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ON TWO-WEIGHT CRITERIA FOR MAXIMAL FUNCTION IN $L^{p(x)}$ SPACES DEFINED ON AN INTERVAL

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In this note we give a necessary and sufficient condition on weight pair (v, w) guaranteeing the boundedness of the maximal operator

$$M_{\alpha(\cdot)}f(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha(x)}} \int_I |f(y)|dy, \quad x \in J, \quad 0 \le \alpha(x) \le \sup_J \alpha < 1,$$

from $L^{p(x)}_w(J)$ to $L^{p(x)}_v(J)$ provided that $(w(\cdot))^{-p'(\cdot)}$ satisfies the doubling condition on J, where $J := [a, b]$ $(-\infty < a < b < +\infty)$ and I denotes an interval in J. The derived criterion is of Sawyer [10] type.

If $\alpha(x) \equiv 0$, then $M_{\alpha(\cdot)}$ is the Hardy-Littlewood maximal function which will be denoted by M.

Weighted inequalities with power-type and oscillating weights for the operator M in $L^{p(x)}$ spaces have been established in [5], [6]. Muckenhoupt-type condition governing the one-weight inequality for M in variable exponent Lebesgue spaces was derived in [7], while two-weight inequalities for the Hardy-Littlewood maximal operator with monotonic radial weights were obtained in [2]. Necessary and sufficient conditions on weight pair (v, w) guaranteeing the boundedness of $M_{\alpha(\cdot)}$ from L^r_w to $L^{q(\cdot)}_v$ (r is constant) were found in [4]. For solution of the two-weight problem for fractional maximal functions with constant parameters in classical Lebesgue spaces we refer to [10], [11], [3, Ch.3] and references therein.

Suppose that p is measurable function on J with the condition

$$1 < p_-(J) \le p(x) \le p_+(J) < \infty,$$

where

$$p_-(J) := \inf_J p; \quad p_+(J) := \sup_J p.$$

Suppose also that ρ is a positive locally integrable function on J, i.e. ρ is a weight. We say that a measurable function $f : J \to \mathbf{R}$ belongs to $L^{p(\cdot)}_\rho(J)$ (or $L^{p(x)}_\rho(J)$) if

$$S_{p,\rho}(f) = \int_J \big|f(x)\rho(x)\big|^{p(x)}dx < \infty.$$

It is known that $L^{p(x)}_\rho(J)$ is a Banach space with the norm

$$\|f\|_{L^{p(x)}_\rho(J)} = \inf\big\{\lambda > 0 : S_{p,\rho}\big(f/\lambda\big) \le 1\big\}.$$

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If $p = \text{const}$, then $L^{p(\cdot)}_\rho$ coincides with the classical Lebesgue space with the weight ρ. Further, if $\rho \equiv 1$, then we use the symbol $L^{p(\cdot)}$ for $L^{p(\cdot)}_\rho$.

For basic properties of $L^{p(\cdot)}$ spaces we refer e.g. to [8], [9].

We say that $p : J \to \mathbf{R}$ satisfies the Dini-Lipschitz condition on J ($p \in DL(J)$) if there exists a positive constant A such that

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}; \quad x, y \in J; \quad |x-y| \le 1/2.$$

It is known that (see [1]) the operator M is bounded in $L^{p(\cdot)}(J)$ if $p \in DL(J)$.

A weight function ρ satisfies the doubling condition on J if there exists a positive constant b such that

$$\int\limits_{I(x,2r)} \rho \le b \int\limits_{I(x,r)} \rho$$

for all $x \in J$ and $r > 0$, where $I(x,r) := (x-r, x+r)$.

Our main result is the following statement:

Theorem 1. *Let $1 < p_-(J) \le p(x) \le p_+(J) < \infty$ and let $0 \le \alpha(x) \le \alpha_+(J) < 1$. Suppose that v and w be weights on J. Suppose also that $p, \alpha \in DL(J)$ and that $(w(\cdot))^{-p'(\cdot)}$ satisfies the doubling condition on J. Then $M_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}_w(J)$ to $L^{p(\cdot)}_v(J)$ if and only if there exists a positive constant c such that for all intervals I, $I \subset J$,*

$$\int\limits_I v^{p(x)} \big[M_{\alpha(x)}(\chi_I(\cdot) w^{-p'(\cdot)}(\cdot))(x)\big]^{p(x)} dx \le c \int\limits_I w^{-p'(x)}(x) dx.$$

In particular, if $\alpha \equiv 0$, then Theorem 1 gives the criterion for the boundedness of the Hardy-Littlewood maximal function M from $L^{p(\cdot)}_w(J)$ to $L^{p(\cdot)}_v(J)$.

As a corollary we have

Theorem 2. *Let $1 < p_-(J) \le p(x) \le p_+(J) < \infty$ and let $0 \le \alpha(x) \le \alpha_+(J) < 1$. Suppose that v be a weight function on J. Assume that $p, \alpha \in DL(J)$. Then $M_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(J)$ to $L^{p(\cdot)}_v(J)$ if and only if*

$$\sup_{\substack{I \\ I \subset J}} \frac{1}{|I|} \int\limits_I v(x)^{p(x)} |I|^{\alpha(x)p(x)} dx < \infty,$$

where the supremum is taken over all subintervals I of J.

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