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Editor's Choice

One-sided operators in $L^{p(x)}$ spaces

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Boundedness of one-sided maximal functions, singular integrals and potentials is established in $L^{p(x)}(I)$ spaces, where I is an interval in $\mathbf{R}$.

1 Introduction

In the paper we study the behavior of one-sided maximal functions, Calderón–Zygmund integrals and potentials in $L^{p(\cdot)}(I)$ spaces, where $I := (a, b)$, $-\infty \le a < b \le \infty$. Namely, we show that if I is a bounded interval, then these operators are bounded in $L^{p(\cdot)}(I)$ if p belongs to a certain class which is larger than the class of all functions satisfying the Dini–Lipschitz (log-Hölder continuity) condition. From these general results we conclude that left-sided (right-sided) operators are bounded in $L^{p(\cdot)}(I)$ if p is non-increasing (resp. non-decreasing). In the case $I = \mathbf{R}_+$ or $I = \mathbf{R}$ we assume, in addition, that p satisfies the "decay condition" at infinity.

Motivation for the study of one-sided operators acting between classical Lebesgue spaces is provided in [30], [22], [11]. Our extension of this study to the setting of variable exponent spaces is not only natural but has the advantage that it shows that one-sided operators may be bounded under weaker conditions on the exponent than were known for two-sided operators.

The paper is organized as follows: in Section 2 we introduce some basic notation and definitions. Sections 3 deals with one-sided maximal function, while in Sections 4 and 5 we study boundedness of the one-sided potentials and one-sided singular integrals respectively.

Constants (often different constants in the same series of inequalities) will generally be denoted by c or C.

2 Preliminaries

Let I be an open set in $\mathbf{R}$. We denote

$$p_E^- = \operatorname*{ess\,inf}_E p, \quad p_E^+ = \operatorname*{ess\,sup}_E p$$

for measurable functions $p : I \to \mathbf{R}$ and measurable sets $E \subseteq I$.

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Let $\mathcal{P}_-(I)$ be the class of all measurable functions $p : I \to \mathbf{R}$ satisfying the conditions:
1)

$$1 < p_I^- \le p(x) \le p_I^+ < \infty; \tag{2.1}$$

2) there exists a positive constant c_1 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < x-y \le 1/2$ the inequality

$$p(x) \le p(y) + \frac{c_1}{\ln(1/(x-y))} \tag{2.2}$$

holds. Further, we say that p belongs to $\mathcal{P}_+(I)$ if (2.1) holds and there exists a positive constant c_2 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < y - x \le 1/2$ the inequality

$$p(x) \le p(y) + \frac{c_2}{\ln(1/(y-x))} \tag{2.3}$$

holds.

It is easy to see that if p is a non-increasing function on I, then condition (2.2) is satisfied, while for non-decreasing p condition (2.3) holds.

Let $1 \le p(x) \le p_I^+ < \infty$. For a measurable function $f : I \to \mathbf{R}$, we say that $f \in L^{p(\cdot)}(I)$ (or $f \in L^{p(x)}(I)$) if

$$S_{p(\cdot)}(f) = \int_I \big|f(x)\big|^{p(x)}\, dx < \infty.$$

It is known that $L^{p(\cdot)}(I)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}(I)} = \inf\big\{\lambda > 0 : S_{p(\cdot)}\big(f/\lambda\big) \le 1\big\}.$$

For properties of $L^{p(\cdot)}$ spaces see e.g. [23], [21], [34], [35], [15], [25], [26], [27], [6], [14].

In the sequel we will use the notation that $p'(\cdot) := p(\cdot)/(p(\cdot)-1)$ for the function p satisfying (2.1).

Theorem A [6] *Let Ω be a bounded domain in $\mathbf{R}^n$. Then the maximal operator*

$$\big(\mathcal{M}_\Omega f\big)(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)\cap\Omega} |f(y)|\, dy, \quad x \in \Omega,$$

is bounded in $L^{p(x)}(\Omega)$ if $p \in \mathcal{P}(\Omega)$, that is,

(a) $1 < p_\Omega^- \le p(x) \le p_\Omega^+ < \infty$;

(b) *p satisfies the Dini–Lipschitz (log-Hölder continuity) condition ($p \in DL(\Omega)$): there exists a positive constant A such that for all $x, y \in \Omega$ with $0 \le |x-y| \le \frac{1}{2}$ the inequality*

$$\big|p(x) - p(y)\big| \le \frac{A}{\log \frac{1}{|x-y|}}$$

holds.

In the same paper, L. Diening proved the following statement:

Proposition A *Let Ω be a bounded domain in $\mathbf{R}^n$. Then $p \in DL(\Omega)$ if and only if there exists a positive constant C such that*

$$\big|B\big|^{p_-(B)-p_+(B)} \le C$$

for all balls B in $\mathbf{R}^n$ such that $|B \cap \Omega| > 0$.

The boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}(\mathbf{R}^n)$ was established in [24], [5] under the conditions that p belong to $\mathcal{P}(\Omega)$ and satisfies the "decay condition" at infinity (see [6] for the case when p is constant outside some ball). In particular the following statement holds:

Theorem B [5] *Let Ω be an open subset of $\mathbf{R^n}$. Then M_Ω is bounded in $L^{p(\cdot)}(\Omega)$ if*
(i) $p \in \mathcal{P}(\Omega)$*;*
(ii)

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)} \tag{2.4}$$

for all $x, y \in \Omega$, $|y| \geq |x|$.

Definition A We say that $p \in \mathcal{P}_\infty(I)$ if (2.1) and (2.4) hold.

In [9], [4] the boundedness of the Calderón–Zygmund singular integral was established in $L^{p(x)}(\mathbf{R}^n)$, while Sobolev-type theorems for the Riesz potentials have been obtained in [26], [27], [7], [4]. Weighted inequalities with power-type weights for the Hardy transforms, Hardy–Littlewood maximal functions, singular and fractional integrals were established in [18], [19], [13], [29], [32], [31], [20], [12], [10] and for general-type weights in [8], [17], [12] (see also [28], [16]).

Let

$$I_+(x,h) := [x, x+h] \cap I, \quad I_-(x,h) := [x-h, x] \cap I;$$
$$I(x,h) := [x-h, x+h] \cap I.$$

Observe that either $I_+(x,h) = \emptyset$ or $|I_+(x,h)| > 0$ because I is an open set. The same conclusion is true for $I_-(x,h)$ and $I(x,h)$.

Proposition B *Let p satisfy* (2.1)*. The following conditions are equivalent:*

(a) *condition* (2.2) *holds;*

(b) *there exists a positive constant C_1 such that for a.e. $x \in I$ and all r with $0 < r \leq \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$ the inequality*

$$r^{p^-_{I_-(x,r)} - p(x)} \leq C_1 \tag{1.2$'$}$$

holds;

(c) *the inequality*

$$r^{p(x) - p^+_{I_+(x,r)}} \leq C_2$$

holds, for a.e. $x \in I$ and all r with $0 < r \leq 1/2$ and $I_+(x,r) \neq \emptyset$.

P r o o f. Let us show that (a) is equivalent to (b). The fact (a) $\Leftrightarrow$ (c) can be obtained in a similar way. We follow [5]. Let (1.2$'$) be fulfilled and let us take $x, y \in I$ so that $0 < x - y \leq 1/2$. We choose r with $0 < r/2 \leq x - y \leq r$. Then

$$C_1 \geq r^{p^-_{I_-(x,r)} - p(x)} \geq c_p \left(\frac{1}{x-y}\right)^{p(x) - p^-_{I_-(x,r)}},$$

where $c_p = 2^{p^-_I - p^+_I}$. Hence

$$p(x) \leq p^-_{I_-(x,r)} + \frac{c}{\ln(1/(x-y))}.$$

Consequently, (2.2) holds.

Conversely, suppose that (2.2) holds and let us take r so that $0 < r \le 1/2$ and $I_-(x,r) \neq \emptyset$. Observe that if

$$S_{r,x} := (1/2) \operatorname*{ess\,sup}_{y \in I_-(x,r)} (p(x) - p(y)) \le 0,$$

then $p(x) \le p(y)$ for a.e. $y, y \in I_-(x,r)$. Therefore $p(x) \le p^-_{I_-(x,r)}$ and, consequently, $(1.2')$ holds for such r and x. Further, if $S_{r,x} > 0$, then we take x_0, $x_0 \in I_-(x,r)$, so that

$$0 < (1/2) S_{r,x} \le p(x) - p(x_0).$$

Hence

$$r^{p^-_{I_-(x,r)} - p(x)} \le \left(\frac{1}{x - x_0}\right)^{2(p(x) - p(x_0))} \le \left(\frac{1}{x - x_0}\right)^{2c/\ln(1/(x - x_0))} \le C. \qquad \square$$

The next statement can be proved in a similar manner; therefore we omit the proof.

Proposition B′ *Suppose that p satisfies* (2.1). *The following conditions are equivalent:*
(a) *condition* (2.3) *holds;*
(b) *the inequality*

$$r^{p^-_{I_+(x,r)} - p(x)} \le C_1$$

holds for a.e. $x \in I$ and all r with $0 < r \le \frac{1}{2}$ and $I_+(x,r) \neq \emptyset$;
(c) *the inequality*

$$r^{p(x) - p^+_{I_-(x,r)}} \le C_2$$

holds, for all $x \in I$ and all r satisfying $0 < r \le \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$.

Remark 2.1 Let I be a bounded interval in $\mathbf{R}$ and let p be continuous on I. Then $\mathcal{P}(I) = \mathcal{P}_-(I) \cap \mathcal{P}_+(I)$. Proposition B implies

Proposition C
a) $p' \in \mathcal{P}_-(I)$ *if and only if* $p \in \mathcal{P}_+(I)$*;*
b) *Let s be a positive constant. If p satisfies* (2.2) (*resp.* (2.3))*, then $s \cdot p$ also satisfies* (2.2) (*resp.* (2.3)).

Let us introduce the following maximal operators:

$$\big(\mathcal{M} f\big)(x) = \sup_{h>0} \frac{1}{2h} \int\limits_{I(x,h)} |f(t)|\, dt,$$

$$\big(\mathcal{M}_- f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_-(x,h)} |f(t)|\, dt,$$

$$\big(\mathcal{M}_+ f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_+(x,h)} |f(t)|\, dt,$$

where I is an open set in $\mathbf{R}$ and $x \in I$.

Let

$$R_r(x) := (e + |x|)^{-r}; \quad R(x) := R_1(x).$$

Lemma A **([5], [3])** *Let r and s be nonnegative functions on a set $G \subseteq \mathbf{R}$. Assume that β is a measurable function on G with values in $\mathbf{R}$. Suppose that*

$$0 \le s(x) - r(x) \le \frac{C}{\log(e + |\beta(x)|)}$$

for a.e. $x \in G$. Then there exists a positive constant C_r such that for every function f,

$$\int_G |f(x)|^{r(x)}\,dx \le C_r \int_G |f(x)|^{s(x)}\,dx + \int_G R_r(\beta(x))^{r_G^-}\,dx.$$

Lemma B ([3]) *Let r and s be nonnegative functions on a set $G \subseteq \mathbf{R}$. Suppose that for a.e. $x \in G$,*

$$|s(x) - r(x)| \le \frac{C}{\log(e + |x|)}.$$

Then there exists a positive constant C_r such that for every function f such that $f(x) \le 1$, $x \in G$,

$$\int_G |f(x)|^{r(x)}\,dx \le C_r \int_G |f(x)|^{s(x)}\,dx + \int_G R_r(x)^{r_G^-}\,dx.$$

Definition 2.2 Let $I = \mathbf{R}_+$ or $I = \mathbf{R}$. Suppose that p is a constant, $1 < p < \infty$. We say that $w \in A_p^+(I)$ if there exists $c > 0$ such that

$$\left(\frac{1}{h}\int_{x-h}^{x} w(t)\,dt\right)^{1/p}\left(\frac{1}{h}\int_{x}^{x+h} w^{1-p'}(t)\,dt\right)^{1/p'} \le c, \quad h, x > 0, \quad h < x,$$

for $I = \mathbf{R}_+$ and

$$\left(\frac{1}{h}\int_{x-h}^{x} w(t)\,dt\right)^{1/p}\left(\frac{1}{h}\int_{x}^{x+h} w^{1-p'}(t)\,dt\right)^{1/p'} \le c; \quad x \in \mathbf{R}, \quad h > 0,$$

for $I = \mathbf{R}$, where $p' = \frac{p}{p-1}$.

We say that $w \in A_1^+(I)$ if there exists $c > 0$ such that $(\mathcal{M}_- w)(x) \le c w(x)$ for a.e. $x \in \mathbf{R}$ when $I = \mathbf{R}$ and for a.e. $x \in \mathbf{R}_+$ whenever $I = \mathbf{R}_+$.

Further, $w \in A_p^-(I)$ if there exists $c > 0$ such that

$$\left(\frac{1}{h}\int_{x}^{x+h} w(t)\,dt\right)^{1/p}\left(\frac{1}{h}\int_{x-h}^{x} w^{1-p'}(t)\,dt\right)^{1/p'} \le c, \quad h, x > 0, \quad h < x,$$

for $I = \mathbf{R}_+$ and

$$\left(\frac{1}{h}\int_{x}^{x+h} w(t)\,dt\right)^{1/p}\left(\frac{1}{h}\int_{x-h}^{x} w^{1-p'}(t)\,dt\right)^{1/p'} \le c; \quad x \in \mathbf{R}, \quad h > 0,$$

for $I = \mathbf{R}$, where $p' = \frac{p}{p-1}$.

We say that $w \in A_1^-(I)$ if there exists $c > 0$ such that $(\mathcal{M}_+ w)(x) \le c\, w(x)$ for a.e. $x \in \mathbf{R}$ when $I = \mathbf{R}$ and for a.e. $x \in \mathbf{R}_+$ whenever $I = \mathbf{R}_+$.

It is easy to verify that $A_1^+(I) \subset A_p^+(I)$, $p > 1$ (see also [33] for $I = \mathbf{R}$).

Let ρ be locally integrable a.e. positive function (weight) on an interval I. Suppose that $1 < r < \infty$, where r is a constant. We denote by $L_\rho^r(I)$ the Lebesgue space with weight ρ, which is the space of all measurable functions $f : I \to \mathbf{R}$ for which

$$\|f\|_{L_\rho^r(I)} = \left(\int_I |f(x)|^r \rho(x)\,dx\right)^{1/r} < \infty.$$

The following statements can be found in [33] for $\mathbf{R}$ and [2] for $\mathbf{R}_+$.

Theorem 2.3 *Let $I = \mathbf{R}$ or $I = \mathbf{R}_+$. Suppose that p is a constant and that $1 < p < \infty$. Then*
(i) *$\mathcal{M}_+$ is bounded in $L^p_w(I)$ if and only if $w \in A^+_p(I)$;*
(ii) *$\mathcal{M}_-$ is bounded in $L^p_w(I)$ if and only if $w \in A^-_p(I)$.*

We shall also need

Definition 2.4 Let p and q be constants such that $1 < p < \infty$, $1 < q < \infty$. We say that $\mathcal{U} \in A^+_{pq}(\mathbf{R}_+)$ if

$$\sup_{0<h\le x}\left(\frac{1}{h}\int_{x-h}^{x}\mathcal{U}^q(t)\,dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int_{x}^{x+h}\mathcal{U}^{-p'}(t)\,dt\right)^{\frac{1}{p'}}<\infty.$$

Further, $\mathcal{U} \in A^-_{pq}(\mathbf{R}_+)$ if

$$\sup_{0<h\le x}\left(\frac{1}{h}\int_{x}^{x+h}\mathcal{U}^q(t)\,dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int_{x-h}^{x}\mathcal{U}^{-p'}(t)\,dt\right)^{\frac{1}{p'}}<\infty.$$

Theorem 2.3′ ([2]) *Let p and α be constants. Suppose that $1 < p < \frac{1}{\alpha}$ and $q = \frac{p}{1-\alpha p}$. Then the Weyl operator W_α given by*

$$W_\alpha f(x) = \int_x^\infty f(t)(t-x)^{\alpha-1}\,dt, \quad x \in \mathbf{R}_+,$$

is bounded from $L^p_{\mathcal{U}^p}(\mathbf{R}_+)$ to $L^q_{\mathcal{U}^q}(\mathbf{R}_+)$ if and only if $\mathcal{U} \in A^+_{pq}(\mathbf{R}_+)$. Further, the Riemann–Liouville operator

$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1}\,dt, \quad x \in \mathbf{R}_+,$$

is bounded from $L^p_{\mathcal{U}^p}(\mathbf{R}_+)$ to $L^q_{\mathcal{U}^q}(\mathbf{R}_+)$ if and only if $\mathcal{U} \in A^-_{pq}(\mathbf{R}_+)$.

Now we prove a one-sided version of Rubio de Francia's extrapolation theorem for variable exponent Lebesgue spaces. For a related statement in the two-sided case see [4].

Theorem 2.5 *Let $I = \mathbf{R}_+$ or $I = \mathbf{R}$. Let $\mathcal{F}$ be a family of pairs of nonnegative functions such that for some p_0 and q_0 with $0 < p_0 \le q_0 < \infty$ the inequality*

$$\left(\int_I f(x)^{q_0} w(x)\,dx\right)^{\frac{1}{q_0}} \le c_0 \left(\int_I g(x)^{p_0} w(x)^{p_0/q_0}\,dx\right)^{\frac{1}{p_0}} \tag{2.5}$$

holds for all $(f,g) \in \mathcal{F}$, where $w \in A^+_1(I)$ (resp. $A^-_1(I)$) and the positive constant c_0 depends on the $A^+_1(I)$ constant of the weight w. Given p satisfying (2.1) and also the condition $p_0 < p^-_I \le p^+_I < \frac{p_0 q_0}{q_0 - p_0}$, define a function q by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in I. \tag{2.6}$$

If $\mathcal{M}_-$ $\big(resp.\ \mathcal{M}_+\big)$ is bounded in $L^{(q(\cdot)/q_0)'}(I)$, then for all $(f,g) \in \mathcal{F}$ such that $f \in L^{q(\cdot)}(I)$ the inequality

$$\|f\|_{L^{q(\cdot)}(I)} \le c\,\|g\|_{L^{p(\cdot)}(I)}$$

holds.

P r o o f. Let us prove the theorem for $I = \mathbf{R}_+$ and $w \in A_1^+(I)$. The proof for other cases is the same. First notice that q satisfies (2.1). Let $\bar{p}(x) := \frac{p(x)}{p_0}$ and $\bar{q}(x) := \frac{q(x)}{q_0}$. Observe that $1 < (\bar{q}')_I^- \le (\bar{q}')_I^+ < \infty$. By assumption, $\mathcal{M}_+$ is bounded in $L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)$, i.e.,

$$\|\mathcal{M}_- f\|_{L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)} \le B\,\|f\|_{L^{(\bar{q})'}(\mathbf{R}_+)}.$$

Let us define $\mathcal{H}$ on $L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)$ as follows:

$$\mathcal{H}\phi(x) = \sum_{k=0}^{+\infty} \frac{\big(\mathcal{M}_-^{(k)}\phi\big)(x)}{2^k B^k},$$

where,

$$\mathcal{M}_-^{(k)} = \underbrace{\mathcal{M}_- \circ \mathcal{M}_- \circ \cdots \circ \mathcal{M}_-}_{k}; \quad \mathcal{M}_-^{(0)} = Id.$$

From the definition it follows that

(a) if $\phi \ge 0$, then $\phi(x) \le \big(\mathcal{H}\phi\big)(x)$;

(b)

$$\|\mathcal{H}\phi\|_{L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)} \le 2\,\|\phi\|_{L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)};$$

(c)

$$\mathcal{M}_-\big(\mathcal{H}\phi\big)(x) \le 2B\,\mathcal{H}\phi(x)$$

for every $x \in \mathbf{R}_+$.

The last implies that $\mathcal{H}\phi \in A_1^+(\mathbf{R}_+)$ with an $A_1^+(\mathbf{R})$ constant independent of ϕ.

Further, by the definition and elementary properties of $L^{p(\cdot)}$ spaces (see e.g. [21]) we have

$$\|f\|^{q_0}_{L^{q(\cdot)}(\mathbf{R}_+)} = \|\,|f|^{q_0}\|_{L^{\bar{q}(\cdot)}(\mathbf{R}_+)} \le \sup \int_{\mathbf{R}_+} |f(x)|^{q_0} h(x)\,dx,$$

where the supremum is taken over all nonnegative $h \in L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)$ with $\|h\|_{L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)} = 1$. Let us fix such an h. We will show that

$$\int_{\mathbf{R}_+} |f|^{q_0} h(x)\,dx \le c\,\|g\|^{q_0}_{L^{p(\cdot)}(\mathbf{R}_+)},$$

where c is independent of h and $f \in L^{q(\cdot)}(\mathbf{R})$. By (a), (b) and Hölder's inequality for $L^{p(\cdot)}$ spaces (see e.g. [21]) we have

$$\begin{aligned}\int_{\mathbf{R}_+} |f|^{q_0} h(x)\,dx &\le \int_{\mathbf{R}_+} |f|^{q_0}\ \mathcal{H}h(x)\,dx\\ &\le 2\,\|\,|f|^{q_0}\|_{L^{\bar{q}}(\mathbf{R}_+)}\|\mathcal{H}h\|_{L^{(\bar{q})'}(\mathbf{R}_+)}\\ &\le 2c\,\|f\|^{q_0}_{L^{q(\cdot)}(\mathbf{R}_+)}\|h\|_{L^{(\bar{q})'(\cdot)}(\mathbf{R}_+)}\\ &= 2c\,\|f\|^{q_0}_{L^{q(\cdot)}(\mathbf{R}_+)}\\ &< \infty.\end{aligned}$$

Using the fact that the $A_1^+(I)$ constant of $\mathcal{H}h$ is bounded by $2B$, applying (2.5) and Hölder's inequality with respect to $\bar{p}$ we find that

$$\int_{\mathbf{R}_+} |f|^{q_0}\mathcal{H}h(x)\,dx \le c\left[\int_{\mathbf{R}_+} g(x)^{p_0}\big(\mathcal{H}h(x)\big)^{\frac{p_0}{q_0}}\,dx\right]^{\frac{q_0}{p_0}} \le$$

$$\leq c\,\|g^{p_0}\|_{L^{\bar p}(\mathbf{R}_+)}^{\frac{q_0}{p_0}}\big\|(\mathcal{H}h)^{\frac{p_0}{q_0}}\big\|_{L^{(\bar p)'}(\mathbf{R}_+)}^{\frac{q_0}{p_0}}$$
$$= c\,\|g\|_{L^{p(\cdot)}(\mathbf{R}_+)}^{q_0}\big\|(\mathcal{H}h)^{\frac{p_0}{q_0}}\big\|_{L^{(\bar p)'}(\mathbf{R}_+)}^{\frac{q_0}{p_0}}.$$

Taking into account these estimates, it remains to show that

$$\big\|(\mathcal{H}h)^{\frac{p_0}{q_0}}\big\|_{L^{(\bar p)'}(\mathbf{R}_+)}^{\frac{q_0}{p_0}} \leq c,$$

where c is independent of h. From (2.6) we have

$$(\bar p)'(x) = \frac{p(x)}{p(x)-p_0} = \frac{q_0}{p_0}\,\frac{q(x)}{q(x)-q_0} = \frac{q_0}{p_0}(\bar q)'(x)$$

for $x \in \mathbf{R}_+$. Hence by (b) we conclude that

$$\big\|(\mathcal{H}h)^{\frac{p_0}{q_0}}\big\|_{L^{(\bar p)'(\cdot)}(\mathbf{R}_+)}^{\frac{q_0}{p_0}} = \|\mathcal{H}h\|_{L^{(\bar q)'(\cdot)}(\mathbf{R}_+)} \leq c\,\|h\|_{L^{(\bar q)'(\cdot)}(\mathbf{R}_+)} = c,$$

where c does not depend on h. □

3 One-sided maximal functions

In this section we establish the boundedness of one-sided maximal functions in $L^{p(x)}$ spaces. According to the next statement, a jumping exponent p implies the failure of the boundedness for the operator $\mathcal{M}$ in $L^{p(\cdot)}(I)$ but one of the one-sided maximal operators is bounded in the same space. In particular, we have

Proposition 3.1 *Let $I = [0, b]$ be a bounded interval. Then*

(a) *there exists a discontinuous function p on I such that $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$ but $\mathcal{M}$ is not bounded in $L^{p(\cdot)}(I)$.*

(b) *there exists a discontinuous function p on I such that $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(I)$ but $\mathcal{M}$ is not bounded in $L^{p(\cdot)}(I)$.*

Proof. Let p_1 and p_2 be constants such that $1 < p_2 < p_1 < \infty$ and let

$$p(x) = \begin{cases} p_1, & x \in (0, \beta], \\ p_2, & x \in (\beta, b], \end{cases}$$

where $0 < \beta < b$.

It is easy to see that the operator $\mathcal{M}_+$ (and consequently $\mathcal{M}$) is not bounded in $L^{p(\cdot)}(I)$. Indeed, let $f(x) = (x-\beta)^{-1/p_1}\chi_{(\beta,b)}(x)$. Then $\int_0^b (f(x))^{p(x)}\,dx < \infty$, while $\int_0^b \big(\mathcal{M}_+ f\big)^{p(x)}(x)\,dx = \infty$ since

$$\mathcal{M}_+ f(x) = \sup_{\beta - x \leq h \leq b - x} F(h) = F((\beta - x)p_1) = c\,(\beta - x)^{-1/p_1}$$

for $x \in (0, \beta]$, where the positive constant c depends only on p_1.

We shall now show that $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$. Let $\|f\|_{L^{p(\cdot)}(I)} \leq 1$ and let us represent f as follows: $f = f_1 + f_2$, where $f_1(x) = \chi_{(0,\beta]}(x)f(x)$, $f_2(x) = f(x) - f_1(x)$. Then we have

$$\int_0^b \big(\mathcal{M}_- f\big)^{p(x)}(x)\,dx \leq c\Bigg[\int_0^\beta \big(\mathcal{M}_- f_1\big)^{p_1}(x)\,dx + \int_\beta^b \big(\mathcal{M}_- f_1\big)^{p_2}(x)\,dx$$
$$+ \int_0^\beta \big(\mathcal{M}_- f_2\big)^{p_1}(x)\,dx + \int_\beta^b \big(\mathcal{M}_- f_2\big)^{p_2}(x)\,dx\Bigg]$$
$$:= c\sum_{i=1}^4 \mathrm{I}_i.$$

By the boundedness of $\mathcal{M}_L$ on $L^{p_1}(I)$, we have

$$\mathrm{I}_1 \leq \int_0^b \big(\mathcal{M}_- f_1\big)^{p_1}(x)\,dx \leq c \int_0^b |f(x)|^{p_1}\,dx \leq c \int_0^b |f(x)|^{p(x)}\,dx \leq c.$$

Further, it is easy to check that $\big(\mathcal{M}_- f_1\big)(x) \leq \sup_{x-\beta\leq h\leq x} \frac{(\beta-x+h)^{1/p_1'}}{h} = c\,(x-\beta)^{-1/p_1'}$ when $x \in (\beta, b)$. Consequently, since $p_2 < p_1$, we have $\mathrm{I}_2 < \infty$.

It is also obvious that $\mathrm{I}_3 = 0$, while due to the boundedness of $\mathcal{M}_-$ in $L^{p_2}(I)$, we see that

$$\mathrm{I}_4 \leq \int_c^b \big(\mathcal{M}_- f_2\big)^{p_2}(x)\,dx \leq c \int_c^b |f(x)|^{p_2}\,dx \leq c.$$

Analogously we can prove part (b). □

Proposition 3.1 motivates us to establish the boundedness of one-sided maximal function under a condition on $p(\cdot)$ which is weaker than the log-Hölder condition.

Theorem 3.2 *Let I be a bounded interval and let $p \in \mathcal{P}_-(I)$. Then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

P r o o f. We use the arguments from [6]. For simplicity let us assume that $I = (0, b)$. First we show the inequality

$$\big(\mathcal{M}_{-,h} f\big)^{p(x)}(x) \leq C(p) \left(\frac{1}{h} \int_{I_-(x,h)} |f(t)|^{p(t)}\,dt + 1\right), \quad 0 < h < x, \tag{3.1}$$

holds for all f with $\|f\|_{L^{p(\cdot)}} \leq 1$, where

$$\big(\mathcal{M}_{-,h} f\big)(x) := \frac{1}{h} \int_{I_-(x,h)} |f(y)|\,dy$$

and the positive constant $C(p)$ depends only on p.

If $h \geq \frac{1}{2}$, then

$$\begin{aligned}
\big(\mathcal{M}_{-,h} f\big)^{p(x)}(x) &= \left(\frac{1}{h} \int_{I_-(x,h)} |f(y)|\,dy\right)^{p(x)} \\
&\leq \left(\frac{1}{h} \int_{I_-(x,h)\cap\{|f|\geq 1\}} |f(y)|^{p(y)}\,dy + 1\right)^{p(x)} \\
&\leq \left(\frac{1}{h} \int_{I_-(x,h)} |f(y)|^{p(y)}\,dy + 1\right)^{p(x)} \\
&\leq (2+1)^{p(x)} \\
&\leq 3^{p_I^+}
\end{aligned}$$

which proves (3.1) for this case.

Let $h < 1/2$. Then using Hölder's inequality we have

$$\begin{aligned}\big(\mathcal{M}_{-,h}f\big)^{p(x)}(x) &\le \left(\frac{1}{h}\int\limits_{I_-(x,h)} |f(y)|^{p^-_{I_-(x,h)}}\,dy\right)^{\frac{p(x)}{p^-_{I_-(x,h)}}}\\ &\le \left(\frac{1}{h}\int\limits_{I_-(x,h)\cap\{|f|\ge 1\}} |f(y)|^{p(y)}\,dy + 1\right)^{\frac{p(x)}{p^-_{I_-(x,h)}}}\\ &\le h^{-\frac{p(x)}{p^-_{I_-(x,h)}}}\left(\int\limits_{I_-(x,h)} |f(y)|^{p(y)}\,dy + h\right)^{\frac{p(x)}{p^-_{I_-(x,h)}}}.\end{aligned}$$

Since $\int_0^b |f(x)|^{p(x)}\,dx \le 1$ and $0 < h < \frac{1}{2}$, we have that $\frac{1}{2}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}\,dy + \frac{1}{2}h \le 1$.

Consequently, taking into account the last estimate and the condition $p \in \mathcal{P}_-(I)$ we find that

$$\begin{aligned}(\mathcal{M}_{-,h})^{p(x)}(x) &\le Ch^{-\frac{p(x)}{p^-_{I_-(x,h)}}}\left(\frac{1}{2}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}\,dy + \frac{1}{2}h\right)\\ &= Ch^{\frac{p^-_{I_-(x,h)}-p(x)}{p^-_{I_-(x,h)}}}\left(\frac{1}{h}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}\,dy + 1\right)\\ &\le C\big(\mathcal{M}_{-,h}\big(|f|^{p(\cdot)}\big)(x) + 1\big).\end{aligned}$$

Thus (3.1) has been proved. Inequality (3.1) immediately implies

$$\big(\mathcal{M}_-f\big)^{p(x)}(x) \le C(p)\big[\big(\mathcal{M}_-\big(|f|^{p(\cdot)}\big)\big)(x) + 1\big]. \tag{3.2}$$

Suppose now that $q(x) = \frac{p(x)}{p_-}$. Then using the fact $q \in \mathcal{P}_-(I)$, inequality (3.2) and the boundedness of $\mathcal{M}_L$ in $L^{p_-}(I)$ we find that

$$\int_0^b \big(\mathcal{M}_-f(x)\big)^{p(x)}\,dx \le c\int_0^b \big(\mathcal{M}_-\big(|f|^{q(\cdot)}(x)\big)\big)^{p_-}\,dx + C \le C\int_0^b |f(x)|^{p(x)}\,dx + C \le C.$$

□

The next theorem follows analogously. Therefore we omit the proof.

Theorem 3.3 *Let I be a bounded interval and let $p \in \mathcal{P}_+(I)$. Then $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(I)$.*

Now we investigate the boundedness of one-sided maximal functions in $L^{p(x)}$ spaces defined on unbounded intervals.

We have the following one-sided version of Theorem 4.1 of [3] (see also Lemmas 2.3 and 2.5 of [5] for the two-sided case).

Proposition 3.4 *Let I be an open subset of $\mathbf{R}$. Suppose that $p \in \mathcal{P}_+(I) \cap \mathcal{P}_\infty(I)$. Suppose also that $S_{p(\cdot)}(f) \le 1$. Then there exists a positive constant C such that*

$$\big(\mathcal{M}_+f(x)\big)^{p(x)} \le C\big(\mathcal{M}_+\big(|f(\cdot)|^{p(\cdot)/p^-_I}\big)(x)\big)^{p^-_I} + S(x) \tag{3.3}$$

for a.e. $x \in I$, where $S \in L^1(\mathbf{R})$.

P r o o f. We use the arguments of Lemmas 2.3 and 2.5 in [5] and Theorem 4.1 in [3].

Let $f \geq 0$. We shall see that there exists a positive constant C such that for a.e. $x \in I$ and all $h > 0$,

$$\left(\frac{1}{h}\int\limits_{I_+(x,h)} f(t)\,dt\right)^{p(x)} \leq C\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{p(t)/p_I^-}\,dt\right)^{p_I^-} + S(x).$$

Let us denote

$$\mathcal{M}_{+,h}f(x) := \frac{1}{h}\int\limits_{I_+(x,h)} f(t)\,dt.$$

We divide the proof into two parts:

(a) $f(x) \geq 1$ or $f(x) = 0$, $x \in I$;

(b) $f(x) \leq 1$ on I.

P r o o f o f (a). **Case 1** ($h < |x|/4$)**.** Denote $\bar{p}(x) = p(x)/p_I^-$. Then it is obvious that $\bar{p} \in \mathcal{P}_+(I) \cap \mathcal{P}_\infty(I)$. It is also clear that $\bar{p}(x) \geq 1$ a.e. on I. Further, let us see that for a.e. $t \in I_+(x,h)$,

$$0 \leq \bar{p}(t) - p^-_{I_+(x,h)} \leq \frac{C}{\log(e + |t|)}. \tag{3.4}$$

Indeed, if $z \in I_+(x,h)$ and $|z| \geq |t|$, then

$$\bar{p}(t) - \bar{p}(z) \leq C/\log(e + |t|) \tag{3.5}$$

On the other hand, if $|z| < |t|$ we observe that

$$|t| \leq h + |x| \leq 5(|x| - 3h) \leq 5|z|.$$

Hence $|z| > |t|/5$. Consequently, by the condition $p \in \mathcal{P}_\infty(I)$,

$$\bar{p}(t) - \bar{p}(z) \leq C/\log(e + |z|) \leq C/\log(e + |t|).$$

Taking the infimum in (3.5) with respect to z we will find that (3.4) holds.

Further, Hölder's inequality and Lemma A yield $\big($here $r(\cdot) \equiv \bar{p}^-_{I_+(x,h)}$, $s(t) = \bar{p}(t)$, $\beta(t) = t$, $r = 1\big)$

$$\begin{aligned}
\big(\mathcal{M}_{+,h}f(x)\big)^{p(x)} &\leq \left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}^-_{I_+(x,h)}}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}}\\
&\leq \left(\frac{C}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt + \frac{1}{h}\int\limits_{I_+(x,h)} R(t)^{\bar{p}^-_{I_+(x,h)}}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}}\\
&\leq \left(\frac{C}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt + C(R(x))^{\bar{p}^-_{I(x,h)}}\right)^{p(x)/\bar{p}^-_{I_+(x,h)}}\\
&\leq C\left(\frac{C}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}} + C(R(x))^{p(x)}.
\end{aligned}$$

Moreover, by Hölder's inequality and the condition $S_{p(\cdot)}(f) \le 1$ we have

$$\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}} = \left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p_I^-}\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}-p_I^-}$$
$$\le \left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{p(t)}\,dt\right)^{(p(x)/\bar{p}^-_{I_+(x,h)}-p_I^-)/p_I^-}\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p_I^-}.$$

Now observe that

$$-\frac{1}{p_I^-}\left[\frac{p(x)}{\bar{p}^-_{I_+(x,h)}} - p_I^-\right] = p(x)\left[\frac{1}{p(x)} - \frac{1}{p^-_{I_+(x,h)}}\right] = p(x)\left[\frac{p^-_{I_+(x,h)} - p(x)}{p(x)p^-_{I_+(x,h)}}\right] \le 0.$$

Hence

$$A(x,h) := h^{-(p(x)/\bar{p}^-_{I_+(x,h)}-p_I^-)/p_I^-} \le 1$$

for $h \ge 1$, while by Proposition B$'$,

$$A(x,h) \le h^{(p^-_{I_+(x,h)}-p(x))p_I^+/(p_I^-)^2} \le C$$

when $h \le 1$. In addition,

$$\left(\int\limits_{I_+(x,h)} (f(t))^{p(t)}\,dt\right)^{(p(x)/\bar{p}^-_{I_+(x,h)}-p_I^-)/p_I^-} \le 1$$

because $S_{p(\cdot)}(f) \le 1$ and $\big(p(x)/\bar{p}^-_{I_+(x,h)}\big) - p_I^- \ge 0$. Consequently,

$$\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}} \le C\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p_I^-}$$

and the desired inequality follows.

Case 2 ($|x| \le 1$ and $r \ge |x|/4$). In this case, it is easy to check that

$$0 \le \bar{p}(t) - \bar{p}^-_{I_+(x,h)} \le \bar{p}_I^+ - \bar{p}_I^- \le \frac{C}{\log(e+|x|)},$$

where $t \in I_+(x,h)$, because $|x| \le 1$.

Consequently, Hölder's inequality and Lemma A yield $\big($with $r(\cdot) \equiv \bar{p}^-_{I_+(x,x+h)}$, $s(\cdot) = \bar{p}(\cdot)$, $\beta(\cdot) \equiv x$ and $r = 1\big)$

$$\big(\mathcal{M}_{+,h}f(x)\big)^{p(x)} \le \left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}^-_{I_+(x,h)}}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}}$$
$$\le \left(\frac{C}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt + \frac{1}{h}\int\limits_{I_+(x,h)} R(x)^{\bar{p}^-_{I_+(x,h)}}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}}$$
$$\le C\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\,dt\right)^{p(x)/\bar{p}^-_{I_+(x,h)}} + CR(x)^{p(x)}.$$

Now using the arguments from Case 1 we obtain the desired estimate.

Case 3 ($|x| \geq 1$ and $h \geq |x|/4$). By the conditions $S_{p(\cdot)}(f)$, $f \geq 1$ or $f = 0$, we have

$$\big(\mathcal{M}_{+,h} f(x)\big)^{p(x)} \leq h^{-p(x)} \left(\int\limits_{I_+(x,h)} (f(y))^{p(y)}\, dy \right)^{p(x)} \leq h^{-p(x)} \leq C|x|^{-p(x)} \leq CR(x)^{p(x)}.$$

Proof of (b). The proof is the same as in the previous argument except for Case 3 because the condition $f \geq 1$ or $f = 0$ was used only in this case. Assume that $|x| \geq 1$ and $h \geq |x|/4$. We have

$$\big(\mathcal{M}_{+,h} f(x)\big)^{p(x)} \leq C \left(\frac{1}{h} \int\limits_{I_+(x,h)\cap I(0,|x|)} f(t)\, dt \right)^{p(x)} + C \left(\frac{1}{h} \int\limits_{I_+(x,h)\setminus I(0,|x|)} f(t)\, dt \right)^{p(x)} := I_1 + I_2.$$

Let $E := I_+(x,h) \setminus I(0,|x|)$. By the condition $p \in \mathcal{P}_\infty(I)$ we find that

$$|\bar{p}(t) - \bar{p}(z)| \leq |\bar{p}(t) - \bar{p}(x)| + |\bar{p}(z) - \bar{p}(x)| \leq \frac{C}{\log(e + |x|)}$$

when $t, z \in E$ because in this case $|x| \leq |y|$ and $|x| \leq |z|$. Hence

$$0 \leq \bar{p}(t) - \bar{p}_E^- \leq \frac{C}{\log(e + |x|)}$$

for all $t \in E$. Consequently, by Hölder's inequality and Lemma A with $r(\cdot) \equiv \bar{p}_E^-$, $s(\cdot) = \bar{p}(\cdot)$, $\beta(\cdot) \equiv x$ and $r = 1$ we find that

$$\begin{aligned}
\left(\frac{1}{h} \int\limits_E f(t)\, dt \right)^{p(x)} &\leq \left(\frac{1}{h} \int\limits_E (f(t))^{\bar{p}_E^-}\, dt \right)^{p(x)/\bar{p}_E^-} \\
&\leq \left(\frac{C}{h} \int\limits_E (f(t))^{\bar{p}(t)}\, dt + \frac{1}{h} \int\limits_E (R(x))^{\bar{p}_E^-}\, dt \right)^{p(x)/\bar{p}_E^-} \\
&\leq C \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(y))^{\bar{p}(t)}\, dt \right)^{p(x)/\bar{p}_E^-} + C(R(x))^{p(x)} \\
&:= S(x,h) + C(R(x))^{p(x)}.
\end{aligned}$$

Notice that $\bar{p}(x) \geq \bar{p}_E^-$ for a.e. $x \in E$. Now we use arguments from Case 1. We have

$$\begin{aligned}
S(x,h) &= \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{p_I^-} \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{(p(x)/\bar{p}_E^-) - p_I^-} \\
&= h^{-(p(x)/\bar{p}_E^-) - p_I^-} \left(\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{(p(x)/\bar{p}_E^-) - p_I^-} \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{p_I^-}.
\end{aligned}$$

Observe that since $-\big(p(x)/\bar{p}_E^-\big) + p_I^- \leq 0$ we have

$$h^{-(p(x)/\bar{p}_E^-) + p_I^-} \leq 1.$$

Indeed, for h with $h \geq 1$, the inequality is obvious, while for $h < 1$, using Proposition B′, we find that

$$h^{-(p(x)/\bar{p}_E^-)+p_I^-} = h^{(p_I^-/p_E^-)(p_E^- - p(x))} \leq h^{(p_I^-/p_I^+)(p^-_{I_+(x,h)} - p(x))} \leq C.$$

Consequently,

$$I_2 \leq C \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{p_I^-} + C(R(x))^{p(x)}.$$

To estimate I_1, we denote $F := I(0,|x|) \cap I_+(x,h)$. Using again the condition $p \in \mathcal{P}_\infty(I)$ we see that

$$|\bar{p}(x) - \bar{p}(t)| \leq \frac{C}{\log(e + |t|)},$$

because if $t \in F$, then $|t| \leq |x|$. Applying Hölder's inequality and Lemma B with $r(\cdot) \equiv \bar{p}(x)$, $s(t) = \bar{p}(t)$ and $r = 1$, we see that

$$\begin{aligned}
\left(\frac{1}{h} \int\limits_F f(t)\, dt \right)^{p(x)} &\leq \left(\frac{1}{h} \int\limits_F (f(t))^{\bar{p}(x)}\, dt \right)^{p(x)/\bar{p}(x)} \\
&\leq \left(\frac{C}{h} \int\limits_F (f(t))^{\bar{p}(t)}\, dt + \frac{1}{h} \int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}\, dt \right)^{p_I^-} \\
&\leq C \left(\frac{1}{h} \int\limits_F (f(t))^{\bar{p}(t)}\, dt \right)^{p_I^-} + C \left(\frac{1}{h} \int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}\, dt \right)^{p_I^-} \\
&\leq \left(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}\, dt \right)^{p_I^-} + C \left(\frac{1}{|x|} \int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}\, dt \right)^{p_I^-}
\end{aligned}$$

because $h > |x|/4$, $F \subset I_+(x,h)$ and $F \subset I(0,|x|)$.

Further, let us take r so that $1 < r < p_I^-$. Then by Hölder's inequality,

$$\left(\frac{1}{|x|} \int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}\, dt \right)^{p_I^-} \leq |x|^{-p_I^-/r} \left(\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)r}\, dt \right)^{p_I^-/r}.$$

Now observe that $\bar{p}(x)r \geq \bar{p}_I^- r > 1$ and $R(t) \leq 1$. Therefore simple estimates give us

$$\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)r}\, dt \leq \int\limits_{I(0,|x|)} (R(t))^{\bar{p}_I^- r}\, dt \leq C.$$

Further, since $|x| > 1$ we see that

$$|x|^{-p_I^-/r} \leq C(e + |x|)^{-p_I^-/r} = CR_{p_I^-/r}(x).$$

Since the last function is in $L^1(\mathbf{R})$, we finally have the desired result. □

Proposition 3.5 *Let I be an open subset of* $\mathbf{R}$. *Suppose that $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$. Suppose also that $S_{p(\cdot)}(f) \le 1$. Then there exists a positive constant C such that*

$$\big(\mathcal{M}_- f(x)\big)^{p(x)} \le C\Big(\mathcal{M}_-\big(|f(\cdot)|^{p(\cdot)/p_I^-}\big)(x)\Big)^{p_I^-} + S(x)$$

for a.e. $x \in I$, where $S \in L^1(\mathbf{R})$.

The proof of this statement is similar to that of Proposition 3.4. In this case we need Proposition B instead of Proposition B′. The proof is omitted.

Proposition 3.6 *Let I be an open set in* $\mathbf{R}$. *Suppose that $p \in \mathcal{P}_+(I) \cap \mathcal{P}_\infty(I)$. Then the operator $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(\mathbf{R}_+)$.*

P r o o f. By inequality (3.3) and the boundedness of the operator $\mathcal{M}_+$ in the Lebesgue space with constant exponent p_I^- we have the desired result. □

In a similar way there follows

Proposition 3.7 *Let I be an open set in* $\mathbf{R}$. *Suppose that $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$. Then the operator $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(\mathbf{R}_+)$.*

Theorem 3.8 *Let $I = \mathbf{R}_+$. Suppose that $p \in \mathcal{P}_+(I)$. Assume also that there is a positive number a such that $p \in \mathcal{P}_\infty((a,\infty))$. Then $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(\mathbf{R}_+)$.*

P r o o f. Since $\mathcal{M}_+$ is positive and sublinear, it is sufficient to show that $\|\mathcal{M}_+ f\|_{L^{p(\cdot)}(\mathbf{R})} < \infty$ if $\|f\|_{L^{p(\cdot)}(\mathbf{R})} < \infty$. Let $f_1(x) = \chi_{[0,a]}(x) f(x)$, $f_2(x) = f(x) - f_1(x)$. Then we have

$$\begin{aligned}\int_0^\infty \big(\mathcal{M}_+ f\big)^{p(x)}(x)\,dx &\le c\Bigg[\int_0^a \big(\mathcal{M}_+ f_1\big)^{p(x)}(x)\,dx + \int_a^\infty \big(\mathcal{M}_+ f_1\big)^{p(x)}(x)\,dx \\ &\qquad + \int_0^a \big(\mathcal{M}_+ f_2\big)^{p(x)}(x)\,dx + \int_a^\infty \big(\mathcal{M}_+ f_2\big)^{p(x)}(x)\,dx\Bigg] \\ &:= c\sum_{k=1}^4 \mathrm{I}_k.\end{aligned}$$

Since $\int_0^a |f_1(x)|^{p(x)}\,dx \le \int_0^\infty |f(x)|^{p(x)}\,dx < \infty$ and $p \in \mathcal{P}_+([0,a])$, using Theorem 3.3 we have that $\mathrm{I}_1 \le c$.
It is obvious that $\mathrm{I}_2 = 0$.
Let us evaluate I_3. Notice that if $0 < h \le a - x$, then $\frac{1}{h}\int_x^{x+h} |f_2(t)|\,dt = 0$, while for $h > a - x > 0$, we have

$$\frac{1}{h}\int_x^{x+h} |f_2(t)|\,dt = \frac{1}{h}\int_a^{x+h} |f(t)|\,dt \le \frac{1}{x+h-a}\int_a^{x+h} |f(t)|\,dt \le \big(\mathcal{M}_+ f\big)(a).$$

Due to Theorem 3.3 we have that $\big(\mathcal{M}_+ f\big)(x) < \infty$ a.e. on every finite interval. Thus we can take a so that $\big(\mathcal{M}_+ f\big)(a) < \infty$. Hence $\big(\mathcal{M}_+ f_2\big)(x) \le \big(\mathcal{M}_+ f\big)(a) < \infty$ when $x \in [0,a]$ and, consequently, $\mathrm{I}_3 \le a\big(\mathcal{M}_+ f\big)^{p_-([0,a])}(a) < \infty$ if $\big(\mathcal{M}_+ f\big)(a) \le 1$; $\mathrm{I}_3 \le a\big(\mathcal{M}_+ f\big)^{p_+([0,a])}(a) < \infty$ if $\big(\mathcal{M}_+ f\big)(a) > 1$.

The boundedness of $\mathcal{M}_+$ in $L^{p(\cdot)}((a,\infty))$ (see Proposition 3.6) yields

$$\mathrm{I}_4 = \int_a^\infty \big(\mathcal{M}_+ f_2\big)^{p(x)}(x)\,dx < \infty.$$

□

Corollary 3.9 *Let $I = \mathbf{R}_+$. Suppose that p satisfies condition* (2.1) *and is non-decreasing on I. Suppose also that there exists a positive number a such that*

$$p(x) \le p(y) + \frac{C}{\log(e+y)}, \quad a < y < x.$$

Then $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(\mathbf{R}_+)$.

This follows from Theorem 3.8 and the fact that for non-decreasing p the condition (2.2) is satisfied.

Theorem 3.10 *Let $I = \mathbf{R}_+$ and let $p \in \mathcal{P}_-(I)$. Suppose that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive a. Then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

P r o o f. Keeping the notation of Theorem 3.8 we have (we assume that $\|f\|_{L^{p(\cdot)}(\mathbf{R}_+)} < \infty$)

$$\int_0^\infty \big(\mathcal{M}_- f\big)^{p(x)}(x)\,dx \le c\left[\sum_{k=1}^4 \mathbf{I}_k\right].$$

It is obvious that $\mathbf{I}_1 \le c$ because of Theorem 3.2. Further,

$$\mathbf{I}_2 = \int_a^\infty \big(\mathcal{M}_- f_1\big)^{p(x)}(x)\,dx = \int_a^\infty \left(\sup_{x-a\le h\le x} h^{-1}\int_{x-h}^x |f_1(y)|\,dy\right)^{p(x)} dx = \int_a^{2a} + \int_{2a}^\infty := \mathbf{I}_{21} + \mathbf{I}_{22}.$$

Notice that for $x \in [a, 2a]$,

$$\sup_{x-a\le h\le x} h^{-1}\int_{x-h}^x |f(y)|\,dy = \sup_{x-a\le h\le x} h^{-1}\int_{x-h}^a |f(y)|\,dy \le \big(\mathcal{M}_- f\big)(a).$$

By Theorem 3.2 we can assume that $\big(\mathcal{M}_- f\big)(a) < \infty$. Consequently, $\mathbf{I}_{21} \le a\big(\mathcal{M}_- f\big)^{p^-_{[a,2a]}}(a) < \infty$ if $\big(\mathcal{M}_- f\big)(a) \le 1$ and $\mathbf{I}_{21} \le a\big(\mathcal{M}_- f\big)^{p^+_{[a,2a]}}(a) < \infty$ if $\big(\mathcal{M}_- f\big)(a) > 1$.

Let us now estimate $\mathbf{I}_{22}$. Assume that $a > 1$. Then for $x - a \le h < x$ we have

$$\frac{1}{h}\int_{x-h}^a |f_1| \le \frac{1}{h}\|f\|_{L^{p(\cdot)}(\mathbf{R}_+)}\|\chi_{(x-h,a)}(\cdot)\|_{L^{p'(\cdot)}(\mathbf{R}_+)} \le Ca^{1/(p')^-_I}/(x-a).$$

Hence, since $a > 1$, we have

$$\mathbf{I}_{22} \le c\int_{2a}^\infty (x-a)^{-p^-_I}\,dx = c\int_a^\infty x^{-p^-_I}\,dx < \infty.$$

Further, it is clear that $\mathbf{I}_3 = 0$, while Proposition 3.7 yields

$$\mathbf{I}_4 \le \int_a^\infty \big(\mathcal{M}_- f_2\big)^{p(x)}(x)\,dx < \infty.$$

□

Corollary 3.11 *Let $I = \mathbf{R}_+$. Suppose that p satisfies condition* (2.1) *and is non-increasing on I. Suppose also that there exists a positive number a such that*

$$p(x) \le p(y) + \frac{C}{\log(e+x)}, \quad a < x < y.$$

Then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(\mathbf{R}_+)$.

Theorem 3.12 *Let $I = \mathbf{R}$ and let $p \in \mathcal{P}_+(I)$. Suppose that there is a positive number a such that $p \in \mathcal{P}_\infty(\mathbf{R} \setminus [-a,a])$. Then $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(I)$.*

Proof. Let $\|f\|_{L^{p(\cdot)}(\mathbf{R})} < \infty$. We have

$$\begin{aligned}\int\limits_{\mathbf{R}} (\mathcal{M}_+ f(x))^{p(x)}\,dx &\le c \int\limits_{-a}^{a} (\mathcal{M}_+ f_1)^{p(x)}(x)\,dx + c \int\limits_{-a}^{a} (\mathcal{M}_+ f_2)^{p(x)}(x)\,dx \\ &\quad + c \int\limits_{\mathbf{R}\setminus[-a,a]} (\mathcal{M}_+ f_1)^{p(x)}(x)\,dx + c \int\limits_{\mathbf{R}\setminus[-a,a]} (\mathcal{M}_+ f_2)^{p(x)}(x)\,dx \\ &\equiv c \sum_{k=1}^{4} \mathbf{I}_k\end{aligned}$$

where $f_1 = f\chi_{[-a,a]}$, $f_2 = f\chi_{\mathbf{R}\setminus[-a,a]}$.

It is easy to see that by the definition of $\mathcal{M}_+$ we have

$$\mathbf{I}_2 = \int\limits_{-a}^{a} (\mathcal{M}_+(f\chi_{(a,\infty)}(x))^{p(x)}\,dx;$$

$$\mathbf{I}_3 = \int\limits_{-\infty}^{-a} (\mathcal{M}_+(f_1(x))^{p(x)}\,dx.$$

To evaluate $\mathbf{I}_2$, observe that when $x \in (-a,a)$,

$$\big(\mathcal{M}_+ f_3\big)(x) = \sup_{r>a-x} \frac{1}{r} \int\limits_{a}^{x+r} |f(t)|\,dt \le \sup_{r>a-x} \frac{1}{x+r-a} \int\limits_{a}^{x+r} |f(t)|\,dt \le \big(\mathcal{M}_+ f\big)(a) < \infty.$$

Further, $\big(\mathcal{M}_+ f\big)(a) < \infty$ because we can always choose such an a.

Hence

$$\mathbf{I}_2 \le a \begin{cases} a\big(\mathcal{M}_+ f\big)^{p^-_{[-a,a]}}(a), & \text{if } \ \big(\mathcal{M}_+ f\big)(a) \le 1; \\ a\big(\mathcal{M}_+ f\big)^{p^+_{[-a,a]}}(a), & \text{if } \ \big(\mathcal{M}_+ f\big)(a) > 1. \end{cases}$$

This implies that $\mathbf{I_2} < \infty$.

Further,

$$\mathbf{I}_3 \le \int\limits_{-\infty}^{-2a} (\mathcal{M}_+ f_1(x))^{p(x)}\,dx + \int\limits_{-2a}^{-a} (\mathcal{M}_+ f_1(x))^{p(x)}\,dx := \mathbf{I}_3^{(1)} + \mathbf{I}_3^{(2)}.$$

By Hölder's inequality and simple calculations we have (we can assume that $a > 1$)

$$\begin{aligned}\mathbf{I}_3^{(1)} &\le \int\limits_{-\infty}^{-2a} (-a-x)^{p(x)} \left(\int\limits_{-a}^{a} |f(t)|\,dt \right)^{p(x)} dx \\ &\le \int\limits_{-\infty}^{-2a} (-a-x)^{p_I^-} \big\|\chi_{(-a,a)} f\big\|_{L^{p(\cdot)}}^{p(x)} \big\|\chi_{(-a,a)}\big\|_{L^{p'(\cdot)}}^{p(x)}\,dx \\ &\le c \int\limits_{a}^{\infty} \frac{dt}{t^{p_I^-}} \\ &\le C < \infty,\end{aligned}$$

where the positive constant C depends on a, f and p.

Notice that

$$\mathbf{I}_3^{(2)} \leq \int_{-2a}^{a} (\mathcal{M}_+ f_1(x))^{p(x)}\, dx < \infty$$

because $\|f_1\|_{L^{p(\cdot)}([-2a,a])} < \infty$ and $p \in \mathcal{P}_+([-2a,a])$.

Finally, Theorem 3.3 and Proposition 3.6 yield respectively

$$\mathbf{I}_1 < \infty; \quad \mathbf{I}_4 < \infty. \qquad \square$$

Theorem 3.13 *Let $I = \mathbf{R}$ and let $p \in \mathcal{P}_-(I)$. Suppose that there exists a positive number a such that $p \in \mathcal{P}_\infty(\mathbf{R} \setminus [-a,a])$. Then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

The proof of this statement is similar to that of Theorem 3.12 and is therefore omitted.

4 One-sided potentials

In this section we assume that $I = [0,b)$, where $0 < b \leq \infty$ and let

$$\big(\mathcal{I}_{\alpha(\cdot)} f\big)(x) = \int_0^b f(t)|x-t|^{\alpha(x)-1}\, dt, \quad x \in (0,b),$$

$$\big(\mathcal{R}_{\alpha(\cdot)} f\big)(x) = \int_0^x f(t)(x-t)^{\alpha(x)-1}\, dt, \quad x \in (0,b),$$

$$\big(\mathcal{W}_{\alpha(\cdot)} f\big)(x) = \int_x^b f(t)(t-x)^{\alpha(x)-1}\, dt, \quad x \in (0,b),$$

where $0 < \alpha(x) < 1$.

If $\alpha(x) := \alpha = const$, then we denote $\mathcal{I}_{\alpha(\cdot)}$, $\mathcal{R}_{\alpha(\cdot)}$, $\mathcal{W}_{\alpha(\cdot)}$ by $\mathcal{I}_\alpha$, $\mathcal{R}_\alpha$ and $\mathcal{W}_\alpha$ respectively.

We analyze these operators in much the same way as the maximal operators were handled earlier.

Proposition 4.1 *Let $I = [0,b]$ be a bounded interval and let $\alpha \in (0,1)$ be a constant. Then*

(a) *there exists a discontinuous function p on I such that $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ and $\mathcal{I}_\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$, where $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $0 < \alpha < 1/p_I^+$;*

(b) *there exists a discontinuous function p on I such that $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ and $\mathcal{I}_\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$, where $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $0 < \alpha < 1/p_I^+$.*

P r o o f. We prove part (a). The proof of (b) is similar; therefore it is omitted.

Let

$$p(x) = \begin{cases} p_1, & 0 \leq x \leq a, \\ p_2, & a < x \leq b, \end{cases}$$

where p_1 and p_2 are constants, $a \in I$, $q_2 < p_1$ and $q_i = \frac{p_i}{1-\alpha p_i}$, $i = 1,2$.

It is clear that $p_2 < q_2 < p_1$. Let $f \ge 0$ and let $\|f\|_{L^{p(\cdot)}([0,b])} \le 1$. We have

$$\begin{aligned}
&\int_0^b \left(\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \right)^{q(x)} dx \\
&\quad \le c\left[\int_0^a \left(\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}}\,dt \right)^{q_1} dx + \int_0^a \left(\int_0^x \frac{f_2(t)}{(x-t)^{1-\alpha}}\,dt \right)^{q_1} dx \right.\\
&\qquad \left. + \int_a^b \left(\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}}\,dt \right)^{q_2} dx + \int_a^b \left(\int_0^x \frac{f_2(t)}{(x-t)^{1-\alpha}}\,dt \right)^{q_2} dx \right] \\
&\quad := c\left[\sum_{k=1}^{4} \mathrm{I}_k \right],
\end{aligned}$$

where $f_1 = f\chi_{(0,a)}$ and $f_2 = f\chi_{[a,b)}$.

It is obvious that $\mathrm{I}_1 \le c$ because $\int_0^a (f_1(t))^{p_1}\,dt \le 1$ and consequently, $\mathcal{R}_\alpha$ is bounded from $L^{p_1}([0,a])$ to $L^{q_2}([0,a])$. It is also clear that $\mathrm{I}_2 = 0$. Now let $x \in (a,b)$. Then

$$\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}}\,dt \le c\,x^\alpha \big(\mathcal{M}_- f_1\big)(x).$$

Hence by the boundedness of $\mathcal{M}_-$ in $L^{p_2}(I)$ and Hölder's inequality we have

$$\mathrm{I}_3 \le c\,b^{\alpha p_2} \int_0^b (\mathcal{M}_- f_1)^{p_2}(x)\,dx \le c\left(\int_0^b (f(t))^{p(t)}\,dt \right)^{\frac{p_2}{p_1}} \le c.$$

Using the boundedness of $\widetilde{\mathcal{R}_\alpha}$ from $L^{p_2}([a,b])$ to $L^{q_2}([a,b])$ (see e.g. [30]), where

$$\big(\widetilde{\mathcal{R}_\alpha}\big)(x) = \int_a^x f(t)(x-t)^{\alpha-1}\,dt, \quad x \in (a,b),$$

we have $\mathrm{I}_4 < \infty$ because $\int_a^b (f_2(t))^{p_2}\,dt \le \int_0^b (f(t))^{p(t)}\,dt \le 1$.

Let us now prove that $\mathcal{W}_\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. Let $f(x) = \chi_{[a,b)}(x)(x-a)^\lambda$, where $\lambda = -\alpha - \frac{1}{q_1}$. Then $\int_0^b (f(x))^{p(x)}\,dx < \infty$, because $-\alpha - \frac{1}{q_1} = -\frac{1}{p_1} > -\frac{1}{p_2}$.

On the other hand, it is easy to see that, for $x \in (0,a)$, we have $\big(\mathcal{W}_\alpha f\big)(x) \ge c(a-x)^{\lambda+\alpha}$. Hence $\|\mathcal{W}_\alpha f\|_{L^{p(\cdot)}(I)} = \infty$.

Finally we conclude that $\mathcal{W}_\alpha$ is not bounded from $L^{p(\cdot)}([0,b])$ to $L^{q(\cdot)}([0,b])$ and, consequently, $\mathcal{I}_\alpha$ is not bounded from $L^{p(\cdot)}([0,b])$ to $L^{q(\cdot)}([0,b])$. □

Theorem 4.2 *Let $I = \mathbf{R}_+$ and let $p \in \mathcal{P}_+(I)$. Suppose that there exists a positive constant a such that $p \in \mathcal{P}_\infty((a,\infty))$. Suppose that α is a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Proof. By Proposition C we have that the condition $p \in \mathcal{P}_+(I)$ implies $\bar{q}' \in \mathcal{P}_-(I)$, where $\bar{q}(x) = \frac{q(x)}{q_0}$ and q_0 is a constant such that $1 < q_0 < q_I^-$. Let us choose p_0 so that $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p(x)} - \frac{1}{q(x)} = \alpha$. Then $p_I^+ < \frac{1}{\alpha} = \frac{p_0 q_0}{q_0 - p_0}$. It is clear that $p_0 = \frac{q_0}{\alpha q_0 + 1} < \frac{q_I^-}{\alpha q_I^- + 1} = p_I^-$.

It remains to apply Theorems 2.3′, 2.5 and 3.10 together with the fact that $\rho^{q_0} \in A_1^+(I) \Rightarrow \rho \in A^+_{p_0 q_0}(\mathbf{R}_+)$ (see Section 2). □

Theorem 4.3 *Let $I = \mathbf{R}_+$ and let $p \in \mathcal{P}_+(I)$. Let α be a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and let $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive number a. Then $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

The proof of this theorem is similar to that of the previous one.

Theorem 4.4 *Let $I := [0,b]$ be a bounded interval, $p \in \mathcal{P}_+(I)$, $0 < \alpha_I^-$ and let $(\alpha p)_I^+ < 1$. Suppose that $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $\mathcal{W}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Remark 4.5 Notice that if $p \in \mathcal{P}_+([0,b])$, then there exists a positive constant c such that for a.e. $x \in [0,b]$ and all r with $0 < r < 1/2$ and $I_+(x,r) \neq \emptyset$, the inequality

$$r^{\frac{1}{\left(p^-_{I_+(x,r)}\right)'} - \frac{1}{p'(x)}} \leq c$$

holds.

To prove Theorem 4.4 we need

Lemma 4.6 *Let $I = [0,b]$ be bounded and let $\|f\|_{L^{p(\cdot)}(I)} \leq 1$. Suppose that $p \in \mathcal{P}_+(I)$, $0 < \alpha < \frac{1}{p_I^+}$ and $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then there exists a positive constant c depending only on p and α such that*

$$\mathcal{W}_{\alpha(\cdot)}(|f|)(x) \leq c\big[(\mathcal{M}_+ f)(x)\big]^{\frac{p(x)}{q(x)}}, \quad x \in I.$$

P r o o f. For the sake of simplicity we assume that $b = 1$, i.e., $I = [0,1]$. We have

$$\begin{aligned}
\mathcal{W}_{\alpha(\cdot)}(|f|)(x) &\leq \int\limits_{0\leq t-x\leq 1} \frac{|f(t)|}{(t-x)^{1-\alpha(x)}}\,dt \\
&\leq c \int\limits_{0\leq t-x\leq 1} |f(t)| \left(\int\limits_{t-x}^{2(t-x)} r^{\alpha(x)-2} dr \right) dt \\
&\leq c \int\limits_0^2 r^{\alpha(x)-2} \left(\int\limits_{0\leq t-x\leq \min\{r,1\}} |f(t)|\,dt \right) dr \\
&= c\int\limits_0^\varepsilon (\cdot) + c\int\limits_\varepsilon^2 (\cdot) \\
&:= c(\mathrm{I}_1 + \mathrm{I}_2),
\end{aligned}$$

where ε will be chosen later (if $\varepsilon > 2$ we assume that $\mathrm{I}_2 = 0$). It is easy to check that

$$\mathrm{I}_1 = \int\limits_0^\varepsilon r^{\alpha(x)-1} \left(\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|\,dt \right) dr.$$

Further, if $x + r \leq 1$, then

$$\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|\,dt \leq \frac{1}{r} \int\limits_x^{x+r} |f(t)|\,dt \leq \mathcal{M}_+ f(x);$$

if $x + r > 1$, then

$$\frac{1}{r}\int\limits_{[x,x+r]\cap(0,1)} |f(t)|\,dt \le \frac{1}{1-x}\int\limits_x^1 |f(t)|\,dt \le \mathcal{M}_+ f(x).$$

So, for all $0 < r < 2$ we have

$$\frac{1}{r}\int\limits_{[x,x+r]\cap(0,1)} |f(t)|\,dt \le \mathcal{M}_+ f(x).$$

Taking into account the latter we find that

$$\mathrm{I}_1 \le \mathcal{M}_+ f(x)\frac{\varepsilon^{\alpha(x)}}{\alpha(x)} \le c_\alpha \mathcal{M}_+ f(x)\varepsilon^{\alpha(x)}.$$

Now by Hölder's inequality for variable Lebesgue spaces (see e.g. [21]) and elementary properties of $L^{p(\cdot)}$ spaces together with Remark 4.5 we find that

$$\begin{aligned}
\mathrm{I}_2 &\le 2\int\limits_\varepsilon^2 r^{\alpha(x)-2}\big\|\chi_{[x,x+r]}f\big\|_{L^{p(\cdot)}([0,1])}\big\|\chi_{[x,x+r]}\big\|_{L^{p'(\cdot)}([0,1])}\,dr \\
&\le c\int\limits_\varepsilon^2 r^{\alpha(x)-2} r^{\frac{1}{(p^-_{[x,x+r]})'}}\,dr \\
&\le c\int\limits_\varepsilon^2 r^{\alpha(x)-2+\frac{1}{p'(x)}}\,dr \\
&= c\varepsilon^{\alpha(x)-\frac{1}{p(x)}}.
\end{aligned}$$

Taking $\varepsilon = \big(\mathcal{M}_+ f(x)\big)^{-p(x)}$ we have

$$\mathcal{W}_{\alpha(\cdot)}(|f|)(x) \le c_{\alpha,p}\big[(\mathcal{M}_+ f)(x)\big]^{\frac{p(x)}{q(x)}}. \qquad \square$$

P r o o f o f T h e o r e m 4.4. Let $\|f\|_{L^{p(\cdot)}([0,b])} \le 1$ which is equivalent to say that $\int\limits_0^b |f(x)|^{p(x)}\,dx \le 1$. By Lemma 4.6 and Theorem 3.3 we have

$$\int\limits_0^b |\mathcal{W}_{\alpha(\cdot)} f(x)|^{q(x)}\,dx \le c\int\limits_0^b \big(\mathcal{M}_+ f(x)\big)^{p(x)}\,dx \le c.$$

The theorem has been proved. $\square$

The next statement follows analogously.

Theorem 4.7 *Let $I = [0,b]$ be a bounded interval and let $p \in \mathcal{P}_-(I)$. Suppose that $0 < \alpha_-$. Assume also that $(\alpha p)^+_I < 1$ and $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

5 Calderón–Zygmund operators

We begin this section with the following definition:

Definition 5.1 We say that a function k in $L^1_{loc}\big(\mathbf{R}\setminus\{0\}\big)$ is a *Calderón–Zygmund kernel* if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left| \int\limits_{\varepsilon<|x|<N} k(x)\,dx \right| \le B_1$$

for all ε and all N, with $0 < \varepsilon < N$, and furthermore

$$\lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|x|<N} k(x)\,dx$$

exists;

(b) there exists a positive constant B_2 such that

$$\big|k(x)\big| \le \frac{B_2}{|x|}, \quad x \ne 0;$$

(c) there exists a positive constant B_3 such that for all x and y with $|x| > 2|y| > 0$ the inequality

$$|k(x-y) - k(x)| \le B_3 \frac{|y|}{|x|^2}$$

holds.

It is known (see [22], [1]) that conditions (a)–(c) are sufficient for the boundedness of the operators:

$$T^*f(x) = \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big|;$$
$$Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x),$$

where

$$T_\varepsilon f(x) = \int\limits_{|x-y|>\varepsilon} k(x-y)f(y)\,dy,$$

in $L^r(\mathbf{R})$, $1 < r < \infty$.

It is clear that $Tf(x) \le T^*f(x)$.

The following example shows that there exists a nontrivial Calderón–Zygmund kernel with support contained in $(0, +\infty)$.

Example 5.2 The function

$$k(x) = \frac{1}{x}\,\frac{\sin(\ln x)}{\ln x}\chi_{(0,+\infty)}(x)$$

is a Calderón–Zygmund kernel (for details see e.g. [22], [1]).

There exists also a nontrivial Calderón–Zygmund kernel supported in $(-\infty, 0)$.

The next results are well-known (see [22], [1]).

Theorem 5.3 *Let p be a constant, $1 < p < \infty$, and let k be a Calderón–Zygmund kernel with support in $(-\infty, 0)$. Then the condition $w \in A_p^+(\mathbf{R})$ implies the inequality*

$$\int\limits_{\mathbf{R}} \big|T^*f(x)\big|^p w(x)\,dx \le c \int\limits_{\mathbf{R}} \big|f(x)\big|^p w(x)\,dx, \quad f \in L^p_w(\mathbf{R}).$$

Theorem 5.4 *Let k be a Calderón–Zygmund kernel with support in $(0, +\infty)$ and let p be a constant, $1 < p < \infty$. If $w \in A_p^-(\mathbf{R})$, then it follows that T^* is bounded in $L^p_w(\mathbf{R})$.*

Theorems 2.5, 3.12, 3.13, 5.2, 5.3 and Proposition D yield our main results of this section:

Theorem 5.5 *Let $I = \mathbf{R}$ and let $p \in \mathcal{P}_+(I)$. Suppose that $p \in \mathcal{P}_\infty(\mathbf{R} \setminus [-a, a])$ for some positive number a. Then T^*, with kernel k supported in $(-\infty, 0)$, is bounded in $L^{p(\cdot)}(I)$.*

Theorem 5.6 *Let $I = \mathbf{R}$ and let $p \in \mathcal{P}_-(I)$. Assume that $p \in \mathcal{P}_\infty(\mathbf{R} \setminus [-a, a])$ for some positive number a. Then T^*, with kernel k supported in $(0, +\infty)$, is bounded in $L^{p(\cdot)}(I)$.*

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