x)|^{p_n} \, \mathrm{d}v_n\right)^{1/p_n}\right)^{p_1/p_2}\right)^{1/p_1} < \infty. \tag{5.1}$$

The definition of $F^{\bar{p},\theta}_{\bar{\beta},\bar{w}}$ is similar to that given in Section 3. We have to use the norm (5.1)

Let $\mathbb{J}$ be a decomposition of $\mathbb{R}^n \backslash \{0\}$ of the type defined in Section 3 and let $J \in \mathbb{J}$.

THEOREM 5.1 *Let* $1 < p_i < q_i < \infty$, $1 < \theta < \infty$ *and let* $(v_i, w_i) \in W^{q_i,p_i}_{\alpha_i}(\mathbb{R})$, $i = 1, \dots, n$. *Suppose that a function m is represented in the form*

$$m(\lambda) = \int_{-\infty}^{\lambda_1} \cdots \int_{-\infty}^{\lambda_n} \prod_{j=1}^{n} (\lambda_j - t_j)^{-\alpha_j} \, \mathrm{d}\mu_J, \quad \lambda \in J, \tag{5.2}$$

where μ_J *are finite measures and*

$$\sup_{J \in J} \operatorname{var} \mu_J \le M. \tag{5.3}$$

Then $m \in \mathcal{M}(F^{\bar{p},\theta}_{\bar{\beta},\bar{w}}, F^{\bar{q},\theta}_{\bar{\beta},\bar{v}})$.

For the rest of this section we shall assume that

$$(1 + |x|)^{-q_i + \varepsilon} \in L^1_{v_i}(\mathbb{R})$$

and

$$(1 + |x|)^{-p_i + \varepsilon} \in L^1_{w_i}(\mathbb{R})$$

for some $\varepsilon > 0$.

THEOREM 5.2 *Let* $1 < p_i < q_i < \infty$, $1 < \theta < \infty$ *and let* $(v_i, w_i) \in U_{\alpha_i}^{q_i,p_i}$, *where* $0 < \alpha_i < 1$ $(i = 1, \dots, n)$. *Assume that the function m in each J is represented by*

$$m(\lambda) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} ((\lambda_i - t_i)_+^{-\alpha_i} + \exp\{i\alpha_i\pi\}(t_i - \lambda_i)^{-\alpha_i})\, \mathrm{d}\mu_J(t), \quad (5.4)$$

where μ_J, $J \in \mathbb{J}$ *satisfy the condition* (5.3). *Then* $m \in \mathcal{M}(F_{\beta,\bar{w}}^{\bar{p},\theta}, F_{\beta,\bar{v}}^{\bar{q},\theta})$.

THEOREM 5.3 *Let* $1 < p_i < q_i < \infty$, $1 < \theta < \infty$ *and* $(v_i, w_i) \in \tilde{U}_{\alpha_i}^{q_i,p_i}$, $(i = 1, \dots, n)$. *Let the function m be represented as*

$$m(\lambda) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} ((\lambda_i - t_i)_+^{-\alpha_i} + \exp\{-i\alpha_i\pi\}(t_i - \lambda_i)_-^{-\alpha_i})\, \mathrm{d}\mu_J(t)$$

under the condition (5.3). *Then* $m \in \mathcal{M}(F_{\beta,\bar{w}}^{\bar{p},\theta}, F_{\beta,\bar{v}}^{\bar{q},\theta})$.

The idea of the proof is similar to the one-dimensional case; however, we should make some remarks.

Let

$$l(\lambda) = \prod_{i=1}^{n} (\lambda_i)_+^{-\alpha},$$

$$h(\lambda) = \prod_{i=1}^{n} ((\lambda_i)_+^{-\alpha_i} + \exp\{i\alpha_i\pi\}(\lambda_i)_-^{-\alpha_i})$$

and

$$\gamma(\lambda) = \prod_{i=1}^{n} ((\lambda_i)_+^{-\alpha_i} + \exp\{-i\alpha_i\pi\}(\lambda_i)_-^{-\alpha_i}).$$

It is well-known (see [27]) that the Fourier transform of a direct product is given by direct product of Fourier images of factors.

The Fourier preimage of l is a linear combination of the products

$$(x_1)_+^{\alpha_1-1}\times\cdots\times(x_i)_+^{\alpha_i-1}\times(x_{i+1})_-^{\alpha_{i+1}-1}\times\cdots\times(x_n)_-^{\alpha_n-1}.$$

Analogously,

$$\check{h}(x)=\prod_{j=1}^{n}C(\alpha_j)(x_j)_+^{\alpha_j-1}$$

and

$$\check{\gamma}(x)=\prod_{j=1}^{n}D(\alpha_j)(x_j)_-^{\alpha_j-1}.$$

These lead to the integral operators:

$$I_{\bar\alpha}f(x)=\int_{\mathbb{R}^n}\prod_{j=1}^{n}|x_j-y_j|^{\alpha_j-1}f(y)\,\mathrm{d}y,$$
$$\mathcal{R}_{\bar\alpha}f(x)=\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}\prod_{j=1}^{n}(x_j-y_j)^{\alpha_j-1}f(y)\,\mathrm{d}y$$

and

$$\mathcal{W}_{\bar\alpha}f(x)=\int_{x_1}^{+\infty}\cdots\int_{x_n}^{+\infty}\prod_{j=1}^{n}(y_j-x_j)^{\alpha_j-1}f(y)\,\mathrm{d}y,$$

where $0<\alpha_j<1$ $(i=1,\dots,n)$, $\bar\alpha=(\alpha_1,\dots,\alpha_n)$.

Now making n-fold applications of appropriate one-dimensional two-weighted inequalities and using Minkowski's inequality we derive

PROPOSITION 5.1 *Let* $1<p_i<q_i<\infty$, $(i=1,\dots,n)$. *The following statements hold:*

(i) *If* $(v_i,w_i)\in W_{\alpha_i}^{q_i,p_i}$ $(i=1,\dots,n)$, *then* $I_{\bar\alpha}$ *is bounded from* $L_{\bar w}^{\bar p}(\mathbb{R}^n)$ *to* $L_{\bar v}^{\bar q}(\mathbb{R}^n)$

(ii) *If* $(v_i,w_i)\in U_{\alpha_i}^{q_i,p_i}$ $(i=1,\dots,n)$, *then* $\mathcal{R}_{\bar\alpha}$ *is bounded from* $L_{\bar w}^{\bar p}(\mathbb{R}^n)$ *to* $L_{\bar v}^{\bar q}(\mathbb{R}^n)$

(iii) *When* $(v_i, w_i) \in \tilde{U}^{q_i,p_i}_{\alpha_i}$ $(i = 1, \dots, n)$, *then* $\mathcal{W}_{\bar{\alpha}}$ *acts boundedly from* $L^{\bar{p}}_{\bar{w}}(\mathbb{R}^n)$ *to* $L^{\bar{q}}_{\bar{v}}(\mathbb{R}^n)$.

From Theorem 5.1 follows a two-weighted version of Mikhlin–Lizorkin type multipliers.

THEOREM 5.4 *Let m be continuous outside the coordinate planes and have there continuous derivatives*

$$\frac{\partial^k m}{\partial \lambda_1^{k_1} \cdots \partial \lambda_n^{k_n}}, \quad 0 \leq k_1 + k_2 + \cdots + k_n = k \leq n, \quad k_j = 0, 1.$$

Moreover assume that

$$\left| \lambda_1^{k_1+\alpha_1} \cdots \lambda_n^{k_n+\alpha_n} \cdot \left(\frac{\partial^k m}{\partial \lambda_1^{k_1} \cdots \partial \lambda_n^{k_n}} \right) \right| \leq M.$$

Then the following statements hold:

(i) *When* $1 < p_i < q_i < \infty$, $0 < \alpha_i < 1$, $i = 1, \dots, n$, $1 < \theta < \infty$ *and* $(v_i, w_i) \in W^{q_i,p_i}_{\alpha_i}$, *then* $m \in \mathcal{M}(F^{\bar{q},\theta}_{\beta,\bar{w}}, F^{\bar{p},\theta}_{\beta,\bar{v}})$.

(ii) *If* $p_i = q_i$, $\alpha_i = 0, j = 1, \dots, n$, $1 < \theta < \infty$ *and* (v_i, w_i) *satisfy the condition of Theorem* 5.1, *then* $m \in \mathcal{M}(F^{\bar{p},\theta}_{\beta,\bar{w}}, F^{\bar{p},\theta}_{\beta,\bar{v}})$.

Finally, in addition, if $w_i \in A_{pi}(\mathbb{R})$ and $v_i \in A_{qi}(\mathbb{R})$ $(i = 1, \dots, n)$, then we obtain $(L^{\bar{p}}_{\bar{w}}, L^{\bar{q}}_{\bar{v}})$ multiplier statements. The n-dimensional weighted version of the Littlewood–Paley theorem must be applied. The proof of the last result (see [1]) works in weighted Lebesgue spaces with mixed norms as well.

6 EXAMPLES

Here on the basis of the previous sections' results we derive various examples of pairs of weights ensuring validity of two-weight estimates for appropriate multipliers.

PROPOSITION 6.1 *Let $1 < p$, $0 < \infty$ and m be a function of the form*

$$m(\lambda) = \int_{-\infty}^{\lambda} \mathrm{d}\mu_I \quad \lambda \in I,$$

where I are intervals of decomposition of $\mathbb{R}$ and the finite measures μ_I are such that

$$\sup \operatorname{var} \mu_I = M < \infty.$$

Assume that

$$w(x) = \begin{cases} |x|^{p-1} \ln^p \dfrac{1}{|x|} & \text{when } |x| \leq \exp\{-p'\} \\ (p')^p \exp\{-p + \beta p'\}|x|^{\beta} & \text{when } |x| > \exp\{-p'\} \end{cases}$$

and

$$v(x) = \begin{cases} |x|^{p-1} & \text{when } |x| \leq \exp\{-p'\} \\ \exp\{-\gamma p' - p\}|x|^{\gamma} & \text{when } |x| > \exp\{-p'\}, \end{cases}$$

where $0 < \gamma \leq \alpha < p - 1$. Then $m \in \mathcal{M}(F^{p,0}_{\beta,w}, F^{p,0}_{\beta,v})$.

It is easy to show that the pair (v, w) satisfies the condition a_p from Definition 4.1 and thus from Theorem 4.1 we obtain Proposition 6.1.

Note that these weights do not belong to the A_p class. On the other hand, the conditions

$$\int_{\mathbb{R}} w(x)(1 + |x|)^{-p+\varepsilon} \,\mathrm{d}x < \infty \tag{6.1}$$

and

$$\int_{\mathbb{R}} v(x)(1 + |x|)^{-p+\varepsilon} \,\mathrm{d}x < \infty \tag{6.2}$$

are satisfied, so that $S \subset F^{p,0}_{\beta,w} \cap F^{p,0}_{\beta,v}$.

PROPOSITION 6.2 *Let* $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$. *Assume that a function* m *is represented by the formula*

$$m(\lambda) = \int_{-\infty}^{\lambda} (\lambda - t)^{-\alpha}\, d\mu_I,$$

where μ_I *satisfy the condition indicated in previous proposition. If the function* w *satisfies the* A_{pq} *condition, i.e.*

$$\sup\left(\frac{1}{|I|}\int_I w^q(x)\,dx\right)^{1/q}\left(\frac{1}{|I|}\int_I w^p(x)\,dx\right)^{1/p} < \infty, \tag{6.3}$$

where the supremum is taken over all one-dimensional intervals I, *then*

$$m \in \mathcal{M}(F^{p,\theta}_{\beta,w^p}, F^{q,\theta}_{\beta,w^q}).$$

In particular, if $0 < \beta < p - 1$, $w(x) = |x|^{\beta}$ *and* $v(x) = |x|^{\beta q/p}$ *then the condition* (6.3) *is satisfied and, consequently,* $(v, w) \in W^{q,p}_{\alpha}$.

Example 6.1 Suppose that $1 < p < q < \infty$, $1/p < \alpha < 1/q'$.

Let

$$w(x) = \begin{cases} |x|^{p-1}\ln^p \dfrac{1}{|x|} & \text{when } |x| \le \exp\{-p'\} \\ \exp\{p'\lambda\}(p')^p|x|^{\lambda} & \text{when } |x| > \exp\{-p'\} \end{cases}$$

and

$$v(x) = \begin{cases} |x|^{\gamma} & \text{when } |x| \le \exp\{-p'\} \\ |x|^{\beta}\exp\{p'(\beta - \gamma)\} & \text{when } |x| > \exp\{-p'\}, \end{cases}$$

where $0 < \gamma = q - q\alpha - 1$, $0 < \lambda < p - 1$, $0 < \beta < (1-\alpha)q + \lambda q/p - q/p' - 1$. Then the function m from the previous theorem belongs to $\mathcal{M}(F^{p,\theta}_{\beta,w}, F^{q,\theta}_{\beta,v})$.

The above-defined pair (v, w) satisfies the conditions (1.9), (1.10).

Note also that weight functions v and w satisfy the conditions (6.1) and (6.2) for p and q respectively.

Example 6.2 Let $\alpha \in (0,1)$, $1-p<\mu_0<p/q-\alpha p$, $\alpha p-1 \leq \mu_1<p-1$, $q/p-1<\varepsilon<q/p-1+q$, $\gamma=-\alpha q+q/p-\varepsilon-\mu_0 q/p$, $w(x)=(1+|x|)^{-\mu_0-\mu_1}\ |x|^{\mu_1}$, $v(x)=(1+|x|)^{\gamma}|x|^{-q/p+\varepsilon}$. Then $\mathcal{R}_\alpha$ is bounded from $L^p_w(\mathbb{R})$ to $L^q_v(\mathbb{R})$ (see [28], p. 93) and $w \in A_p(\mathbb{R})$, $v \in A_q(\mathbb{R})$.

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