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Mathematics

Maximal and Potential Operators in Variable Morrey Spaces Defined on Nondoubling Quasimetric Measure Spaces

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ABSTRACT. The boundedness of modified maximal operator and potentials in variable Morrey spaces defined on quasimetric measure spaces, where the doubling condition is not needed, is established. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *maximal operator, potential, non-homogeneous spaces, variable Morrey space, boundedness.*

Let $X := (X, \rho, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a non-negative real-valued function (quasimetric) d on $X \times X$ satisfying the conditions:

(i) $d(x, y) = 0$ for all $x, y \in X$;

(ii) $d(x, y) > 0$ for all $x \neq y,\ x, y \in X$;

(iii) there exists a constant $a_1 > 0$, such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$;

(iv) there exists a constant $a_0 > 0$, such that $d(x, y) \le a_0(d(y, x)$ for all $x, y \in X$.

We assume that the balls $B(a, r) := \{x \in X : \rho(a, x) < r\}$ are measurable, for all $a \in X$ and $r > 0$, and $0\, \mu(B(a, r)) < \infty$; for every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also suppose that $\mu(X) = \infty$ and $\mu\{a\} = 0$ for all $a \in X$.

The triple (X, ρ, μ) will be called quasimetric measure space. If μ satisfies the doubling condition

$$\mu(B(x, 2r)) \le c\mu(B(x, r)),$$

where the positive constant c does not depend on $x \in X$ and $r > 0$, then (X, d, μ) is called a space of homogeneous type (*SHT*). A quasimetric measure space, where the doubling condition might be failed, is also called a non-homogeneous space.

We say that the measure μ satisfies the growth condition ($\mu \in GC$) if there is a positive constant b such that for all $a \in X$ and $r > 0$,

$$\mu(B(a, r)) \le br \qquad (1)$$

The boundedness of maximal and potential operators in Lebesgue spaces on non-homogeneous spaces was established in [1] (for Euclidean spaces), [2-5] (see also [6, Ch. 6]).

Suppose that p is a μ-measurable function on X. Denote

$$p_-(E) := \inf_E p\ ;\quad p_+(E) := \sup_E p$$

for a μ-measurable set $E \subset X$;

$$p_- := p_-(X);\ p_+ := p_+(X).$$

Assume that $1 < p_- \le p_+ < \infty$. The Lebesgue space with variable exponent $L^{p(\cdot)}(X)$ (or $L^{p(x)}(X)$) is the class of all μ-measurable functions f on X for which

$$S_p(f) := \int_X |f(x)|^{p(x)} d\mu(x) < \infty .$$

The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\{\lambda > 0 : S_p(f/\lambda) \le 1\}.$$

It is known (see e.g. [7-10]) that $L^{p(\cdot)}$ space is a Banach space. For other properties of $L^{p(\cdot)}$ we refer to [7,25] etc.

The boundedness of the Hardy-Littlewood maximal and potential operators in $L^{p(x)}(\Omega)$ ($\Omega \subseteq R^n$) spaces was established in [12-15]. The same problem on an SHT was studied in [10,17-20] etc.

Definition 1. Let $N \ge 1$ be a constant. We say that $p \in P(N)$ if there is a positive constant C such that

$$\mu(B(x, Nr))^{p_-(B(x,r)) - p_+(B(x,r))} \le C \tag{2}$$

for all $x \in X$ and $r > 0$.

Now we are ready to define variable exponent Morrey spaces.

Definition 2. Let $N \ge 1$ be a constant. Suppose that $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. We say that $f \in M^{p(\cdot)}_{q(\cdot)}(X)_N$ if

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N} := \sup_{x \in X, r > 0} \mu(B(x, Nr))^{\frac{1}{p(x)} - \frac{1}{q(x)}} \|f\|_{L^{q(\cdot)}(B(x,r))} < \infty .$$

It is obvious that $\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N} = \|f\|_{L^{p(\cdot)}(X)}$ if $p(x) \equiv q(x)$.

For some properties of the spaces $M^{p(\cdot)}_{q(\cdot)}(\Omega)$, where Ω is a bounded domain in R^n see [21]. For variable exponent Morrey spaces on an SHT we refer e.g. [20].

The boundedness problem for maximal and fractional integrals in classical Morrey spaces ($p \equiv const$, $q \equiv const$) defined on Euclidean spaces was studied in [22-24]. The same problem for constant exponents in the case of quasimetric metric spaces was investigated in [5,26].

In [21] the boundedness of the Hardy-Littlewood maximal and Riesz potential operator in $M^{p(\cdot)}_{q(\cdot)}(\Omega)$ on a bounded domain $\Omega \subset R^n$ defined with respect to the Lebesgue measure was obtained. In [20] the authors have shown that maximal and Calderón-Zygmund operators on an SHT with finite measure and diameter are bounded in $L^{p(\cdot)}_{q(\cdot)}(X)$ provided that p satisfies log-Hölder continuity condition on X.

Let $N \ge 1$ be a constant and let

$$Mf(x) = \sup_{x \in X, r > 0} \frac{1}{\mu(B(x, Nr))} \int_{B(x,r)} |f(y)| d\mu(y) < \infty ;$$

$$I_{\alpha(x)} f(x) = \int_X \frac{f(y)}{d(x, y)^{1-\alpha(x)}} d\mu(y),\quad x \in X,\quad 0 < \alpha_- \le \alpha_+ < 1,$$

be the modified maximal and fractional integral operators respectively on a quasimetric measure space (X, d, μ).

Now we formulate the main results of the paper.

Theorem 1. *Let* $1 < p_- \le p_+ < \infty$ *and let* $N := a_1(1+2a_0)$, *where* a_0 *and* a_1 *are from the definition of the quasimetric d. If there exists a positive constant C such that for all* $x \in X$ *and* $r > 0$, *the inequality*

$$\mu(B(x, Nr))^{\,p_-(B(x,r)) - p(x)} \le c \tag{3}$$

holds, then M is bounded in $L^{p(\cdot)}(X)$.

Remark: Notice that condition (2) implies condition (3).

To formulate the next results we need the notation

$$\bar{a} := a_1(a_1(a_0+1)+1).$$

Theorem 2. *Let* $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. *Suppose that* $N := a_1(1+2a_0)$ *and* $p, q \in P(N)$. *Then M is bounded from* $L^{p(\cdot)}_{q(\cdot)}(X)_N$ *to* $L^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}$.

Theorem 3. *Let* $N := a_1(1+2a_0)$, $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$, $1 < t_- \le t(x) \le s(x) \le s_+ < \infty$. *Suppose that* $0 < \alpha_- \le \alpha_+ < \dfrac{1}{p_-}$, $s(x) = \dfrac{p(x)}{1-\alpha(x)p(x)}$, $\dfrac{t(x)}{s(x)} = \dfrac{q(x)}{p(x)}$ *and that* $p, q, \alpha \in P(N)$. *Suppose also that the measure* μ *satisfies condition* (1). *Then the operator* $I_{\alpha(x)}$ *is bounded from* $M^{p(\cdot)}_{q(\cdot)}(X)_N$ *to* $M^{s(\cdot)}_{t(\cdot)}(X)_{N\bar{a}}$.

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მათემატიკა

მაქსიმალური და პოტენციალური ოპერატორები გაორმაგების პირობის გარეშე კვაზიმეტრიკულ ზომად სივრცეებზე ცვლადმაჩვენებლიან მორის სივრცეებში

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სტატიაში მოდიფიცირებული მაქსიმალური ფუნქციებისა და პოტენციალებისათვის, რომლებიც განსაზღვრული არიან გაორმაგების პირობის გარეშე კვაზიმეტრიკულ ზომად სივრცეებზე, დამტკიცებულია შემოსაზღვრულობა ცვლადმაჩვენებლიან მორის სივრცეებში.

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