Journal of Mathematical Sciences, Vol. 178, No. 6, November, 2011

WEIGHTED CRITERIA FOR THE HARDY TRANSFORM UNDER THE B_p CONDITION IN GRAND LEBESGUE SPACES AND SOME APPLICATIONS

A. Meskhi

I. Javakhishvili Tbilisi State University
A. Razmadze Mathematical Institute
2, University St., Tbilisi 0186, Georgia

Georgian Technical University
77 Kostava Street, Tbilisi 0175, Georgia
meskhi@rmi.ge

UDC 517.9

We show that the Hardy operator

$$Hf(x) = \frac{1}{x}\int\limits_0^x f(t)dt$$

from $L^{p),\theta}_{\mathrm{dec},w}(I)$ *to* $L^{p),\theta}_w(I)$, $0<p<\infty$, $\theta>0$, $I=(0,1)$, *is bounded if and only if the weight* w *belongs to the well–known class* B_p *restricted to the interval* I. *This result is applied to derive a similar assertion for the Riemann–Liouville fractional integral operator and to establish criteria for the boundedness of the Hardy–Littlewood maximal operator in the weighted grand Lorentz space* $\Lambda^{p),\theta}_w$. *Bibliography*: 23 *titles.*

Introduction

The paper is devoted to one weight criteria in grand Lebesgue spaces for the Hardy transform on the cone of nonincreasing functions. In particular, we show that the operator

$$(Hf)(x) = \frac{1}{x}\int\limits_0^x f(t)dt$$

from $L^{p),\theta}_{\mathrm{dec},w}(I)$ to $L^{p),\theta}_w(I)$ ($0<p<\infty$, $\theta>0$, $I=(0,1)$) is bounded if and only if the weight w belongs to the well–known B_p class defined on the interval I. The proof of this result is based on the extrapolation for B_p weights. As a consequence, we derive a similar statement for the

Translated from *Problems in Mathematical Analysis* **60**, September 2011, pp. 53–64.

1072-3374/11/1786-0622

Riemann–Liouville transform

$$(R_\alpha f)(x) = x^{-\alpha} \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad 0 < \alpha < 1.$$

The one weight result for the operator H is applied to establish criteria for the boundedness of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q f(y) dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x, in the weighted grand Lorentz space $\Lambda_w^{p)}$.

In the last decade, the theory of grand Lebesgue spaces introduced by Iwaniec and Sbordone [1] is one of the intensively developing directions of the modern analysis. The necessity to investigate these spaces emerged from their rather essential role in various fields, in particular, in the integrability problem of Jacobian under minimal hypothesis (cf. [1] for details).

As was proved in [2], the Hardy–Littlewood maximal operator is bounded in the weighted grand Lebesgue spaces $L_w^{p)}$ if and only if the weight w belongs to the Muckenhoupt class A_p. The same phenomena was noticed in [3] for the Hilbert transform. We refer to [4, 5] for one–weight results regarding maximal and singular integrals of various type in these spaces. In [6], the author studied the boundedness of the fractional integral operator in weighted grand Lebesgue spaces from the one weight viewpoint.

For the weight theory of Hardy type transforms and fractional integrals in classical Lebesgue spaces we refer, for example, to [7]–[11] and the references therein. The one weight problem from nonincreasing functions viewpoint for the kernel operators involving Hardy type transforms was studied, in particular, in [12]–[14].

The paper is organized as follows. In Section 1, we give definitions and some well–known properties of Hardy transforms, Hardy–Littlewood maximal functions, and grand Lebesgue spaces. In Section 2, we prove the one weight inequality of general type for pairs of decreasing functions in grand Lebesgue spaces. Section 3 deals with one weight criteria for the operators H for nondecreasing functions in grand Lebesgue spaces. In Section 4, we apply the result of Section 3 to derive one weight criteria for the operator R_α for nonincreasing functions in grand Lebesgue spaces and to establish necessary and sufficient conditions for the boundedness of M in weighted grand Lorentz spaces $\Lambda_w^{p),\theta}(I)$.

Finally, we point out that constants (often different constants in the same series of inequalities) are generally denoted by c or C. The expression $f(x) \approx g(x)$ means that $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$, where the positive constants c_1 and c_2 are independent of x.

1 Preliminaries

Let $0 < p < \infty$ and let $\theta > 0$. Throughout the paper, we use the notation $I := (0, 1)$. Suppose that w is integrable a.e. nonnegative function (i.e., a weight) on I. We assume that

$$\int\limits_0^r w(x) dx > 0 \quad \forall\, r \in I.$$

We denote by $L_w^{p),\theta}(I)$ the generalized grand Lebesgue space. This is the class of all measurable functions $f: I \to \mathbb{R}$ for which

$$\|f\|_{L_w^{p),\theta}(I)} := \sup_{0<\varepsilon\leqslant\varepsilon_0} \left(\varepsilon^\theta \int_0^1 |f(t)|^{p-\varepsilon} w(t)dt \right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $\varepsilon_0 = p-1$ if $p>1$ and $\varepsilon_0 \in (0,p)$ if $0<p\leqslant 1$.

If $w \equiv \text{const}$, then we have unweighted space $L^{p),\theta}(I)$.

It is easy to check that

$$\|f\|_{L_w^{p),\theta}(I)} \approx \sup_{0<\varepsilon\leqslant\sigma_0} \varepsilon^{\theta/p} \|f\|_{L_w^{p-\varepsilon}(I)}$$

for some small positive number σ_0.

It turns out that the generalized grand Lebesgue spaces $L^{p),\theta}$ are appropriate for solving the existence, uniqueness, and regularity problems for various nonlinear partial differential equations. The space $L^{p),\theta}$ (defined on domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in the paper [15], where the existence and uniqueness of the inhomogeneous n-harmonic equation $\operatorname{div} A(x,\nabla u) = \mu$ was studied.

If $\theta = 1$, then $L^{p),\theta}$ coincides with the Iwaniec–Sbordone space, denoted by $L^{p)}(I)$.

For structural properties of grand Lebesgue spaces we refer, for example, to [16, 17].

Let us denote by D the class of all nonnegative nonincreasing functions on I. We denote by $L^{p),\theta}_{\mathrm{dec},w}(I)$ the intersection $L_w^{p),\theta} \cap D$.

Note that $L^{p),\theta}_{\mathrm{dec},w}(I) \neq L^{p}_{\mathrm{dec},w}(I)$. Indeed, for example, if $w \equiv \text{const}$, then the function $f(x) = x^{-1/p}$ belongs to $L^{p)}_{dec}(I)$, but does not belong to $L^{p}_{dec}(I)$.

Let us mention the following continuous embeddings of grand Lebesgue spaces:

$$L_w^p(I) \hookrightarrow L_w^{p),\theta_1}(I) \hookrightarrow L_w^{p),\theta_2}(I) \hookrightarrow L_w^{p-\varepsilon}(I),$$

where $0<\varepsilon\leqslant p-1$ and $\theta_1 < \theta_2$.

For the following statement we refer to [18] (cf. also [9]).

Theorem A. *Let $0<p<\infty$, and let w be nonnegative function on $\mathbb{R}_+$. Then the following inequality holds*:

$$\int_0^\infty \big(Hf(x)\big)^p w(x)dx \leqslant c \int_0^\infty (f(x))^p w(x)dx$$

for all nonnegative and nonincreasing functions f on $\mathbb{R}_+$ if and only if $w \in B_p$, i.e., there is a positive constant B such that for all $r>0$

$$r^p \int_r^\infty \frac{w(x)}{x^p} dx \leqslant B \int_0^r w(x)dx.$$

Let (cf. [19])

$$\|w\|_{B_p} = \inf \left\{ C>0 : \int_0^r w(t)dt + r^p \int_r^\infty \frac{w(t)}{t^p} dt \leqslant C \int_0^r w(t)dt \quad \forall r>0 \right\}.$$

Note that $\|w\|_{B_p} > 1$.

The class B_p possesses the following remarkable properties (cf. [18, 9]):

(i) if $p < q$, then $w \in B_p \Rightarrow w \in B_q$,

(ii) if $w \in B_p$, then there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$; moreover,

$$\|w\|_{B_{p-\varepsilon}} \leqslant \frac{C_0 \|w\|_{B_p}}{1 - \varepsilon \alpha^p \|w\|_{B_p}}, \tag{1.1}$$

where C_0 and α $(0 < \alpha < 1)$ are universal constants and $\varepsilon < \dfrac{1}{\alpha^p \|w\|_{B_p}}$.

Proposition A ([18])**.** *Let $1 < p < \infty$. Suppose that w is decreasing on $\mathbb{R}_+$. Then $w \in B_p$ if and only if*

$$\sup_{t>0} \frac{1}{r} \left[\int_0^r w(x) dx \right]^{1/p} \left[\int_0^r w^{1-p'}(x) dx \right]^{1/p'} < \infty, \quad p' = \frac{p}{p-1}.$$

The classical Lorentz space Λ_w^p is defined as the set of functions g on $\mathbb{R}^n$ such that

$$\|g\|_{\Lambda_w^p} = \left[\int_0^\infty [g^*(x)]^p w(x) dx \right]^{1/p} < \infty,$$

where g^* is the decreasing rearrangement of g:

$$g^*(t) = \inf\{\lambda : |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}| \leqslant t\}.$$

The following statement provides a solution of the one weight problem for the operator M in weighted Lorentz spaces Λ_w^p defined on $\mathbb{R}^n$ (cf. [18], [9]).

Theorem B. *Let $0 < p < \infty$, and let w be a nonnegative function on $\mathbb{R}_+$. Then the operator M is bounded in Λ_w^p if and only if $w \in B_p$.*

Let supp $w \subset I$. Together with grand Lebesgue spaces, we are interested in the space $\Lambda_w^{p),\theta}$ defined as follows:

$$\Lambda_w^{p),\theta} := \left\{ g : \|g\|_{\Lambda_w^{p),\theta}} := \sup_{0<\varepsilon\leqslant\varepsilon_0} \left(\varepsilon^\theta \int_0^\infty [g^*(x)]^{p-\varepsilon} w(x) dx \right)^{1/(p-\varepsilon)} \right.$$

$$\left. = \sup_{0<\varepsilon\leqslant\varepsilon_0} \left(\varepsilon^\theta \int_0^1 [g^*(x)]^{p-\varepsilon} w(x) dx \right)^{1/(p-\varepsilon)} < \infty \right\},$$

where $\varepsilon_0 = p - 1$ if $p > 1$ and $0 < \varepsilon_0 < p$ if $p \leqslant 1$.

It is easy to see that if $w = \text{const}$ on I, then $\Lambda_w^{p),\theta}$ is the generalized grand Lebesgue space $L^{p),\theta}(\Omega)$, where $|\Omega| = 1$.

Remark. The grand Lorentz–Karamata spaces $\mathcal{L}_b^{q),p}$, introduced in [20, 21], are defined by the quasinorm (we suppose that $\mathcal{L}_b^{q),p}$ is defined on I)

$$\|f\|_{\mathcal{L}_b^{q),p}(I)} := \sup_{0<\sigma<\sigma_0} \sigma^{1/p} \left(\int_0^1 t^{p/(q-\sigma)} \Big[b(t) f^*(t) \Big]^p \frac{dt}{t} \right)^{1/p},$$

where $0 < p, q < \infty$ and b is a positive continuous function defined on $\mathbb{R}_+$ and slowly varying in the sense of Karamata. In particular, if $p = q$, then $\mathcal{L}_b^{q),p}(I)$ is the grand Lebesgue–Karamata space defined by the quasinorm (cf. [21])

$$\|f\|_{\mathcal{L}_b^{q),p}(I)} = \|f\|_{\mathcal{L}_b^{p)}(I)} = \sup_{0<\varepsilon<\sigma_0} \varepsilon^{1/p} \left(\int_0^1 \Big[b(t) f^*(t) \Big]^{p-\varepsilon} \frac{dt}{t} \right)^{1/(p-\varepsilon)}.$$

It is clear that, in general, the spaces $\mathcal{L}_b^{p)}$ and $\Lambda_w^{p)}$ are different.

Let φ be a nonnegative nonincreasing locally integrable function on $\mathbb{R}_+$. Suppose that

$$\Phi(x) = \int_0^x \varphi(t) dt.$$

Let S_φ be the Hardy type transform given by

$$(S_\varphi f)(x) = \frac{1}{\Phi(x)} \int_0^x f(t) \varphi(t) dt. \tag{1.2}$$

Lemma A (cf. [14, 19]). *Let $0 < p < \infty$, and let w be a nonnegative function on $\mathbb{R}_+$. Then the inequality*

$$\|S_\varphi f\|_{L_w^p(\mathbb{R}_+)} \leqslant A \|f\|_{L^p(\mathbb{R}_+)}$$

holds for all nonnegative and nondecreasing functions f if and only if

$$\int_0^r w(x) dx + \Phi(r)^p \int_r^\infty \frac{w(x)}{\Phi(x)} dx \leqslant A^p \int_0^r w(x) dx.$$

The following extrapolation theorem is taken from [19].

Theorem C. *Suppose that $0 < p < \infty$. Let ψ be a nonnegative nondecreasing function on $\mathbb{R}_+$, and let w be a nonnegative function on $\mathbb{R}$. Assume that (f, g) is a pair of nonnegative nonincreasing functions on $\mathbb{R}_+$. Assume also that for every $w \in B_{p_0}$*

$$\int_0^\infty f^{p_0} w \leqslant \psi(\|w\|_{B_{p_0}}) \int_0^\infty g^{p_0} w.$$

Then for every $p > 0$ and $w \in B_p$

$$\int_0^\infty f^p w \leqslant \widetilde{\psi}(\|w\|_{B_p}) \int_0^\infty g^p w,$$

where the function $\widetilde{\psi}$ is defined as follows:

$$\widetilde{\psi}(t) = \inf_{0<\varepsilon<\frac{p_0}{tp\alpha^p}} \psi^{p/p_0}\Big(\frac{p_0}{\varepsilon}\Big) \frac{C_0 t}{1-\varepsilon t \frac{p}{p_0}\alpha^p}, \tag{1.3}$$

C_0 and α are universal constants defined in (1.1).

2 General Type Theorem

In this section, we prove the general type theorem for couples of nonincreasing functions in weighted grand Lebesgue spaces $L_w^{p),\theta}(I)$, where w is a weight on I. We need the following slight modification of Theorem C for the interval I.

Definition. Let $0 < p < \infty$. We say that a nonnegative integrable function w on I belongs to the class $B_p(I)$ if there is a positive constant $\widetilde{B}$ such that for all $0 < r \leqslant 1$

$$r^p \int_r^1 \frac{w(t)}{t^p} dt \leqslant \widetilde{B} \int_0^r w(t)dt.$$

We use the following notation:

$$\|w\|_{B_p(I)} := \inf \Big\{ C > 0 : \int_0^r w(t)dt + r^p \int_r^1 \frac{w(t)}{t^p} dt \leqslant C \int_0^r w(t)dt \quad \forall 0 < r \leqslant 1 \Big\},$$

$$\widetilde{w}(x) := \begin{cases} w(x), & x \in I, \\ 0, & x > 1. \end{cases}.$$

It is clear that $\|w\|_{B_p(I)} > 1$.

The proof of the following lemma can be checked immediately.

Lemma 2.1. *Let $0 < p < \infty$, and let $w \in B_p(I)$. Then $\widetilde{w} \in B_p$. Moreover,*

$$\|w\|_{B_p(I)} = \|w\|_{B_p}.$$

Corollary 2.1. *Let $0 < p < \infty$. Suppose that $w \in B_p(I)$. Then there is a positive number ε such that $w \in B_{p-\varepsilon}(I)$. Moreover,*

$$\|w\|_{B_{p-\varepsilon}(I)} \leqslant \frac{C_0 \|w\|_{B_p(I)}}{1 - \varepsilon \alpha^p \|w\|_{B_p(I)}}, \tag{2.1}$$

where the constants C_0 and α are the same as in (1.1).

It is also clear that if $p < q$, then $w \in B_p(I)$ implies $w \in B_q(I)$.

The following statement is a restricted version of Lemma 2.3 in [19]. For the sake of completeness, we give the proof.

Lemma 2.2. *Let $0 < p_0 < \infty$, and let ψ be a nondecreasing function on I. Suppose that (f, g) is a pair of nonincreasing functions on I. Suppose that for every $w \in B_{p_0}(I)$*

$$\int_0^1 fw \leqslant \psi(\|w\|_{B_{p_0}(I)}) \int_0^1 gw. \tag{2.2}$$

Then for every $0 < \varepsilon < p_0$ and every $t \in I$

$$\int_0^t f(\tau)\tau^{p_0-1-\varepsilon} d\tau \leqslant \psi\Big(\frac{p_0}{\varepsilon}\Big) \int_0^t g(\tau)\tau^{p_0-1-\varepsilon} d\tau. \tag{2.3}$$

Proof. We take $w(t) = v(t)t^{p_0-1-\varepsilon}$, where v is a nonnegative nonincreasing function on I. Then

$$\int_0^r w(t)dt + r^{p_0}\int_r^1 \frac{w(t)}{t^{p_0}}dt = \int_0^r w(t)dt + r^{p_0}\int_r^1 \frac{v(t)}{t^{1+\varepsilon}}dt \leqslant \int_0^r w(t)dt + \varepsilon^{-1}v(r)r^{p_0-\varepsilon}$$
$$= \int_0^r w(t)dt + \frac{p_0-\varepsilon}{\varepsilon}v(r)\int_0^r t^{p_0-\varepsilon-1}dt \leqslant \frac{p_0}{\varepsilon}\int_0^r w(t)dt.$$

Consequently, $w \in B_{p_0}(I)$; moreover,

$$\|w\|_{B_{p_0}(I)} \leqslant \frac{p_0}{\varepsilon}.$$

Taking $v(t) = \chi_{(0,s)}(t)$ $(t \in I)$ and using (2.2), we obtain (2.3). □

Lemma 2.3. *Let $0 < p < \infty$. Suppose that S_φ is defined by (1.2). Then the inequality*

$$\|S_\varphi f\|_{L^p_w(I)} \leqslant A\|f\|_{L^p_w(I)}$$

holds for all nonnegative and nondecreasing functions f if and only if

$$\int_0^r w(x)dx + \Phi(r)^p \int_r^1 \frac{w(x)}{\Phi(x)}dx \leqslant A^p \int_0^r w(x)dx.$$

The proof of this statement follows immediately if we take the weight function $\widetilde{w}$ instead of w in Lemma A and extend f by 0 outside I.

The following statement is a restricted version of Theorem C on I.

Lemma 2.4. *Let ψ be a nonnegative and nondecreasing function on I. Assume that (f,g) is a pair of nonnegative nonincreasing functions on I. Let $0 < p_0 < \infty$. Suppose that for every $w \in B_{p_0}(I)$*

$$\int_0^1 f^{p_0} w \leqslant \psi(\|w\|_{B_{p_0}(I)}) \int_0^1 g^{p_0} w.$$

Then for every $p > 0$ and $w \in B_p(I)$

$$\int_0^1 f^{p} w \leqslant \widetilde{\psi}(\|w\|_{B_{p}(I)}) \int_0^1 g^{p} w,$$

where the function $\widetilde{\psi}$ is defined by (1.3).

Proof. Let $w \in B_p(I)$, and let $0 < \varepsilon < p$. Suppose that

$$\varphi(t) := t^{p_0 - 1 - \varepsilon}.$$

Using Lemma 2.2 and the fact that f is decreasing, we see that

$$\begin{aligned}\int_0^1 f^p(t) w(t) dt &\leqslant \int_0^1 \left(\frac{p_0 - \varepsilon}{t^{p_0 - \varepsilon}} \int_0^t f^{p_0}(s) s^{p_0 - 1 - \varepsilon} ds \right)^{p/p_0} w(t) dt \\ &\leqslant \psi^{p/p_0}(p_0/\varepsilon) \int_0^1 \left(\frac{p_0 - \varepsilon}{t^{p_0 - \varepsilon}} \int_0^t g^{p_0}(s) s^{p_0 - 1 - \varepsilon} ds \right)^{p/p_0} w(t) dt \\ &= \psi^{p/p_0}(p_0/\varepsilon) \int_0^1 \left(S_\varphi g^{p_0}(t) \right)^{p/p_0} w(t) dt.\end{aligned}$$

On the other hand, by Lemma 2.3, it suffices to compute a constant A such that

$$\int_0^r w(x) dx + r^{\frac{(p_0 - \varepsilon)p}{p_0}} \int_r^1 w(x) x^{-\frac{(p_0 - \varepsilon)p}{p_0}} dx \leqslant A \int_0^r w(x) dx. \tag{2.4}$$

Note that (2.4) is equivalent to the condition $w \in B_{(p_0 - \varepsilon)p/p_0}(I)$ with $A = \|w\|_{B_{(p_0 - \varepsilon)p/p_0}}(I)$.

Since $w \in B_p(I)$, there is a positive number η such that $w \in B_{p-\eta}(I)$. Now, we choose ε so small that

$$p - \eta = \frac{(p_0 - \varepsilon)p}{p_0}.$$

Then $\eta = \varepsilon p / p_0$. By (2.1), we have

$$A = \|w\|_{B_{p-\eta}(I)} \leqslant \frac{C \|w\|_{B_p(I)}}{1 - \varepsilon \frac{p}{p_0} \alpha^p \|w\|_{B_p(I)}}.$$

Summarizing the above–derived inequalities and taking into account that η can be taken sufficiently small, we conclude that

$$\int_0^1 f^p(t)w(t)dt \leqslant \psi^{p/p_0}\Big(\frac{p_0}{\varepsilon}\Big)\frac{C\|w\|_{B_p(I)}}{1-\varepsilon\frac{p}{p_0}\alpha^p\|w\|_{B_p(I)}}\int_0^1 g^p(t)w(t)dt$$

for every

$$0<\varepsilon<\frac{p_0}{p\alpha^p\|w\|_{B_p(I)}}. \qquad \square$$

Now, we are ready to prove the main result of this section.

Theorem 2.1. *Suppose that $\theta>0$ and ψ is a nonnegative nonincreasing function on I. Assume that $0<p_0<\infty$. Let (f,g) be a pair of nonnegative nonincreasing functions on I, and let for every $w\in B_{p_0}(I)$*

$$\int_0^1 f^{p_0}w \leqslant \psi(\|w\|_{B_{p_0}(I)})\int_0^1 g^{p_0}w.$$

Then for every $p>0$ and $w\in B_p(I)$

$$\|f\|_{L_w^{p),\theta}(I)} \leqslant c\|g\|_{L_w^{p),\theta}(I)},$$

where the positive constant c depends only on p, θ, and w.

Proof. We prove the theorem for $1<p<\infty$. The proof for the case $0<p\leqslant 1$ is similar. Let $w\in B_p(I)$. Then there is a positive number σ such that $w\in B_{p-\sigma}(I)$. Hence $w\in B_{p-\varepsilon}(I)$ for all $0<\varepsilon\leqslant\sigma$. We can assume that $\sigma<p-1$. Further, if $\varepsilon\in(\sigma,p-1]$, then, by the Hölder inequality,

$$\|f\|_{L_w^{p-\varepsilon}(I)} \leqslant \Bigg(\int_0^1 f^{q-\sigma}(x)w(x)dx\Bigg)^{\frac{1}{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}$$

because

$$\Big(\frac{p-\sigma}{p-\varepsilon}\Big)'=\frac{p-\sigma}{\varepsilon-\sigma}.$$

Further, the conditions $\sigma<p-1$ and $\sigma<\varepsilon<p-1$ yield

$$0<\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}<\frac{p-1-\sigma}{p-\sigma}. \tag{2.5}$$

Simple calculations, (2.5), Lemma 2.4 and the fact that w is integrable on I yield that

$$\begin{aligned}\|f\|_{L^{p),\theta}_w(I)} &= \max\left\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(I)}, \sup_{\sigma<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(I)} \right\} \\ &\leqslant \max\left\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(I)}, \sup_{\sigma<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \right\} \\ &\leqslant \max\left\{1, \sup_{\sigma<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \right\} \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(I)} \\ &\leqslant \max\left\{1, \left[\sup_{\sigma<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}}\right] \sigma^{-\frac{\theta}{p-\sigma}} \big(1+w(I)\big)^{\frac{p-1-\sigma}{p-\sigma}} \right\} \\ &\quad \times \sup_{0<\varepsilon\leqslant\sigma} \overline{\psi}_\varepsilon(\|w\|_{B_{p-\varepsilon}(I)}) \varepsilon^{\frac{\theta}{p-\varepsilon}} \|g\|_{L^{p-\varepsilon}_w([0,1])} \\ &\leqslant \left[c(p,\theta,w,\sigma) \sup_{0<\varepsilon\leqslant\sigma} \overline{\psi}_\varepsilon(\|w\|_{B_{p-\varepsilon}(I)})\right] \|g\|_{L^{p),\theta}_w(I)} =: D\|g\|_{L^{p),\theta}_w(I)},\end{aligned}$$

where

$$c(p,\theta,w,\sigma) := \max\left\{1, \left[\sup_{\sigma<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}}\right] \sigma^{-\frac{\theta}{p-\sigma}} \big(1+w(I)\big)^{\frac{p-1-\sigma}{p-\sigma}} \right\}$$

and the function $\overline{\psi}_\varepsilon$ is defined by

$$\overline{\psi}_\varepsilon(t) = \inf_{0<\eta<\frac{p}{t(p-\varepsilon)\alpha^{p-\varepsilon}}} \psi^{\frac{1}{p}}\left(\frac{p}{\eta}\right) \left[\frac{C_0 t}{1-\eta t \frac{p-\varepsilon}{p} \alpha^{p-\varepsilon}}\right]^{\frac{1}{p-\varepsilon}}.$$

Here C_0 and α are universal constant defined in (1.1).

Observe now that since $\|w\|_{B_{p-\varepsilon}(I)} \leqslant \|w\|_{B_{p-\sigma}(I)}$, the following estimate holds:

$$D \leqslant c(p,\theta,,\sigma,w) \inf_{0<\eta<\frac{1}{\alpha^{p-\sigma}\|w\|_{B_{p-\sigma}(I)}}} \psi^{\frac{1}{p}}\left(\frac{p}{\eta}\right) \left[1 + \frac{C_0\|w\|_{B_{p-\sigma}(I)}}{1-\eta\alpha^{p-\sigma}\|w\|_{B_{p-\sigma}(I)}}\right]^{\frac{1}{p-\sigma}}. \tag{2.6}$$

□

3 Hardy Transforms

In this section, we derive criteria for the one weight inequality for the Hardy transform H in grand Lebesgue spaces $L^{p),\theta}_w(I)$ for decreasing functions.

Our main result is the following statement.

Theorem 3.1. *Let $0 < p < \infty$, and let $\theta > 0$. Suppose that w is a weight on I. Then the inequality*

$$\|Hf\|_{L^{p),\theta}_w(I)} \leqslant c\|f\|_{L^{p),\theta}_{\mathrm{dec},w}(I)} \tag{3.1}$$

holds if and only if $w \in B_p(I)$.

Proof. *Sufficiency.* Since (Hf, f) is a pair of nonnegative and nonincreasing functions, where f is nonnegative and nonincreasing, the result follows from Theorem 2.1. By Lemma 2.3 (taking $\varphi(x) \equiv 1$) and (2.6), the following inequality holds for the best possible constant c in (3.1):

$$c \leqslant c(p, \theta, , \sigma, w) \inf \left(\frac{p}{\eta}\right)^{\frac{1}{p}} \left[1 + \frac{C_0 \|w\|_{B_{p-\sigma}(I)}}{1 - \eta\alpha^{p-\sigma}\|w\|_{B_{p-\sigma}(I)}}\right]^{\frac{1}{p-\sigma}}.$$

where the infimum is taken over η such that $0 < \eta < \dfrac{1}{\alpha^{p-\sigma}\|w\|_{B_{p-\sigma}(I)}}$, the positive constant $c(p, \theta, \sigma, w)$ depends only on p, θ, σ, w, and σ is a small positive number taken in such a way that $w \in B_{p-\sigma}(I)$.

Necessity. We consider the case $1 < p < \infty$. The proof of the case $0 < p \leqslant 1$ is the same. Suppose that (3.1) holds. Since

$$0 < \int_0^r w(x)dx \quad \forall\, r > 0,$$

there is a number $\varepsilon_r \in (0, p-1]$ such that

$$\sup_{0<\varepsilon\leqslant p-1} \left(\varepsilon^\theta \int_0^r w(x)dx\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon_r^\theta \int_0^r w(x)dx\right)^{\frac{1}{p-\varepsilon_r}}.$$

Hence, taking $f_r(x) = \chi_{(0,r)}(x)$, we have

$$\|f\|_{L^{p),\theta}_{\mathrm{dec},w}(I)} = \varepsilon_r^{\frac{\theta}{p-\varepsilon_r}} \left(\int_0^r w(x)dx\right)^{\frac{1}{p-\varepsilon_r}}.$$

On the other hand, for f_r we have

$$\|Hf\|_{L^{p),\theta}_w(I)} \geqslant \sup_{0<\sigma\leqslant p-1} r\sigma^{\frac{\theta}{p-\sigma}} \left(\int_r^1 \frac{w(x)}{x^{p-\sigma}}dx\right)^{\frac{1}{p-\sigma}} \geqslant r\varepsilon_r^{\frac{\theta}{p-\varepsilon_r}} \left(\int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}}dx\right)^{\frac{1}{p-\varepsilon_r}}.$$

Taking (3.1) into account, we find

$$r^{p-\varepsilon_r} \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}}dx \leqslant c \int_0^r w(x)dx,$$

where the positive constant c is independent of r.

Consequently,

$$r^p \int_r^1 \frac{w(x)}{x^p}dx = r^{p-\varepsilon_r+\varepsilon_r} \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r+\varepsilon_r}}dx \leqslant r^{p-\varepsilon_r} \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}}dx \leqslant c \int_0^r w(x)dx.$$

Finally, we conclude that $w \in B_p(I)$. □

To prove the following statement we need some lemmas.

Lemma 3.1. *Let $1 \leqslant p < \infty$, and let w be a weight on I. Then the following conditions are equivalent*:

(i) *the inequality*

$$w(\{x \in I : Hf(x) > \lambda\}) \leqslant \frac{C}{\lambda^p} \int_0^1 f^p(t) w(t) dt$$

holds for all nonnegative f;

(ii) *the inequality*

$$\int_0^1 (Hf)^p(t) w(t) dt \leqslant C \int_0^1 f^p(t) w(t) dt,$$

holds for all nonnegative f;

(iii) *the condition*

$$\sup_{0<r<s<1} \left[\frac{1}{s^p} \int_r^s w(x) dx \right]^{1/p} \left[\int_0^r w^{1-p'}(x) dx \right]^{1/p'} < \infty, p' = \frac{p}{p-1}, \tag{3.2}$$

is satisfied.

The proof of this lemma can be found in [22] in the case $\mathbb{R}_+$, but the arguments remain also valid for I. We omit details.

Lemma 3.2. *Let $1 < p < \infty$. Suppose that w is decreasing on I. Then $w \in B_p(I)$ if and only if*

$$\sup_{0<r<1} \frac{1}{r} \left[\int_0^r w(x) dx \right]^{1/p} \left[\int_0^r w^{1-p'}(x) dx \right]^{1/p'} < \infty, p' = \frac{p}{p-1}. \tag{3.3}$$

Proof. *Sufficiency.* First of all, we note that (3.3) implies (3.2). Now Lemma 3.1 completes the proof of Sufficiency.

Necessity. Let us now show that if w is decreasing on I, then the condition $w \in B_p(I)$ implies (3.3). For this purpose it suffices to show that the boundedness of H in $L^p_{\mathrm{dec},w}(I)$ yields (3.3). We follow the proof of Theorem 1.10 in [18]. Suppose that

$$f_{n,r}(x) = w_n^{1-p'}(x) \chi_{(0,r)}(x),$$

where

$$w_n(x) := \begin{cases} w(x), w(x) > 1/n, \\ 1/n, \quad w(x) \leqslant 1/n \end{cases}.$$

Since $f_{n,r}$ is nonincreasing, it follows that $Hf_{n,r}$ is nonincreasing. Hence, by using the boundedness of H in $L^p_{\mathrm{dec},w}(I)$ we find that

$$\left[\frac{1}{r} \int_0^r w_n^{1-p'}(x) dx \right]^p \int_0^r w(x) dx \leqslant \int_0^r (Hf_{n,r}(x))^p w(x) dx \leqslant c \int_0^r w_n^{1-p'}(x) dx.$$

Consequently,

$$\frac{1}{r^p}\left[\int_0^r w_n^{1-p'}(x)dx\right]^{p-1}\int_0^r w(x)dx \leqslant C.$$

Passing now to the limit as $n \to \infty$ we have the desired result. □

Corollary 3.1. *Let $1 < p < \infty$, and let w be a decreasing weight on I. Then the inequality* (3.1) *holds if and only if* (3.3) *is satisfied.*

4 Applications to Fractional Integrals and Maximal Functions

In this section, we apply the main result of Section 3 to derive criteria for the boundedness of R_α from $L^{p),\theta}_{\mathrm{dec},w}(I)$ to $L^{p),\theta}_{w}(I)$ and to establish necessary and sufficient conditions for the boundedness of M in weighted grand Lorentz spaces $\Lambda^{p),\theta}_w$.

Theorem 4.1. *Assume that $0 < p < \infty$, $\theta > 0$, and $0 < \alpha < 1$. Suppose that w is a weight on I. Then the inequality*

$$\|R_\alpha f\|_{L^{p),\theta}_w(I)} \leqslant C\|f\|_{L^{p),\theta}_{\mathrm{dec},w}(I)}$$

holds if and only if $w \in B_p(I)$.

Proof. The proof of this theorem is based on Theorem 3.1 and the pointwise estimates

$$c_1 R_\alpha f(x) \leqslant Hf(x) \leqslant c_2 R_\alpha f(x), \quad f\downarrow, f \geqslant 0, \tag{4.1}$$

where the positive constants c_1 and c_2 are independent of x and f.

For the sake of completeness, we prove (4.1). The left–hand side estimate in (4.1) follows from the inequalities

$$(R_\alpha f)(x) = x^\alpha \int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}}dt + x^\alpha \int_{x/2}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}dt$$

$$\leqslant 2^{1-\alpha}\frac{1}{x}\int_0^x f(t)dt + \frac{2^{-\alpha}}{\alpha}f(x/2) \leqslant C\left(Hf(x) + \frac{f(x/2)}{x}\int_0^{x/2} dt\right) \leqslant CHf(x),$$

where the positive constant C depends only on α.

The right–hand side estimate in (4.1) can be checked immediately since $(x-t)^{\alpha-1} \geqslant x^{\alpha-1}$ for $0 < t < x$. In fact, $c_2 = 1$. □

Theorem 4.2. *Let $0 < p < \infty$. Suppose that w is a nonnegative function on $\mathbb{R}_+$ such that* $\operatorname{supp} w \subset I$ *and $w \in L(I)$. Then the operator M is bounded in $\Lambda^{p),\theta}_w$ if and only if $w \in B_p(I)$.*

Proof. *Sufficiency* follows from the pointwise inequality (cf., for example, [23, P. 122]):

$$(Mg)^*(x) \leqslant A_n(Hg^*)(x), \quad x > 0,$$

and Theorem 3.1.

Necessity follows from Theorem 3.1 and the fact that if f is a nonincreasing function on $\mathbb{R}_+$, then

$$(Mg)^*(t) \geqslant 4^{-n} A \cdot Hf(t), \quad t > 0,$$

for $g(x) = f(A|x|^n)$, $x \in \mathbb{R}^n$, where A is the volume of the unit sphere in $\mathbb{R}^n$ (cf., for example, [18] for details). □

Acknowledgement

The work was partially supported by the Rustaveli National Science Foundation Grant (Project No. GNSF/ST09_23_3-100).

The author expresses his gratitude to Professor V. Kokilashvili for his valuable remarks and suggestions.

References

1. T. Iwaniec and C. Sbordone, "On the integrability of the Jacobian under minimal hypotheses," *Arch. Ration. Mech. Anal.* **119**, 129–143 (1992).
2. A. Fiorenza, B. Gupta, and P. Jain, "The maximal theorem in weighted grand Lebesgue spaces," *Studia Math.* **188**, No. 2, 123-133 (2008).
3. V. Kokilashvili and A. Meskhi, "A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces," *Georgian Math. J.* **16**, No. 3, 547–551 (2009).
4. V. Kokilashvili, "Boundedness criteria for singular integrals in weighted grand Lebesgue spaces," *J. Math. Sci. (New York)* **170**, No. 1, 20–33 (2010).
5. V. Kokilashvili, "Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem," *Proc. A. Razmadze Math. Inst.* **151**, 129-133 (2009).
6. A. Meskhi, *Criteria for the Boundedness of Potential Operators in Grand Lebesgue Spaces* arXiv:1007.1185v1 [math.FA] 7 July, 2010.
7. V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin etc. (1984).
8. A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co, Singapore etc. (2003).
9. M. J. Carro, J. A. Raposo, and J. Soria, "Recent developments in the theory of Lorentz spaces and weighted inequalities," *Mem. Am. Math. Soc.* **187**, No. 877 (2007).
10. V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co, Singapore etc. (1991).
11. D. E. Edmunds, V. Kokilashvili, and A. Meskhi, *Bounded and Compact Integral Operators*, Kluwer Academic Publishers, Dordrecht etc. (2002).

12. S. Lai, "Weighted norm inequalities for general operators on monotone functions," *Trans. Am. Math. Soc.* **340**, No. 2, 811–836 (1993).

13. C. Andersen, "Weighted generalized Hardy inequalities for nonincreasing functions," *Can. J. Math.* **42**, No. 6, 1121–1135 (1991).

14. M. J. Carro and J. Soria, "Boundedness of some integral operators," *Can. J. Math.* **45**, 195–231 (1993).

15. L. Greco, T. Iwaniec, and C. Sbordone, "Inverting the p-harmonic operator," *Manuscripta Math.* **92**, 249–258 (1997).

16. A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces, *Collect. Math.* **51**, No. 2, 131–148 (2000).

17. C. Capone and A. Fiorenza, "On small Lebesgue spaces," *J. Funct. Spaces Appl.* **3**, No. 1, 73–89 (2005).

18. M. A. Arino and B. Muckenhoupt, "Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions," *Trans. Am. Math. Soc.* **320**, No. 2, 727–735 (1990).

19. M. J. Carro and M. Lorente, "Rubio de Francia's extrapolation theorem for B_p weights," *Proc. Am. Math. Soc.* **138**, No. 2, 629–640 (2010).

20. A. Fiorenza and G. E. Karadzhov, "Grand and small Lebesgue spaces and their analogs," *Z. Anal. Anwend.* **23**, No. 4, 657–681 (2004).

21. C. Capone, A. Fiorenza, and G. E. Karadzhov, "Grand Orlicz spaces and integrability of the Jacobian," *Math. Scand.* **102**, No. 1, 131–148 (2008).

22. K. Andersen and B. Muckenhoupt, "Weighted weak type inequalities with applications to Hilbert transforms and maximal functions," *Studia Math.* **72**, 9–26 (1982).

23. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Orlando (1988).

Submitted on June 11, 2011