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TWO-WEIGHT ESTIMATES IN $L^{P(X)}$ SPACES FOR CLASSICAL INTEGRAL OPERATORS AND APPLICATIONS TO THE NORM SUMMABILITY OF FOURIER TRIGONOMETRIC SERIES

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In this note two-weighted norm inequalities in variable Lebesgue spaces for certain integral operators arising in harmonic analysis are presented. On the basis of these estimates for singular integrals and maximal functions we formulate the norm convergence and a summability by linear methods of Fourier series in a two-weighted setting. Together with this a two-weight version in $L^{p(x)}$ space of the Bernstein inequality is established.

For the boundedness of maximal functions and singular integrals in $L^{p(x)}$ spaces we refer to [2], [21], [1], [3]. Sobolev-type theorems have been established in [23,24], [8,9]. In [14-18], [4], [25] (see also [13], [26]) boundedness criteria were obtained for classical integral operators in weighted Lebesgue spaces with variable exponent when the weights are of power type; conditions sufficient for boundedness for oscillating weight functions are also derived. In the paper [11] criteria are given for a pair of weights (v, w) governing the norm summability of Fourier series in the weighted L^p_v space (p is a constant), for a function from (in general) the narrower class L^p_w.

The complete solution of the two-weight problem for singular integrals is still open in classical L^p spaces. Two-weighted estimates for the above-mentioned operators are known for pairs of weights of power type with fixed singularities (see [10], [5], [6]). Our interest in this problem is motivated by the usefulness of such inequalities in different types of boundary-value problems for partial differential equations.

It is worth mentioning that two-weight inequalities for singular integrals even with special weights are useful, for instance, in boundary- value problems of elliptic-type differential equations in "bad" domains. In [12] is shown that the presence of zero cusp points on the boundary can result in non-existence or non-uniqueness of solutions of Dirichlet and Neumann problems for harmonic functions from Smirnov classes and boundary functions from L^p spaces. In this connection, two-weight estimates for singular integrals enable one to identify, for boundary functions, the weighted Lebesgue spaces for which the problem becomes soluble.

Let Ω be an interval open set in R^n and let q be a measurable function on $\Omega \subseteq \mathbb{R}^n$ such that $1 \le q(x) \le \operatorname*{ess\,sup}_{x\in\Omega} q(x) < \infty$. Suppose that ρ is a weight function on Ω, i.e. ρ is a non-negative, almost everywhere positive function on Ω. By $L^{q(\cdot)}_\rho(\Omega)$ we denote the

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space of all measurable functions f on Ω such that

$$S_{q(\cdot),\rho}(f) := \int_{\Omega} |f(x)|^{q(x)}\rho(x)dx < \infty.$$

This is a Banach space with respect to the norm

$$\|f\|_{L^{q(\cdot)}_{\rho}(\Omega)} := \inf\Big\{\lambda > 0 : S_{q(\cdot),\rho}(f/\lambda) \le 1\Big\}.$$

If $\rho = 1$, then we use the symbol $L^{q(\cdot)}(\Omega)$ instead of $L^{q(\cdot)}_{\rho}(\Omega)$. Note that $L^{q(\cdot)}(\Omega)$ coincides with the classical Lebesgue space $L^{q}(\Omega)$ when $q(\cdot)$ is constant.

The next equality is obvious:

$$\|\rho f\|_{L^{q(\cdot)}(\Omega)} = \|f\|_{L^{q(\cdot)}_{\rho^{q(\cdot)}}(\Omega)}.$$

In the sequel we will use the following notation:

$$p_-(E) := \operatorname*{ess\,inf}_{x\in E}\, p(x);\;\; p_+(E) := \operatorname*{ess\,sup}_{x\in E}\, p(x),$$

where $p(\cdot)$ is a measurable function on $\mathbb{R}^n$ and E is a measurable set in $\mathbb{R}^n$.

Further, we denote:

$$p_0(x) := p_-(\{y : |y| < |x|\});\;\; \widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } |x| \le 1 \\ p_c = \text{const} & \text{if } |x| > 1 \end{cases}.$$

In the sequel we assume that K and M are the Calderón-Zygmund and the Hardy-Littlewood maximal operators respectively defined on $\mathbb{R}^n$.

Definition. We say that a function $p(\cdot)$ satisfies the Dini-Lipschitz condition on Ω ($p(\cdot) \in DL(\Omega)$) if

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{|x-y|}};\;\; 0 < |x-y| \le 1/2;\;\; x, y \in \bar{\Omega}.$$

We shall also need weighted Hardy-type operators:

$$(T_{v,w}f)(x) = v(x) \int_{|y|<|x|} f(y)w(y)dy, \quad x \in \mathbb{R}^n;$$

$$(T'_{v,w}f)(x) = v(x) \int_{|y|>|x|} f(y)w(y)dy, \quad x \in \mathbb{R}^n.$$

Easily verifiable necessary conditions and sufficient conditions governing two-weight inequalities in $L^{p(\cdot)}(\mathbb{R}_+)$ spaces for the Hardy operator defined on the semi-axis $\mathbb{R}_+$ have been established in [7] , where $p(\cdot)$ is a measurable function, provided $1 < p_- \le p(x) \le p_+ < \infty$. (For two-weight criteria for the Hardy operator defined on the interval $[0, 1]$ in Lebesgue spaces with variable exponent provided that $p(\cdot) \in DL([0, 1])$ see [19]).

Now we formulate the main results of this note.

Theorem 1. *Let $1 < p_- \le p(x) \le p_+ < \infty$ and let $p(\cdot) \in DL(\mathbb{R}^n)$ with $p(x) \equiv p_c \equiv$ const for $|x| > 1$. Suppose that v and w are weights on $\mathbb{R}^n$ such that*

a) $T_{v(\cdot)/|\cdot|^n, 1/w(\cdot)}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}^n)$*;*

b) $T'_{v(\cdot), \frac{1}{w(\cdot)|\cdot|^n}}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}^n)$*;*

c) *there exists a positive constant b such that*

$$\operatorname*{ess\,sup}_{|y|/4<|x|<4|y|} w(x) \le bw(y)$$

for almost all $y \in \mathbb{R}^n$.

Then there exists a positive constant c such that for all f with $\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, *the inequality*

$$\|(Nf)v\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds, where N is either the Hardy-Littlewood maximal function M or the Calderón-Zygmund singular integral K.

In particular, using two-weight estimates for the operators $T_{v(\cdot)/|x|^n,1/w(\cdot)}$ and $T'_{v(\cdot),1/(w(\cdot)|\cdot|^n)}$ we have

Theorem 2. *Let* $1 < p_- \leq p(x) \leq p_+ < \infty$. *Suppose that* $p(\cdot) \in DL(\mathbb{R}^n)$ *and* $p(x) \equiv p_c \equiv \mathrm{const}$ *for* $|x| > 1$. *Suppose that v and w are functions increasing on* $\mathbb{R}_+$ *and satisfying*

$$B := \sup_{t>0} \left(\int\limits_{|x|>t} (v(|x|)/|x|^n)^{p(x)} \left(\int\limits_{|y|<t} w^{-(\widetilde{p}_0)'(x)}(|y|)dy \right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx \right) < \infty. \quad (1)$$

Then there exists a positive constant c such that for all f with $\|f(\cdot)w(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$ *the inequality*

$$\|(Nf)(\cdot)v(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c\|f(\cdot)w(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds, where N is either M or K.

Now, based on two-weighted estimates for maximal functions and singular integrals, we study the mean summability of Fourier trigonometric series in variable Lebesgue $L^{p(x)}$ spaces. In the case when the exponent $p(\cdot)$ is constant the norm convergence (summability) in L^p_w holds if and only if w belongs to the Muckenhoupt class A_p (see [20], [22]). The paper [11] treated the situation when the weight w can be outside the A_p class. In this case the norm summability might hold in the sense of a wider space norm. Namely, a necessary and sufficient condition was established for a pair of weights (v, w) which governs the Abel and (C, α) summability in L^p_v for an arbitrary function f from L^p_w.

Let $\mathbb{T}$ denote the interval $(-\pi, \pi)$ and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (2)$$

be the Fourier series of a 2π -periodic function $f \in L^1(\mathbb{T})$.

Let $S_n(x, f)$, $\sigma_n^\alpha(x, f)$ $(0 < \alpha \leq 1)$ and $u_r(x, f)$ denote the partial sums, Cesàro and Abel means of the series (2) respectively.

Let

$$\widetilde{f}(e^{i\theta}) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int\limits_{\varepsilon<|\varphi|\leq\pi} f(e^{i(\theta-\varphi)}) \operatorname{ctg} \frac{\varphi}{2} d\varphi$$

be the conjugate- function operator and let

$$\bar{f}(e^{i\theta}) = \frac{1}{2\pi} \sup_{0<\varepsilon<\pi} \left| \int\limits_{\varepsilon<|\varphi|\leq\pi} f(e^{i(\theta-\varphi)}) \operatorname{ctg} \frac{\varphi}{2} d\varphi \right|$$

be the maximal conjugate function.

Theorem 3. *Let* $1 < p_- \leq p(x) \leq p_+ < \infty$ *and let* $p(\cdot) \in DL(\mathbb{T})$. *Assume that v and w are* 2π *-periodic, positive, increasing functions on* $(0, \pi]$ *that are even on* $\mathbb{T}$.

Let the following condition be satisfied:

$$\sup_{0<t<\pi}\Bigg(\int\limits_{t<|x|<\pi}(v(|x|)|x|^{-1})^{p(x)}\Bigg(\int\limits_{0<|y|<t}w^{-(p_0)'(x)}(|y|)dy\Bigg)^{\frac{p(x)}{(p_0)'(x)}}dx\Bigg)<\infty. \qquad (3)$$

Then

$$\|\bar{f}(\cdot)v(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{T})}\le c\|f(\cdot)w(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{T})}$$

with a constant c independent of f.

Theorem 4. *Let the conditions of Theorem* 3 *be satisfied. Then*

$$\lim_{n\to\infty}\|f(\cdot)-S_n(\cdot,f)\|_{L_v^{p(\cdot)}(\mathbb{T})}=0$$

for any $f\in L_w^p(\mathbb{T})$.

Theorem 5. *Let the conditions of Theorem* 3 *be satisfied. Then*

$$\lim_{n\to\infty}\|f(\cdot)-\sigma_n^\alpha(\cdot,f)\|_{L_v^{p(\cdot)}(\mathbb{T})}=0$$

and

$$\lim_{n\to\infty}\|f(\cdot)-u_r(\cdot,f)\|_{L_v^{p(\cdot)}(\mathbb{T})}=0$$

for arbitrary $f\in L_w^p(\mathbb{T})$.

Two-weight estimates for Cesàro means enable us to obtain the extended Berntein inequality for the derivative of trigonometric polynomial and its conjugate.

Theorem 6. *Let the conditions of Theorem* 3 *be satisfied. Then for an arbitrary trigonometric polynomial $T_n(x)$ and its conjugate $\widetilde{T}_n(x)$ we have*

$$\|T_n'v\|_{L^{p(\cdot)}(\mathbb{T})}\le cn\|T_nw\|_{L^{p(\cdot)}(\mathbb{T})}$$

and

$$\|\widetilde{T}_n'v\|_{L^{p(\cdot)}(\mathbb{T})}\le cn\|T_nw\|_{L^{p(\cdot)}(\mathbb{T})}.$$

Example 1. Let $v(x)=x^{\frac{\alpha}{p(x)}}$, $w(x)=x^\gamma$, where $p(\cdot)$ is a nonincreasing function on $[0,\pi]$ that is even on $\mathbb{T}$; $1<p_-(\mathbb{T})\le p_+(\mathbb{T})<\infty$; $p(\cdot)\in DL(\mathbb{T})$; $-p(\pi)<\alpha<-1$; $0<\gamma<\frac{1}{p'(\pi)}$; $\alpha=\gamma p(\pi)-p(\pi)$. Then condition (3) is satisfied for such v, w and p.

Example 2. Suppose that $n=1$; $p(\cdot)$ is a nonincreasing function on $[0,\infty)$ that is even on $\mathbb{R}$; $1<p_-\le p_+<\infty$; $p(\cdot)\in DL(\mathbb{R})$. Assume that $p(x)\equiv p_c=\text{const}$ when $|x|>1$. Further, let $v(x)=x^{\frac{\alpha}{p(x)}}$, $w(x)=x^\gamma$ on $[0,1]$ and let $v(x)=w(x)=x^\beta$ on $(1,\infty)$, where $-p(1)<\alpha<-1$, $0<\gamma<\frac{1}{p'(1)}$, $\alpha=\gamma p(1)-p(1)$ and $0<\beta<1/p_c$. Then condition (1) for $n=1$ is satisfied.

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