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STRONG AND ITERATED MAXIMAL FUNCTIONS, AND APPLICATIONS TO THE MEAN SUMMABILITY OF THE DOUBLE TRIGONOMETRIC FOURIER SERIES

In this note two-weight criteria for strong and iterated Hardy–Littlewood maximal functions

$$M_S f(x,y) = \sup_{\substack{x\neq t\\ y\neq\tau}} \frac{1}{(x-t)(y-\tau)} \int_t^x \int_\tau^y |f(s,\sigma)| ds d\sigma;\ x,t,y,\tau\in\mathbb{R}. \tag{1}$$

$$(M_1M_2)f(x,y) = \sup_{x\neq t} \frac{1}{x-t} \int_t^x \bigg(\sup_{y\neq\tau} \frac{1}{y-\tau} \int_\tau^y |f(s,\sigma)| d\sigma \bigg) ds;\ x,t,y,\tau\in\mathbb{R}. \tag{2}$$

are established provided that the weights satisfy some additional conditions. Applications to the mean summability problem for double trigonometric Fourier series in weighted Lebesgue spaces are presented.

Let ρ be an almost everywhere positive function on $\mathbb{R}^n$.

We denote by $L^p_w(\mathbb{R}^n)$ $(1<p<\infty)$ the weighted Lebesgue space which is the class of all measurable functions with finite norm

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \bigg(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \bigg)^{1/p}.$$

The one-weight problem for the operator M_S has been studied in [1-2]. The only known result concerning the two-weight inequality for the operator M_S is the following statement due to E. Sawyer (see [7]):

Theorem A. *Let $1<p<\infty$. Then M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$ if and only if*

$$\int_G \Big[M_S(\chi_G w^{1-p'}) \Big]^p v \le c \int_G w^{1-p'} < \infty,\ p' = \frac{p}{p-1},$$

for all bounded open connected sets $G\subset\mathbb{R}^2$, provided that the operator

$$f \to \sup_{I\times J\ni(x,y)} \int_I \int_J |f| d\sigma,$$

where I and J are arbitrary intervals in $\mathbb{R}$ and $\sigma = w^{1-p'}$, is bounded in $L^q_\sigma(\mathbb{R}^2)$ $(1<q<p<\infty)$, or $w(x,y) = w_1(x)w_2(y)$.

Necessary and sufficient conditions for the two-weight inequality for the strong fractional maximal functions

$$M_{\alpha,\beta} = \sup_{\substack{x\neq t\\ y\neq\tau}} \frac{1}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} \int_t^x \int_\tau^y |f(s,\sigma)| ds d\sigma;\ x,t,y,\tau\in R.$$

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has been found in [3] (see also [4]) provided that the weight on the right hand side is of product type.

First we present the criteria guaranteeing the two-weight inequality for the one-dimensional Hardy-Littlewood maximal function

$$Mg(x) = \sup_{x \neq t} \frac{1}{x-t} \int_t^x |g(\tau)| d\tau, \quad x, t \in \mathbb{R}.$$

The two-weight problem for the operator M has been solved in [8]. For more trasparent sufficient conditions for the two-weight inequality for the operator M see [5-6]. We have the following statements:

Proposition 1. *Let $1 < p < \infty$. Suppose that v and w be even and increasing on $(0, \infty)$ weights.*

Then M is bounded from $L^p_w(\mathbb{R})$ to $L^p_v(\mathbb{R})$ if and only if

$$A \equiv \sup_{t>0} \Big(\int_t^\infty v(s) s^{-p} ds \Big)^{1/p} \Big(\int_0^t w^{1-p'}(s) ds \Big)^{1/p'} < \infty.$$

Moreover, there exist positive constants c_1 and c_2 depending only on p such that

$$c_1 A \leq \|M\| \leq c_2 A.$$

Proposition 2. *Let $1 < p < \infty$. Suppose that v and w be even and decreasing on $(0, \infty)$ weights.*

Then M is bounded from $L^p_w(\mathbb{R})$ to $L^p_v(\mathbb{R})$ if and only if

$$A_1 \equiv \sup_{r>0} A_1(r) \equiv \sup_{r>0} \Big(\frac{1}{r} \int_0^r v(s) ds \Big)^{1/p} \Big(\frac{1}{r} \int_0^r w^{1-p'}(s) ds \Big)^{1/p'} < \infty.$$

Moreover, there exist positive constants c_1 and c_2 depending only on p such that

$$c_1 A_1 \leq \|M\| \leq c_2 A_1.$$

To formulate the main results concerning the operator M_S we need

Definition. A Weight function $\rho : \mathbb{R}^2 \to \mathbb{R}^1$ is said to satisfy the doubling condition with respect to y uniformly to x on $\mathbb{R}_+$ ($\rho \in DC_x(\mathbb{R}_+)$) if there exists a positive constant c such that for arbitrary $t > 0$ and almost all $x > 0$ the inequality

$$\int_0^{2t} \rho(x, y) dy \leq c \int_0^t \rho(x, y) dy$$

holds. Analogously it can be defined the class $DC_y(\mathbb{R}_+)$.

Theorem 1. *Let $1 < p < \infty$. Suppose that the two-dimensional weights v and w are even and increasing on $(0, \infty)$ in each variable separately and, in addition, $w^{1-p'} \in DC_x(\mathbb{R})$, $DC_y(\mathbb{R})$.*

Then M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$ if and only if

$$B \equiv \sup_{a,b>0} \Big(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(xy)^p} dx dy \Big)^{1/p} \Big(\int_0^a \int_0^b w^{1-p'}(x,y) dx dy \Big)^{1/p'} < \infty. \quad (3)$$

Theorem 2. *Let $1 < p < \infty$. Suppose that the two-dimensional weights v and w are even and decreasing on $(0, \infty)$ in each variable separately and, in addition, suppose that $w^{1-p'} \in RD_x(\mathbb{R}_+), RD_y(\mathbb{R}_+)$.*

Then M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$ if and only if

$$B_1 \equiv \sup_{a,b>0} B_1(a,b) \equiv \sup_{a,b>0} \left(\frac{1}{ab}\int_0^a\int_0^b v(x,y)dxdy\right)^{1/p} \times$$
$$\times \left(\frac{1}{ab}\int_0^a\int_0^b w^{1-p'}(x,y)dxdy\right)^{1/p'} < \infty. \tag{4}$$

Theorem 3. *Let $1 < p < \infty$. Suppose that the two-dimensional weights v and w are even in each variable separately, increasing on $(0,\infty)$ in the first variable and decreasing on $(0,\infty)$ in the second variable. Suppose also that $w^{1-p'} \in DC_x(\mathbb{R}_+), DC_y(\mathbb{R}_+)$.*
Then M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$ if and only if

$$B_2 \equiv \sup_{a,b>0} \left(\frac{1}{b}\int_a^\infty\int_0^b \frac{v(x,y)}{x^p}dxdy\right)^{1/p} \times$$
$$\times \left(\frac{1}{b}\int_0^a\int_0^b w^{1-p'}(x,y)dxdy\right)^{1/p'} < \infty. \tag{5}$$

Example 1. Let $2 < p < \infty$, $v(x,y) = (|x|+|y|)^{-\alpha}(|xy|)^p$ and $w(x,y) = (|x|+|y|)^\beta$, where $\alpha = 2p-\beta$, $p \le \beta < 2(p-1)$. Then by Theorem 1 we have that M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$.

Example 2. Let $1 < p < \infty$ and $A = \min\{e^{-\frac{\beta}{p-1}}, e^{-\frac{\gamma}{p-1}}\}$, where $\beta = \gamma - 2p$, $\gamma > 2p-1$. Suppose that

$$v(x,y) = \begin{cases} |xy|^{p-1}\ln^\beta \frac{2A}{|xy|}, & |x|,|y| < \sqrt{A} \\ (\ln^\beta 2)A^{p-1-\lambda}|xy|^\lambda, & \text{therwise,} \end{cases}$$

$$w(x,y) = \begin{cases} |xy|^{p-1}\ln^\gamma \frac{2A}{|xy|}, & |x|,|y| < \sqrt{A} \\ (\ln^\gamma 2)A^{p-1-\lambda}|xy|^\lambda, & \text{therwise,} \end{cases}$$

where $-1 < \lambda < p-1$. Then for (v,w) the operator Ms is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$.

Let us now consider the case when the weight on the right hand side is a product of one-dimensional weights.

Theorem 4. *Let $1 < p < \infty$. Suppose that the two-dimensional weight v is even and increasing on $(0,\infty)$ in each variable separately. Suppose also that $w(x,y) = w_1(x)w_2(y)$, where w_1 and w_2 are even and increasing on $(0,\infty)$ weights.*
Then the following statements are equivalent:

(i) *M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*

(ii) *M_1M_2 is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*

(iii) *The condition* (1) *holds.*

Theorem 5. *Let $1 < p < \infty$. Suppose that the two-dimensional weight v is even and decreasing on $(0,\infty)$ in each variable separately. Suppose also that $w(x,y) = w_1(x)w_2(y)$, where w_1 and w_2 are even and decreasing on $(0,\infty)$ weights.*
Then the following statements are equivalent:
(i) *M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*
(ii) *M_1M_2 is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*

(iii) *The condition* (2) *holds.*

Theorem 6. *Let $1 < p < \infty$. Suppose that the two-dimensional weight v is even in each variable separately, increasing on $(0, \infty)$ in the first variable and decreasing on $(0, \infty)$ in the second variable. Further, assume that $w(x, y) = w_1(x)w_2(y)$, where w_1 is even and increasing on $(0, \infty)$; w_2 is even and decreasing on $(0, \infty)$.*

Then the following statements are equivalent:
(i) *M_S is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*
(ii) *M_1M_2 is bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$;*
(iii) *The condition* (3) *holds.*

Example 3. Let $1 < p < \infty$ and let

$$v(x, y) = \begin{cases} |xy|^{p-1} \ln \frac{4\beta}{|xy|}, & |x|, |y| \le \min\left\{1, \frac{2}{e^{\frac{\beta}{2(p-1)}}}\right\} \\ (\ln^\beta 4)|xy|^\lambda, & \text{otherwise,} \end{cases}$$

$$w(x, y) = \begin{cases} |x|^{p-1}|y|^\eta \ln^\gamma \frac{2}{|x|}, & |x|, |y| \le 1, \\ (\ln^\gamma 2)|xy|^\lambda, & \text{otherwise,} \end{cases}$$

where $\beta > -1$, $\gamma > p$, $0 < \eta < p - 1$, $\beta = \gamma - p - 1$. Then it is easy to verify that the pair (v, w) satisfies the conditions of Theorem 4 and consequently the operators M_S and M_1M_2 are bounded from $L^p_w(\mathbb{R}^2)$ to $L^p_v(\mathbb{R}^2)$.

Let $T^2 = (-\pi, \pi) \times (-\pi, \pi)$ and let $f : T^2 \to \mathbb{R}$ be an integrable, 2π-periodic function with respect to each variable separately. Suppose that

$$\sigma(f) = \sum_{m,n=0}^{\infty} \lambda_{mn}(a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + \\ + c_{mn} \cos mx \sin ny + d_{mn} \sin nx \sin my) \quad (6)$$

is the double Fourier series of a function f, where

$$\lambda_{mn} = \begin{cases} 1/4, & m = n = 0 \\ 1/2, & m = 0, n > 0; m > 0, n = 0 \\ 1, & m > 0, n > 0 \end{cases}$$

and a_{mn}, b_{mn}, c_{mn} and d_{mn} denote the Fourier coefficients of $f(x, y)$.

Let

$$\sigma^{(\alpha,\beta)}_{mn}(x, y, f) = \frac{\sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}A_{n-k}S_{jk}(x, y, f)}{A^\alpha_m A^\beta_n} \quad (\alpha, \beta > 0)$$

be the Cesaro (C, α, β) means of (4), where S_{jk} denote the partial sums of (4).

For some information concerning the Fourier trigonometric series see, e.g., [9], p. 464.

Now we formulate the statements concerning the mean summability for the double trigonometric Fourier series in weighted Lebesgue spaces.

Theorem 7. *Let $1 < p < \infty$. Suppose that a pair of weights (v, w) satisfies conditions of one of the Theorems $4 - 6$. Then*

$$\|\sup_{m,n} \sigma^{(\alpha,\beta)}_{mn}(\cdot, \cdot, f)\|_{L^p_v(T^2)} \le c\|f\|_{L^p_w(T^2)}$$

for arbitrary $f \in L^p_w(T^2) \cap L \ln^+ L(T^2)$.

Theorem 8. *Let $1 < p < \infty$ and let (v, w) satisfies the conditions of one of the Theorems $4 - 6$. Then*

$$\lim_{m,n\to\infty} \|\sigma^{(\alpha,\beta)}_{mn}(\cdot, \cdot, f) - f\|_{L^p_v(T^2)} = 0$$

for arbitrary $f \in L^p_w(T^2) \cap L \ln^+ L(T^2)$.

The analogous results for the Abel–Poisson means $U_f(x, y, r, \rho)$ are also valid (see, e.g., [9], p. 464, for the definition).

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