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V. Kokilashvili, A. Meskhi and M. Sarwar

# ONE AND TWO WEIGHT NORM ESTIMATES FOR ONE–SIDED OPERATORS IN $L^{p(\cdot)}$ SPACES

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In this note one and two weight estimates are presented for one-sided maximal functions and potentials in weighted Lebesgue spaces with variable exponent. In particular we present:

1) one-weight inequality for one-sided maximal operators;

2) two-weight estimates (criteria) for one-sided fractional maximal operators;

3) Fefferman–Stein type inequality for one–sided fractional maximal functions;

4) trace inequality for one-sided potentials;

5) a generalization of the Hardy-Littlewood theorem for the Riemann-Liouville and Weyl transforms.

From the results regarding one–sided maximal operators we conclude that the one–weight inequality for these operators automatically holds when both the exponent of the space and the weight are monotonic functions.

One-sided integral operators in $L^{p(\cdot)}$ spaces were studied in [9]. In particular, the authors established the boundedness of one-sided Hardy-Littlewood maximal functions, potentials and singular integrals in $L^{p(\cdot)}(I)$ spaces under the condition on $p$ which is weaker than the Log–Hölder continuity (weak Lipschitz) condition.

For a solution of the two–weight problem under transparent integral conditions on weights for one–sided maximal functions and potentials we refer to the monographs [11], [6] (Ch.2) and also references cited therein.

Necessary and sufficient conditions on a power–type weight guaranteeing weighted estimates for maximal and potential operators in $L^{p(\cdot)}$ spaces were obtained in [16]–[19], [10], [22], [23].

Weighted inequalities for two–sided maximal and potential operators in $L^{p(\cdot)}$ spaces with general weights were derived in [5], [7], [8], [12]–[15], [20], [21].

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In [2] necessary and sufficient conditions on a weight function $v$ governing the boundedness compactness of the generalized Riemann–Liouville transform $R_{\alpha(\cdot)}$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L_v^{q(\cdot)}(\mathbb{R}_+)$, $\alpha_- > 1/p_-$, were derived.

Let $I$ be an open set in $\mathbb{R}$ and let $p$ be a measurable function on $I$. Suppose that

$$1 \leq p_- \leq p_+ < \infty,$$

where $p_-$ and $p_+$ are the infimum and the supremum respectively of $p$ on $I$. We denote by $\|f\|_{L^{p(\cdot)}(I)}$ the norm of a measurable function $f$ on $I$. If $\rho$ is a weight function on $I$, then we define

$$\|f\|_{L^{p(\cdot)}_\rho(I)} := \|f\rho\|_{L^{p(\cdot)}(I)}.$$

Further, we denote

$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p, \qquad E \subset I;$$

$$I_+(x,h) := [x, x+h] \cap I, \quad I_-(x,h) := [x-h, x] \cap I;$$
$$I(x,h) := [x-h, x+h] \cap I.$$

We deal with the following integral operators:

$$\left(M^-_{\alpha(\cdot)} f\right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int\limits_{I_-(x,h)} |f(t)| dt, \quad x \in I;$$

$$\left(M^+_{\alpha(\cdot)} f\right)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int\limits_{I_+(x,h)} |f(t)| dt, \quad x \in I;$$

$$R_{\alpha(\cdot)} f(x) = \int\limits_{-\infty}^{x} \frac{f(t)}{(x-t)^{1-\alpha(x)}} dt;$$

$$W_{\alpha(\cdot)} f(x) = \int\limits_{x}^{\infty} \frac{f(t)}{(t-x)^{1-\alpha(x)}} dt, \quad x \in \mathbb{R},$$

where $0 < \alpha_- \leq \alpha_+ < 1$ and $I$ is an open set in $\mathbb{R}$.

**Definition A** ([9])**.** Let $\mathcal{P}_-(I)$ be the class of all measurable positive functions $p : I \to \mathbb{R}$ satisfying the following condition: there exist a positive constant $C_1$ such that for a.e $x \in I$ and a.e $y \in I$ with $0 < x - y \leq \frac{1}{2}$ the inequality

$$p(x) \leq p(y) + \frac{C_1}{\ln\left(\frac{1}{x-y}\right)} \tag{1}$$

holds. Further, we say that $p$ belongs to $\mathcal{P}_+(I)$ if $p$ is positive function on $I$ and there exists a positive constant $C_2$ such that for a.e $x \in I$ and a.e $y \in I$

with $0 < y - x \le \frac{1}{2}$ the inequality

$$p(x) \le p(y) + \frac{C_2}{\ln\left(\frac{1}{y-x}\right)} \tag{2}$$

is fulfilled.

**Definition B ([3]).** We say that a measurable positive function on $I$ belongs to the class $\mathcal{P}_\infty(I)$ ($p \in \mathcal{P}_\infty(I)$) if

$$|p(x) - p(y)| \le \frac{C}{\ln(e + |x|)} \tag{3}$$

holds for all $x, y \in I$ with $|y| \ge |x|$. .

**Definition C.** Let $p$ be a measurable function on an unbounded open set $I \subset \mathbb{R}$. We say that $p \in \mathcal{G}$ if there is a constant $0 < K < 1$ such that

$$\int_I K^{p(x)p_-/(p(x)-p_-)} dx < \infty.$$

**Theorem A ([9]).** *Let $I$ be a bounded interval in $\mathbb{R}$. Suppose that $1 < p_- \le p_+ < \infty$. Then*
(i) *if $p \in \mathcal{P}_-(I)$, then $M^-$ is bounded in $L^{p(\cdot)}(I)$;*
(ii) *if $p \in \mathcal{P}_+(I)$, then $M^+$ is bounded in $L^{p(\cdot)}(I)$.*

**Theorem B ([9]).** *Let $I$ be an open subset of $\mathbb{R}^n$, $1 < p_- \le p_+ < \infty$ and let (3) hold. Then*
(i) *if $p \in \mathcal{P}_-(I)$, then $M^-$ is bounded in $L^{p(\cdot)}(I)$;*
(ii) *if $p \in \mathcal{P}_+(I)$, then $M^+$ is bounded in $L^{p(\cdot)}(I)$.*

The next statement gives one–weight criteria for one-sided maximal operators in classical Lebesgue spaces (see [1]).

**Theorem C([1]).** *Let $I \subseteq \mathbb{R}$ be an interval. Assume that $0 \le \alpha < 1$ and $1 < p < 1/\alpha$, where $p$ and $\alpha$ are constants ($1/\alpha = \infty$ if $\alpha = 0$). We set $1/q = 1/p - \alpha$.*
(i) *Let $T := M_\alpha^-$. Then the inequality*

$$\left[\int_I |Tf(x)|^q v(x)dx\right]^{1/q} \le C\left[\int_I |f(x)|^p v^{p/q}(x)dx\right]^{1/p} \tag{4}$$

*holds if and only if*

$$\sup_{h>0}\left(\frac{1}{h}\int_{I_+(x,x+h)} v(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int_{I_-(x-h,x)} v^{-p'/q}(t)dt\right)^{\frac{1}{p'}} < \infty. \tag{5}$$

(ii) *Let $T := M_\alpha^+$. Then (4) holds if and only if*

$$\sup_{h>0}\left(\frac{1}{h}\int\limits_{I_-(x-h,x)} v(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int\limits_{I_+(x,x+h)} v^{-p'/q}(t)dt\right)^{\frac{1}{p'}}<\infty. \qquad (6)$$

**Definition D.** Let $I \subseteq \mathbb{R}_+$ be an interval. Suppose that $1 < p < q < \infty$, where $p$ and $q$ are constants. We say that the weight $v \in A^-_{p,q}(I)$ ( resp. $v \in A^+_{p,q}(I)$ ) if (5) ( resp. (6)) holds.

If $p = q$, then we denote the class $A^+_{p,q}(I)$ (resp. $A^-_{p,q}(I)$ ) by $A^+_p(I)$ (resp. $A^-_p(I)$).

Notice that $v \in A^+_{p,q}(I)$ (resp. $v \in A^-_{p,q}(I)$) is equivalent to the condition $v \in A^+_{1+q/p'}(I)$ (resp. $v \in A^-_{1+q/p'}(I)$).

**Definition E.** We say that a measure $\mu$ belongs to the class $RD^{(d)}(\mathbb{R}^n)$ (dyadic reverse doubling condition) if there exists a constant $\delta > 1$, such that for all dyadic cubes $Q$ and $Q'$, $Q \subset Q'$, $|Q| = \frac{|Q'|}{2^n}$, the inequality

$$\mu(Q') \geq \delta\mu(Q)$$

holds.

Now we formulate our main results regarding the one–sided maximal functions.

**Theorem 1.** *Let $I$ be a bounded interval in $\mathbb{R}$ and let $1 < p_- \leq p_+ < \infty$.*

(i) *If $p \in \mathcal{P}_+(I)$ and a weight function $w$ satisfies the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then for all $f \in L^{p(\cdot)}_w(I)$ the inequality*

$$\|(Nf)w\|_{L^{p(\cdot)}(I)} \leq C\|wf\|_{L^{p(\cdot)}(I)} \qquad (7)$$

*holds, where $N = M^+$.*

(ii) *Let $p \in \mathcal{P}_-(I)$ and let $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$. Then inequality* (7) *holds for all $f \in L^{p(\cdot)}_w(I)$, where $N = M^-$.*

The result similar to Theorem 1 has been derived in [20], [21] for the maximal operator defined on $\Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

In the case of unbounded intervals we have the next statement.

**Theorem 2.** *Let $I = \mathbb{R}_+$ and let $1 < p_- \leq p_+ < \infty$. Suppose that there is a positive number $a$ such that $p(x) \equiv p_c \equiv const$ outside $(0, a)$.*

(i) *If $p \in \mathcal{P}_+(I)$ and $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then* (7) *holds for $N = M^+$.*

(ii) *If $p \in \mathcal{P}_-(I)$ and $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$, then* (7) *holds for $N = M^-$.*

**Corollary 1.** *Let $p$ be increasing function on an interval $I = (a, b)$ such that $1 < p(a) \leq p(b) < \infty$. Suppose that $w$ is increasing positive function on $I$. Then the one–weight inequality*

$$\|w^{1/p(\cdot)}(Nf)(\cdot)\|_{L^{p(\cdot)}(I)} \leq c\|w^{1/p(\cdot)}f(\cdot)\|_{L^{p(\cdot)}(I)} \qquad (8)$$

*holds for* $N = M^+$.

**Corollary 2.** *Let* $p$ *be decreasing function on an interval* $I = (a, b)$ *such that* $1 < p(b) \le p(a) < \infty$. *Suppose that* $w$ *is decreasing positive function on* $I$. *Then the inequality* (8) *holds for* $N = M^-$.

**Theorem 3.** *Let* $I$ *be a bounded interval and let* $1 < p_- \le p_+ < \infty$. *Suppose that* $\alpha$ *is constant satisfying* $0 < \alpha < 1/p_+$. *Let* $q(x) = \frac{p(x)}{1-\alpha p(x)}$.

(i) *If* $p \in \mathcal{P}_+(I)$ *and a weight* $w$ *satisfies the condition* $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$. *Then the inequality*

$$\|(N_\alpha f)w\|_{L^{q(\cdot)}(I)} \le C\|wf\|_{L^{p(\cdot)}(I)}, \qquad f \in L^{p(\cdot)}_w(I) \tag{9}$$

*holds for* $N_\alpha = M^+_\alpha$.

(ii) *If* $p \in \mathcal{P}_-(I)$ *and let* $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$. *Then inequality* (9) *holds for* $N_\alpha = M^-_\alpha$.

**Theorem 4.** *Let* $I = \mathbb{R}_+$, $1 < p_- \le p_+ < \infty$ *and let* $p(x) \equiv p_c \equiv$ *const outside some interval* $(0, a)$. *Suppose that* $q(x) = \frac{p(x)}{1-\alpha p(x)}$, *where* $\alpha$ *is constant satisfying* $0 < \alpha < 1/p_+$.

(i) *If* $p \in \mathcal{P}_+(I)$ *and* $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, *then* (9) *holds for* $N_\alpha = M^+_\alpha$.

(ii) *If* $p \in \mathcal{P}_-(I)$ *and* $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$, *then* (9) *holds for* $N_\alpha = M^-_\alpha$.

**Theorem 5.** *Let* $p$, $q$ *and* $\alpha$ *be measurable functions on* $I = \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. *Suppose also that* $p \in \mathcal{G}(I)$. *Further, assume that* $w^{-(p_-)'} \in RD^{(d)}(I)$. *Then* $M^+_{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot)}_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if*

$$B \equiv \sup_{\substack{a\in\mathbb{R}\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(\mathbb{R})}\left\|\chi_{(a,a+h)}w^{-1}\right\|_{L^{(p_-)'}(\mathbb{R})} < \infty. \tag{10}$$

**Theorem 6.** *Let* $p$, $q$ *and* $\alpha$ *be measurable functions on* $I = \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. *Suppose also that* $p \in \mathcal{G}$ *and that* $w^{-(p_-)'} \in RD^{(d)}(I)$. *Then* $M^-_{\alpha(\cdot)}$ *is bounded from* $L^p_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if*

$$\sup_{\substack{a\in\mathbb{R}\\ h>0}} \left\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)-1}v(\cdot)\right\|_{L^{q(\cdot)}(I)}\left\|\chi_{(a-h,a)}w^{-1}\right\|_{L^{(p_-)'}(I)} < \infty. \tag{11}$$

**Corollary 3.** *Let* $I = \mathbb{R}$ *and* $1 < p < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$, *where* $p$ *is constant. Assume that* $w^{-p'} \in RD^{(d)}(\mathbb{R})$. *Then* $M^+_{\alpha(\cdot)}$ *is bounded from* $L^p_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if and only if*

$$\sup_{\substack{a\in\mathbb{R}\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)\; h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(I)}\left\|\chi_{(a,a+h)}w^{-1}\right\|_{L^{p'}(I)} < \infty.$$

**Corollary 4.** *Let $I = \mathbb{R}$ and let $1 < p < q_- \le q_+ < \infty$, where $p$ is constant. Suppose that $\alpha$ is measurable function on $\mathbb{R}$ satisfying $0 < \alpha_- \le \alpha_+ < 1$. Suppose also that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^-_{\alpha(\cdot)}$ is bounded from from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \big\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)-1}v(\cdot)\big\|_{L^{q(\cdot)}(I)}\big\|\chi_{(a-h,a)}w^{-1}\big\|_{L^{p'}(I)} < \infty.$$

**Corollary 5.** *Let $I = \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. Suppose that $p_- = p(\infty)$ and $p \in \mathcal{P}_\infty(I)$. Assume that $w^{-(p_-)'} \in RD^{(d)}(\mathbb{R})$. Then*

(i) *$M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if (10) holds;*

(ii) *$M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if (11) holds.*

In the diagonal case we have

**Theorem 7.** *Let $I = \mathbb{R}$ and let $1 < p < \infty$, where $p$ is constant. Suppose that $0 < \alpha_- \le \alpha_+ < \infty$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^p_v(I)$ if and only if there is a positive constant $C$ such that for all intervals $J \subset \mathbb{R}$,*

$$\int_{\mathbb{R}} v^p(x)\Big(M^+_{\alpha(\cdot)}\Big(w^{-p'}\chi_J\Big)(x)\Big)^p dx \le C\int_J w^{-p'}(x)dx < \infty.$$

**Theorem 8.** *Let $I = \mathbb{R}$ and let $1 < p < \infty$, where $p$ is constant. Suppose that $0 < \alpha_- \le \alpha_+ < \infty$.. Then $M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^p_v(I)$ if and only if*

$$\int_{\mathbb{R}} v^p(x)\Big(M^-_{\alpha(\cdot)}\Big(w^{-p'}\chi_J\Big)(x)\Big)^p dx \le C\int_J w^{-p'}(x)dx < \infty$$

*for all intervals $J \subset \mathbb{R}$.*

**Theorem 9.** *Let $\alpha$, $p$ and $q$ be measurable functions on $I = \mathbb{R}$. Suppose that $1 < p_- < q_- \le q_+ < \infty$ and $0 < \alpha_- \le \alpha_+ < 1/p_-$. Suppose that $p \in \mathcal{G}$. Then the following inequalities hold:*

$$\|\rho(\cdot)(M^+_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \le c\|f(\cdot)(\widetilde{N}^-_{\alpha(\cdot)}\rho)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})},$$

$$\|\rho(\cdot)(M^-_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \le c\|f(\cdot)(\widetilde{N}^+_{\alpha(\cdot)}\rho)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})},$$

*where*

$$\big(\widetilde{N}^-_{\alpha(\cdot)}\rho\big)(x) = \sup_{h>0} h^{-1/p_-}\|\rho(\cdot)h^{\alpha(\cdot)}\chi_{(x-h,x)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})},$$

$$\big(\widetilde{N}^+_{\alpha(\cdot)}\rho\big)(x) = \sup_{h>0} h^{-1/p_-}\|\rho(\cdot)h^{\alpha(\cdot)}\chi_{(x,x+h)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}.$$

Further, we have

**Theorem 10.** *Let $I=\mathbb{R}$ and let measurable functions $p$, $q$, and $\alpha$ satisfy the conditions $1<p_-<q_-\le q_+<\infty$, $0<\alpha_-\le\alpha_+<1$. Further, suppose that $p\in\mathcal{G}(I)$.*

(i) *If*

$$\sup_{J\subset\mathbb{R}}\big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L_v^{q(\cdot)}(\mathbb{R})}|J|^{-\frac{1}{p_-}}<\infty,$$

*where the supremum is taken over all bounded intervals $J\subset\mathbb{R}$, then $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ are bounded from $L^{p(\cdot)}(I)$ to $L_v^{q(\cdot)}(I)$.*

**Theorem 11.** *Let $p$ be constant. Suppose that $1<p<q_-\le q_+<\infty$. Let $0<\alpha_-\le\alpha_+<1$. Then the following are equivalent:*

(i) $R_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L_v^{q(\cdot)}(I)$*;*

(ii) $W_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L_v^{q(\cdot)}(I)$*;*

(iii)

$$\sup_{J\subset\mathbb{R}}\big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L_v^{q(\cdot)}(\mathbb{R})}|J|^{-\frac{1}{p}}<\infty. \tag{12}$$

**Corollary 6.** *Let $I=\mathbb{R}$ and let $p,q$ and $\alpha$ satisfy the conditions of Theorem 11. Then the following are equivalent:*

(i) $R_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$*;*

(ii) $W_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$*;*

(iii) (12) *holds for $v\equiv 1$.*


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Authors' addresses:

V. Kokilashvili and A. Meskhi
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193, Georgia

M. Sarwar
Abdus Salam School of Mathematical Sciences
GC University University
Lahore, Pakistan