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Research Article

Potential Operators on Cones of Nonincreasing Functions

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Necessary and sufficient conditions on weight pairs guaranteeing the two-weight inequalities for the potential operators $(I_\alpha f)(x) = \int_0^\infty (f(t)/|x-t|^{1-\alpha})dt$ and $(\mathcal{I}_{\alpha_1,\alpha_2} f)(x,y) = \int_0^\infty \int_0^\infty (f(t,\tau)/|x-t|^{1-\alpha_1}|y-\tau|^{1-\alpha_2})dtd\tau$ on the cone of nonincreasing functions are derived. In the case of $\mathcal{I}_{\alpha_1,\alpha_2}$, we assume that the right-hand side weight is of product type. The same problem for other mixed-type double potential operators is also studied. Exponents of the Lebesgue spaces are assumed to be between 1 and ∞.

1. Introduction

Our aim is to derive necessary and sufficient conditions on weight pairs governing the boundedness of the following potential operators:

$$\begin{gathered}(I_\alpha f)(x) = \int_0^\infty \frac{f(t)}{|x-t|^{1-\alpha}}dt, \quad 0<\alpha<1,\\ (\mathcal{I}_{\alpha_1,\alpha_2} f)(x,y) = \iint_0^\infty \frac{f(t,\tau)}{|x-t|^{1-\alpha_1}|y-\tau|^{1-\alpha_2}}dt\,d\tau, \quad 0<\alpha_1,\ \alpha_2<1,\end{gathered} \tag{1.1}$$

from L^p_{dec} to L^q, where $1<p,\ q<\infty$.

Historically, necessary and sufficient condition on a weight function u, for which the boundedness of the one-dimensional Hardy transform

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt \tag{1.2}$$

from $L^p_{\mathrm{dec}}(u, \mathbb{R}_+)$ to $L^p(u, \mathbb{R}_+)$ holds, was established in [1]. Two-weight Hardy inequality criteria on cones of nonincreasing functions were derived in the paper [2]. The multidimensional analogues of these results were studied in [3–5]. Some characterizations of the two-weight inequality for the single integral operators involving Hardy-type transforms for monotone functions were given in [6–8]. The same problem for the Riesz potentials

$$(T_\alpha f)(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}dy, \quad 0 < \alpha < n, \tag{1.3}$$

for nonnegative nonincreasing radial functions was studied in [9].

In the paper [10] necessary and sufficient conditions governing the boundedness of the multiple Riemann-Liouville transform

$$(\mathcal{R}_{\alpha_1,\alpha_2} f)(x,y) = \int_0^x \int_0^y \frac{f(t,\tau)}{(x-t)^{1-\alpha_1}(y-\tau)^{1-\alpha_2}} dt\, d\tau, \quad 0 < \alpha_1,\ \alpha_2 < 1, \tag{1.4}$$

from $L^p_{\mathrm{dec}}(w, \mathbb{R}^2_+)$ to $L^p(v, \mathbb{R}^2_+)$ were derived, provided that w is a product of one-dimensional weights. Earlier, the problem of the boundedness of the two-dimensional Hardy transform $H_2 = \mathcal{R}_{1,1}$ from $L^p_{\mathrm{dec}}(w, \mathbb{R}^2_+)$ to $L^p(v, \mathbb{R}^2_+)$ was studied in [4] under the condition that w and v have the following form: $w(x,y) = w_1(x)w_2(y),\ v(x,y) = v_1(x)v_2(y)$.

It should be emphasized that the two-weight problem for the Hardy-type transforms and fractional integrals with single kernels has been already solved. For the weight theory and history of these operators in classical Lebesgue spaces, we refer to the monographs [11–15] and references cited therein.

The monograph [13] is dedicated to the two-weight problem for multiple integral operators in classical Lebesgue spaces (see also the papers [16–18] for criteria guaranteeing trace inequalities for potential operators with product kernels).

Unfortunately, in the case of double potential operator, we assume that the right-hand weight is of product type and the left-hand one satisfies the doubling condition with respect to one of the variables. Even under these restrictions the two-weight criteria are written in terms of several conditions on weights. We hope to remove these restrictions on weights in our future investigations.

Some of the results of this paper were announced without proofs in [19].

Finally we mention that constants (often different constants in the same series of inequalities) will generally be denoted by c or C; by the symbol $Tf \approx Kf$, where T and K are linear positive operators defined on appropriate classes of functions, we mean that there are positive constants c_1 and c_2 independent of f and x such that $(Tf)(x) \le c_1(Kf)(x) \le c_2(Tf)(x)$; $\mathbb{R}_+$ denotes the interval $(0,\infty)$ and p' means the number $p/(p-1)$ for $1 < p < \infty$; $W(x) := \int_0^x w(t)dt$; $W_j(x_j) := \int_0^{x_j} w_j(t)dt$; $W(t_1,\ldots,t_n) := \Pi_{i=1}^n W_i(t_i)$.

2. Preliminaries

We say that a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is nonincreasing if f is nonincreasing in each variable separately.

Let $\mathfrak{D}$ be the class of all nonnegative nonincreasing functions on $\mathbb{R}_+^n$. Suppose that u is measurable a.e. positive function (weight) on $\mathbb{R}_+^n$. We denote by $L^p(u, \mathbb{R}_+^n)$, $0 < p < \infty$, the class of all nonnegative functions on $\mathbb{R}_+^n$ for which

$$\|f\|_{L^p(u,\mathbb{R}_+^n)} := \left(\int_{\mathbb{R}_+^n} f^p(x_1, \ldots, x_n) u(x_1, \ldots, x_n) dx_1 \cdots dx_n\right)^{1/p} = \left(\int_{\mathbb{R}_+^n} f^p(x) u(x) dx\right)^{1/p} < \infty. \tag{2.1}$$

By the symbol $L^p_{\mathrm{dec}}(u, \mathbb{R}_+^n)$ we mean the class $L^p(u, \mathbb{R}_+^n) \cap \mathfrak{D}$.

The next statement regarding two-weight criteria for the Hardy operator H on the cone of nonincreasing functions was proved in [2].

Theorem A. *Let v and w be weight functions on $\mathbb{R}_+$, and let $W(\infty) = \infty$.*

(i) *Suppose that $1 < p \le q < \infty$. Then the inequality*

$$\left[\int_0^\infty (Hf(x))^q v(x) dx\right]^{1/q} \le C\left[\int_0^\infty (f(x))^p w(x) dx\right]^{1/p}, \quad f \in L^p_{dec}(w, \mathbb{R}_+), \tag{2.2}$$

holds if and only if the following two conditions are satisfied:

$$\begin{gathered}\sup_{a>0}\left(\int_0^a v(x) dx\right)^{1/q}\left(\int_0^a w(x) dx\right)^{-1/p} < \infty, \\ \sup_{a>0}\left(\int_a^\infty \frac{v(x)}{x^q} dx\right)^{1/q}\left(\int_0^a W^{-p'}(x) x^{p'} w(x) dx\right)^{1/p'} < \infty.\end{gathered} \tag{2.3}$$

(ii) *Let $1 < q < p < \infty$. Then H is bounded from $L^p_{dec}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if the following two conditions are satisfied:*

$$\begin{gathered}\left[\int_0^\infty \left[\left(\int_0^t v(x) dx\right)^{1/p} W^{-1/p}(t)\right]^r v(t) dt\right]^{1/r} < \infty, \\ \left[\int_0^\infty \left[\left(\int_t^\infty x^{-q} v(x) dx\right)^{1/p}\left(\int_0^t x^{p'} W^{-p'}(x) w(x) dx\right)^{1/p'}\right]^r t^{p'} W^{-p'}(t) w(t) dt\right]^{1/r} < \infty,\end{gathered} \tag{2.4}$$

where $r = pq/(p - q)$.

The following statement was proved in [2] for $n = 1$. For $n \geq 1$ we refer to [4].

Proposition A. *Let $1 < p,\ q < \infty$. Suppose that T is a positive integral operator defined on functions $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, which are nonincreasing in each variable separately. Suppose that T^* is its formal adjoint. Let $w(x_1, \ldots, x_n) = w_1(x_1) \cdots w_n(x_n)$ be a product weight such that $W_i(\infty) = \infty$, $i = 1, \ldots, n$. Let v be a general weight on $\mathbb{R}^n_+$. Then the operator T is bounded from $L^p_{dec}(w, \mathbb{R}^n_+)$ to $L^p(v, \mathbb{R}^n_+)$ if and only if the inequality*

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n_+} \left(\int_0^{x_1} \cdots \int_0^{x_n} T^* g\right)^{p'} W^{-p'}(x_1, \ldots, x_n) w(x_1, \ldots, x_n) dx_1 \cdots dx_n\right)^{1/p'} \\
&\quad \leq c \left(\int_{\mathbb{R}^n_+} g(x)^{q'} v^{1-q'}(x) dx\right)^{1/q'}
\end{aligned} \tag{2.5}$$

holds for all $g \geq 0$.

Let R_α be the Riemann-Liouville transform with single kernel

$$(R_\alpha f)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad x \in \mathbb{R}_+,\ \alpha > 0. \tag{2.6}$$

If $\alpha = 1$, then R_α is the Hardy transform. The $L^p(w, \mathbb{R}_+) \to L^q(v, \mathbb{R}_+)$ boundedness for R_1 was characterized by Muckenhoupt ([20]) for $p = q$, and by Kokilashvili [21] and Bradley [22] for $p < q$ (see also the monograph by Maz'ya [23] for these and relevant results).

In the case when $0 < \alpha < 1$, the Riemann-Liouville transform has singularity. For the results regarding the two-weight problem, in this case we refer, for example, to the monograph [11] and the references cited therein.

The next result deals with the case $\alpha > 1$ (see [24]).

Theorem B. *Let $\alpha > 1$. Then the operator R_α is bounded from $L^p(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\begin{aligned}
&\sup_{t>0} \left(\int_t^\infty (x-t)^{(\alpha-1)q} v(x) dx\right)^{1/q} \left(\int_0^t w^{1-p'}(y) dy\right)^{1/p'} < \infty, \\
&\sup_{t>0} \left(\int_t^\infty v(x) dx\right)^{1/q} \left(\int_0^t (t-x)^{(\alpha-1)p'} w^{1-p'}(y) dy\right)^{1/p'} < \infty,
\end{aligned} \tag{2.7}$$

for $1 < p \leq q < \infty$ and

$$\begin{aligned}
&\left\{\int_0^\infty \left(\int_t^\infty (x-t)^{(\alpha-1)q} v(x) dx\right)^{r/q} \left(\int_0^t w^{1-p'}(y) dy\right)^{r/q'} w^{1-p'}(t) dt\right\}^{1/r} < \infty, \\
&\left\{\int_0^\infty \left(\int_t^\infty v(x) dx\right)^{r/p} \left(\int_0^t (t-y)^{(\alpha-1)p'} w^{1-p'}(y) dy\right)^{r/p'} v(t) dt\right\}^{1/r} < \infty,
\end{aligned} \tag{2.8}$$

for $1 < q < p < \infty$, where r is defined as follows: $1/r = 1/q - 1/p$.

Theorem C (see [10]). *Let $1 < p \le q < \infty$, and let $0 < \alpha_i < 1$, $i = 1,2$. Assume that v and w are weights on $\mathbb{R}^2_+$. Suppose also that $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ for some one-dimensional weights w_1 and w_2 and that $W_i(\infty) = \infty$, $i = 1,2$. Then the following conditions are equivalent:*

(a) *$\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$;*

(b) *the following four conditions hold simultaneously:*

(i)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2} w(t_1,t_2)dt_1dt_2\right)^{-1/p}\left(\int_0^{a_1}\int_0^{a_2}(t_1^{\alpha_1}t_2^{\alpha_2})^q v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty, \tag{2.9}$$

(ii)

$$\begin{aligned}\sup_{a_1,a_2>0}&\left(\int_0^{a_1}\int_0^{a_2}(t_1t_2)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right)^{1/p'}\\ &\times\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty}\left(t_1^{\alpha_1-1}t_2^{\alpha_2-1}\right)^q v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty,\end{aligned} \tag{2.10}$$

(iii)

$$\begin{aligned}\sup_{a_1,a_2>0}&\left(\int_0^{a_1}w_1(t_1)dt_1\right)^{-1/p}\left(\int_0^{a_2}t_2^{p'}W_2^{-p'}(t_2)w_2(t_2)dt_2\right)^{1/p'}\\ &\times\left(\int_0^{a_1}\int_{a_2}^{\infty}t_1^{q\alpha_1}t_2^{q(\alpha_2-1)}v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty,\end{aligned} \tag{2.11}$$

(iv)

$$\begin{aligned}\sup_{a_1,a_2>0}&\left(\int_0^{a_1}t_1^{p'}W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2}w_2(t_2)dt_2\right)^{-1/p}\\ &\times\left(\int_{a_1}^{\infty}\int_0^{a_2}t_1^{q(\alpha_1-1)}t_2^{q\alpha_2}v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty.\end{aligned} \tag{2.12}$$

In particular, Theorem C yields the trace inequality criteria on the cone of nonincreasing functions.

Corollary A (see [10]). *Let $1 < p \le q < \infty$, and let $0 < \alpha_i < 1$, $i = 1,2$. Then the following conditions are equivalent:*

(a) *the boundedness of $\mathcal{R}_{\alpha_1,\alpha_2}$ from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ holds for $w \equiv 1$;*

(b)

$$B_1 := \sup_{a_1,a_2>0}B_1(a_1,a_2) := \sup_{a_1,a_2>0}(a_1a_2)^{-1/p}\left(\int_0^{a_1}\int_0^{a_2}x_1^{q\alpha_1}x_2^{q\alpha_2}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty; \tag{2.13}$$

(c)

$$B_2 := \sup_{a_1,a_2>0} B_2(a_1,a_2) := \sup_{a_1,a_2>0} (a_1a_2)^{1/p'}\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} x_1^{q(\alpha_1-1)}x_2^{q(\alpha_2-1)}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty; \tag{2.14}$$

(d)

$$B_3 := \sup_{a_1,a_2>0} B_3(a_1,a_2) := \sup_{a_1,a_2>0} a_1^{-1/p}a_2^{1/p'}\left(\int_{0}^{a_1}\int_{a_2}^{\infty} x_1^{q\alpha_1}x_2^{q(\alpha_2-1)}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty; \tag{2.15}$$

(e)

$$B_4 := \sup_{a_1,a_2>0} B_4(a_1,a_2) := \sup_{a_1,a_2>0} a_1^{1/p'}a_2^{-1/p}\left(\int_{a_1}^{\infty}\int_{0}^{a_2} x_1^{q(\alpha_1-1)}x_2^{q\alpha_2}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty. \tag{2.16}$$

3. Potentials on $\mathbb{R}_+$

In this section we discuss the two-weight problem for the operator I_α. We begin with the following lemma.

Lemma 3.1. *The following relation holds for nonnegative and nonincreasing function f:*

$$(R_\alpha f)(x) \approx x^\alpha Hf(x), \tag{3.1}$$

where H is the Hardy operator defined above.

Proof. We follow the proof of Proposition 3.1 of [10]. We have

$$(R_\alpha f)(x) = \int_0^{x/2}\frac{f(t)}{(x-t)^{1-\alpha}}dt + \int_{x/2}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt := J_1(x) + J_2(x). \tag{3.2}$$

Observe that if $0 < t < x/2$, then $(x-t)^{\alpha-1} \le 2^{1-\alpha}x^{\alpha-1}$. Hence,

$$J_1(x) \le 2^{1-\alpha}x^{\alpha-1}\int_0^x f(t)dt = 2^{1-\alpha}x^\alpha(Hf)(x). \tag{3.3}$$

Further, since f is nonincreasing, we have that

$$J_2(x) \le \alpha^{-1}\left(\frac{x}{2}\right)^\alpha f\left(\frac{x}{2}\right) \le c_\alpha x^\alpha(Hf)(x). \tag{3.4}$$

Finally we have the upper estimate for R_α.

The lower estimate is obvious because $(x-t)^{\alpha-1} \ge x^{\alpha-1}$ for $t \le x$. □

In the next statement we assume that W_α is the operator given by

$$(W_\alpha f)(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt, \qquad \alpha > 0. \tag{3.5}$$

Lemma 3.2. *Let $1 < p \le q < \infty$, and let $\alpha > 0$. Suppose that $W(\infty) = \infty$. Then the operator W_α is bounded from $L^p_{dec}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\left(\int_0^\infty \left(\int_0^x \frac{g(t)}{(x-t)^{-\alpha}} dt \right)^{p'} W^{-p'}(x) w(x) dx \right)^{1/p'} \le c \left(\int_0^\infty g(t)^{q'} v^{1-q'}(t) dt \right)^{1/q'}, \quad g \ge 0. \tag{3.6}$$

Proof. Taking Proposition A into account (for $n = 1$), an integral operator

$$(Tf)(x) = \int_0^\infty k(x,y) f(y) dy \tag{3.7}$$

is bounded from $L^p_{\mathrm{dec}}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if

$$\left(\int_0^\infty \left(\int_0^x (T^* f)(\tau) d\tau \right)^{p'} W^{-p'}(x) w(x) dx \right)^{1/p'} \le c \left(\int_0^\infty f(t)^{q'} v^{1-q'}(t) dt \right)^{1/q'}, \quad f \ge 0, \tag{3.8}$$

where T^* is a formal adjoint to T.

We have

$$\int_0^x (R_\alpha f)(t) dt = \int_0^x \left(\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \right) dt = \int_0^x f(\tau) \left(\int_0^{x-\tau} \frac{du}{u^{1-\alpha}} \right) d\tau = \frac{1}{\alpha} \int_0^x f(\tau)(x-\tau)^\alpha d\tau. \tag{3.9}$$

Taking $T = W_\alpha$ and $T^* = R_\alpha$, we derive the desired result. □

Now we formulate the main results of this section.

Theorem 3.3. *Let $1 < p \le q < \infty$, and let $0 < \alpha < 1$. Suppose that $W(\infty) = \infty$. Then I_α is bounded from $L^p_{dec}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\sup_{a>0} A_1(a, v, w) := \sup_{a>0} \left(\int_0^a w(t) dt \right)^{-1/p} \left(\int_0^a t^{\alpha q} v(t) dt \right)^{1/q} < \infty, \tag{3.10}$$

$$\sup_{a>0} A_2(a, v, w) := \sup_{a>0} \left(\int_0^a t^{p'} W^{-p'}(t) w(t) dt \right)^{1/p'} \left(\int_a^\infty t^{(\alpha-1)q} v(t) dt \right)^{1/q} < \infty, \tag{3.11}$$

$$\sup_{a>0} A_3(a, v, w) := \sup_{a>0} \left(\int_a^\infty W^{-p'}(x) w(x) (x-a)^{\alpha p'} \, dx \right)^{1/p'} \left(\int_0^a v(x) dx \right)^{1/q} < \infty, \tag{3.12}$$

$$\sup_{a>0} A_4(a, v, w) := \sup_{a>0} \left(\int_0^a w(x) dx \right)^{-1/p} \left(\int_0^a v(x) (a-x)^{\alpha q} dx \right)^{1/q} < \infty. \tag{3.13}$$

Theorem 3.4. *Let $1 < q < p < \infty$, and let $0 < \alpha < 1$. $W(\infty) = \infty$. Then I_α is bounded from $L^p_{dec}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\begin{gathered}
\left[\int_{\mathbb{R}_+} \left[\left(\int_0^t x^{\alpha q} v(x) dx \right)^{1/p} W^{-1/p}(t) \right]^r t^{\alpha q} v(t) dt \right]^{1/r} < \infty, \\
\left[\int_{\mathbb{R}_+} \left[\left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{1/p} \left(\int_0^t \frac{W^{-p'}(x) w(x)}{x^{-p'}} \right)^{1/p'} \right]^r t^{p'} W^{-p'}(t) w(t) dt \right]^{1/r} < \infty, \\
\left[\int_{\mathbb{R}_+} \left[\left(\int_t^\infty \frac{W^{-p'}(x) w(x)}{(x-t)^{-\alpha p'}} \right)^{1/p'} \left(\int_0^t v(x) dx \right)^{1/p} \right]^r v(t) dt \right]^{1/r} < \infty, \\
\left[\int_{\mathbb{R}_+} \left[W^{-1}(t) \int_0^t \frac{v(x)}{(t-x)^{-\alpha q}} dx \right]^{r/q} w(t) dt \right]^{1/r} < \infty,
\end{gathered} \tag{3.14}$$

where $1/r = 1/q - 1/p$.

Proof of Theorems 3.3 and 3.4. By using the representation

$$(I_\alpha f)(x) = (R_\alpha f)(x) + (W_\alpha f)(x), \ \ x > 0, \tag{3.15}$$

the obvious equality

$$\int_t^\infty W^{-p'}(x) w(x) dx = c_p W^{1-p'}(t). \tag{3.16}$$

Theorems A and B and Lemmas 3.1 and 3.2, we have the desired results. □

Corollary 3.5. *Let $1 < p \le q < \infty$, and let $0 < \alpha < 1/p$. Then the operator I_α is bounded from $L^p_{dec}(1, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$B := \sup_{a>0} a^{(\alpha-1/p)} \left(\int_0^a v(t)\, dt \right)^{1/q} < \infty. \tag{3.17}$$

Proof. Necessity follows immediately taking the test function $f_a(x) = \chi_{(0,a)}(x)$ in the two-weight inequality

$$\left(\int_0^\infty v(x) \left(I_\alpha f(x) \right)^q dx \right)^{1/q} \le c \left(\int_0^\infty \left(f(x) \right)^p dx \right)^{1/p} \tag{3.18}$$

and observing that $I_\alpha f_a(x) \ge \int_0^a (dt/|x-t|^{1-\alpha}) \ge a^\alpha$ for $x \in (0,a)$.

Sufficiency. By Theorem 3.3, it is enough to show that

$$\max\{A_1, A_2, A_3, A_4\} \le cB, \tag{3.19}$$

where $A_i := \sup_{a>0} A_i(a, v, 1)$, $i = 1,2,3,4$ (see Theorem 3.3 for the definition of $A_i(a, v, w)$).

The estimates $A_i \le cB$, $i = 1,4$, are obvious. We show that $A_i \le cB$ for $i = 2,3$. We have

$$\begin{aligned}
A_2^q(a, v, 1) &= a^{q/p'} \sum_{k=0}^{\infty} \int_{2^k a}^{2^{k+1} a} t^{(\alpha-1)q} v(t)\, dt \\
&\le a^{q/p'} \sum_{k=0}^{\infty} \left(2^k a \right)^{(\alpha-1)q} \left(\int_{2^k a}^{2^{k+1} a} v(t)\, dt \right) \\
&\le cB^q a^{q/p'} \sum_{k=0}^{\infty} \left(2^k a \right)^{(\alpha-1)q} \left(2^{k+1} a \right)^{(1/p-\alpha)q} \\
&= cB^q a^{q/p'} \left(\sum_{k=0}^{\infty} 2^{-kq/p'} \right) a^{-q/p'} \le cB^q.
\end{aligned} \tag{3.20}$$

Further, by the condition $0 < \alpha < 1/p$, we have that

$$A_3^q(a, v, 1) \le \left(\int_a^\infty x^{(\alpha-1)p'} dx \right)^{1/p'} \left(\int_0^a v(t)\, dt \right)^{1/q} = c_{\alpha,p} a^{\alpha-1/p} \left(\int_0^a v(t)\, dt \right)^{1/q} \le cB. \tag{3.21}$$

□

Definition 3.6. Let ρ be a locally integrable a.e. positive function on $\mathbb{R}_+$. We say that ρ satisfies the doubling condition ($\rho \in \mathrm{DC}(\mathbb{R}_+)$) if there is a positive constant $b > 1$ such that for all $t > 0$ the following inequality holds:

$$\int_0^{2t} \rho(x)\, dx \le b \min\left\{ \int_0^t \rho(x)\, dx, \int_t^{2t} \rho(x)\, dx \right\}. \tag{3.22}$$

Remark 3.7. It is easy to check that if $\rho \in \mathrm{DC}(\mathbb{R}_+)$, then ρ satisfies the reverse doubling condition: there is a positive constant $b_1 > 1$ such that

$$\int_0^{2t} \rho(x)dx \geq b_1 \max\left\{\int_0^t \rho(x)dx, \int_t^{2t} \rho(x)dx\right\}. \tag{3.23}$$

Indeed by (3.22) we have

$$\int_0^{2t} \rho(x)dx \geq \frac{1}{b}\int_0^{2t} \rho(x)dx + \int_t^{2t} \rho(x)dx. \tag{3.24}$$

Then

$$\int_0^{2t} \rho(x)dx \geq \frac{b}{b-1}\int_t^{2t} \rho(x)dx. \tag{3.25}$$

Analogously,

$$\int_0^{2t} \rho(x)dx \geq \frac{b}{b-1}\int_0^{t} \rho(x)dx. \tag{3.26}$$

Finally, we have (3.23).

Corollary 3.8. *Let $1 < p \leq q < \infty$, and let $0 < \alpha < 1$. Suppose that $W(\infty) = \infty$. Suppose also that $v \in DC(\mathbb{R}_+)$. Then I_α is bounded from $L^p_{dec}(w, \mathbb{R}_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if condition* (3.11) *is satisfied.*

Proof. Observe that by Remark 3.7, for $m_0 \in \mathbb{Z}$, the inequality

$$\int_0^{2^{m_0}} v(x)dx \leq b_1^{m_0-k}\int_0^{2^k} v(x)dx \tag{3.27}$$

holds for all $k > m_0$, where b_1 is defined in (3.23).

Let $a > 0$. Then there is $m_0 \in \mathbb{Z}$ such that $a \in [2^{m_0}, 2^{m_0+1})$. By applying (3.27) and the doubling condition for v, we find that

$$\begin{aligned}
&\left(\int_0^a w(t)dt\right)^{-p'/p}\left(\int_0^a t^{\alpha q}v(t)dt\right)^{p'/q} \\
&\quad = c\left(\int_a^\infty W^{-p'}(t)w(t)dt\right)\left(\int_0^a t^{\alpha q}v(t)dt\right)^{p'/q} \\
&\quad \leq c\left(\int_{2^{m_0}}^\infty W^{-p'}(t)w(t)dt\right)\left(\int_0^{2^{m_0+1}} t^{\alpha q}v(t)dt\right)^{p'/q} \\
&\quad \leq c\sum_{k=m_0}^{\infty}\left(\int_{2^k}^{2^{k+1}} W^{-p'}(t)w(t)dt\right)\left(\int_0^{2^{m_0+1}} v(t)dt\right)^{p'/q} 2^{m_0\alpha p'}
\end{aligned}$$

$$\begin{aligned}
&\leq c\sum_{k=m_0}^{\infty}\left(\int_{2^k}^{2^{k+1}} W^{-p'}(t)w(t)dt\right)b_1^{m_0-k-1}\left(\int_0^{2^{k+2}} v(t)dt\right)^{p'/q}2^{m_0\alpha p'}\\
&\leq c\sum_{k=m_0}^{\infty}b_1^{m_0-k-1}\left(\int_{2^k}^{2^{k+1}} W^{-p'}(t)w(t)dt\right)\left(\int_{2^{k+1}}^{2^{k+2}} v(t)dt\right)^{p'/q}2^{k(\alpha-1)p'}2^{kp'}\\
&\leq c\sum_{k=m_0}^{\infty}b_1^{m_0-k-1}\left(\int_{2^k}^{2^{k+1}} t^{p'}W^{-p'}(t)w(t)dt\right)\left(\int_{2^{k+1}}^{2^{k+2}} v(t)t^{(\alpha-1)q}dt\right)^{p'/q}\\
&\leq c\left(\sup_{a>0}A_2(a,v,w)\right)^{p'}\sum_{k=m_0}^{\infty}b_1^{m_0-k-1}\leq c\left(\sup_{a>0}A_2(a,v,w)\right)^{p'}.
\end{aligned}\tag{3.28}$$

So, we have seen that (3.11)⇒(3.10). Let us check now that (3.13)⇒(3.12).

Indeed, for $a>0$, we choose m_0 so that $a\in[2^{m_0},2^{m_0+1})$. Then, by using the condition $v\in DC(\mathbb{R}_+)$ and Remark 3.7,

$$\begin{aligned}
&\left(\int_a^{\infty} W^{-p'}(x)w(x)(x-a)^{\alpha p'}dx\right)\left(\int_0^a v(x)dx\right)^{p'/q}\\
&\quad\leq\left(\int_{2^{m_0}}^{\infty} W^{-p'}(x)w(x)x^{\alpha p'}dx\right)\left(\int_0^{2^{m_0+1}} v(x)dx\right)^{p'/q}\\
&\quad\leq c\sum_{k=m_0}^{\infty}2^{k\alpha p'}\left(\int_{2^k}^{2^{k+1}} W^{-p'}(x)w(x)dx\right)\left(\int_0^{2^{m_0+1}} v(x)dx\right)^{p'/q}\\
&\quad\leq c\sum_{k=m_0}^{\infty}2^{k\alpha p'}\left(\int_{2^k}^{2^{k+1}} W^{-p'}(x)w(x)dx\right)b_1^{m_0-k+2}\left(\int_0^{2^{k-1}} v(x)dx\right)^{p'/q}\\
&\quad\leq c\sum_{k=m_0}^{\infty}b_1^{m_0-k+2}\left(\int_{2^k}^{\infty} W^{-p'}(x)w(x)dx\right)\left(\int_0^{2^k} v(x)\left(2^k-x\right)^{\alpha q}dx\right)^{p'/q}\\
&\quad\leq c\left(\sup_{a>0}A_4(a,v,w)\right)^{p'}\sum_{k=m_0}^{\infty}b_1^{m_0-k+2}\leq c\left(\sup_{a>0}A_4(a,v,w)\right)^{p'}.
\end{aligned}\tag{3.29}$$

Hence, (3.13)⇒(3.12) follows. Implication (3.11)⇒(3.13) follows in the same way as in the case of implication (3.11)⇒(3.10). The details are omitted. □

4. Potentials with Multiple Kernels

In this section we discuss two-weight criteria for the potentials with product kernels $\mathcal{I}_{\alpha_1,\alpha_2}$.

To derive the main results, we introduce the following multiple potential operators:

$$\begin{aligned}\mathcal{W}_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{x_1}^{\infty}\int_{x_2}^{\infty} \frac{f(t_1,t_2)dt_1dt_2}{(t_1-x_1)^{1-\alpha_1}(t_2-x_2)^{1-\alpha_2}},\\ (\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{0}^{x_1}\int_{x_2}^{\infty} \frac{f(t_1,t_2)dt_1dt_2}{(x_1-t_1)^{1-\alpha_1}(t_2-x_2)^{1-\alpha_2}},\\ (\mathcal{W}\mathcal{R})_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{x_1}^{\infty}\int_{0}^{x_2} \frac{f(t_1,t_2)dt_1dt_2}{(t_1-x_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}},\end{aligned} \tag{4.1}$$

where $x_1, x_2 \in \mathbb{R}_+$, $f \geq 0$, and $0 < \alpha_i < 1$, $i = 1, 2$.

Definition 4.1. One says that a locally integrable a.e. positive function ρ on $\mathbb{R}^2_+$ satisfies the doubling condition with respect to the second variable ($\rho \in \mathrm{DC}(y)$) if there is a positive constant c such that for all $t > 0$ and almost every $x > 0$ the following inequality holds:

$$\int_0^{2t} \rho(x,y)dy \leq c \min\left\{\int_0^t \rho(x,y)dy, \int_t^{2t} \rho(x,y)dy\right\}. \tag{4.2}$$

Analogously is defined the class of weights $\mathrm{DC}(x)$.

Remark 4.2. If $\rho \in \mathrm{DC}(y)$, then ρ satisfies the reverse doubling condition with respect to the second variable; that is, there is a positive constant c_1 such that

$$\int_0^{2t} \rho(x,y)dy \geq c_1 \max\left\{\int_0^t \rho(x,y)dy, \int_t^{2t} \rho(x,y)dy\right\}. \tag{4.3}$$

Analogously, $\rho \in \mathrm{DC}(x) \Rightarrow \rho \in \mathrm{RDC}(x)$. This follows in the same way as the single variable case (see Remark 3.7).

Theorem C implies the next statement.

Corollary B. *Let the conditions of Theorem C be satisfied.*

(i) *If $v \in DC(x)$, then for the boundedness of $\mathcal{R}_{\alpha_1,\alpha_2}$ from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$, it is necessary and sufficient that conditions* (2.10) *and* (2.12) *are satisfied.*

(ii) *If $v \in DC(y)$, then $\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only If conditions* (2.10) *and* (2.11) *are satisfied.*

(iii) *If $v \in DC(x) \cap DC(y)$, then $\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if the condition* (2.10) *is satisfied.*

Proof of Corollary B. The proof of this statement follows by using the arguments of the proof of Corollary 3.8 (see Section 2) but with respect to each variable separately (also see Remark 4.2). The details are omitted. □

The following result concerns with the two-weight criteria for the two-dimensional operator $\mathcal{R}_{\alpha_1,\alpha_2}$ with $\alpha_1, \alpha_2 > 1$ (see [25], [13, Section 1.6]).

Theorem D. *Let $1 < p \le q < \infty$, and let $\alpha_1, \alpha_2 \ge 1$.*

(i) *Suppose that $w^{1-p'} \in DC(y)$. Then the operator $\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if*

$$P_1 := \sup_{a,b>0}\left(\int_0^a\int_0^b \frac{w^{1-p'}(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q} < \infty,$$

$$P_2 := \sup_{a,b>0}\left(\int_0^a\int_0^b w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{(x_1-a)^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q} < \infty. \tag{4.4}$$

Moreover, $\|\mathcal{R}_{\alpha_1,\alpha_2}\| \approx \max\{P_1, P_2\}$.

(ii) *Let $w^{1-p'} \in DC(x)$. Then the operator $\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\widetilde{P}_1 := \sup_{a,b>0}\left(\int_0^a\int_0^b \frac{w^{1-p'}(x_1,x_2)}{(b-x_2)^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}}dx_1dx_2\right)^{1/q} < \infty,$$

$$\widetilde{P}_2 := \sup_{a,b>0}\left(\int_0^a\int_0^b w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{(x_2-b)^{(1-\alpha_2)q}x_1^{(1-\alpha_1)q}}dx_1dx_2\right)^{1/q} < \infty. \tag{4.5}$$

Moreover, $\|\mathcal{R}_{\alpha_1,\alpha_2}\| \approx \max\{\widetilde{P}_1, \widetilde{P}_2\}$.

Let us introduce the following multiple integral operators:

$$
\begin{aligned}
(\mathcal{H}\mathcal{R})_{\alpha_1,\alpha_2} f(x_1,x_2) &= x_1^{\alpha_1-1}\int_0^{x_1}\int_0^{x_2}\frac{f(t_1,t_2)dt_1dt_2}{(x_2-t_2)^{1-\alpha_2}},\\
(\mathcal{R}\mathcal{H})_{\alpha_1,\alpha_2} f(x_1,x_2) &= x_2^{\alpha_2-1}\int_0^{x_1}\int_0^{x_2}\frac{f(t_1,t_2)dt_1dt_2}{(x_1-t_1)^{1-\alpha_1}},\\
(\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2} f(x_1,x_2) &= x_1^{\alpha_1-1}\int_0^{x_1}\int_{x_2}^{\infty}\frac{f(t_1,t_2)dt_1dt_2}{(t_2-x_2)^{1-\alpha_2}},\\
(\mathcal{W}\mathcal{H})_{\alpha_1,\alpha_2} f(x_1,x_2) &= x_2^{\alpha_2-1}\int_{x_1}^{\infty}\int_0^{x_2}\frac{f(t_1,t_2)dt_1dt_2}{(t_1-x_1)^{1-\alpha_1}},\\
(\mathcal{H}'\mathcal{R})_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{x_1}^{\infty}\int_0^{x_2}\frac{f(t_1,t_2)dt_1dt_2}{t_1^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}},\\
(\mathcal{R}\mathcal{H}')_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_0^{x_1}\int_{x_2}^{\infty}\frac{f(t_1,t_2)dt_1dt_2}{(x_1-t_1)^{1-\alpha_1}t_2^{1-\alpha_2}},\\
(\mathcal{H}'\mathcal{W})_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{x_1}^{\infty}\int_{x_2}^{\infty}\frac{f(t_1,t_2)dt_1dt_2}{t_1^{1-\alpha_1}(t_2-x_2)^{1-\alpha_2}},\\
(\mathcal{W}\mathcal{H}')_{\alpha_1,\alpha_2} f(x_1,x_2) &= \int_{x_1}^{\infty}\int_{x_2}^{\infty}\frac{f(t_1,t_2)dt_1dt_2}{(t_1-x_1)^{1-\alpha_1}t_2^{1-\alpha_2}}.
\end{aligned}
\tag{4.6}
$$

Now we prove some auxiliary statements.

Proposition 4.3. *Let $1 < p \le q < \infty$, and let $\alpha_1, \alpha_2 \ge 1$. Suppose that either $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ or $v(x_1,x_2) = v_1(x_1)v_2(x_2)$ for some one-dimensional weights w_1, w_2, v_1, and v_2.*

(i) *The operator $(\mathcal{R}\mathcal{H})_{\alpha_1,\alpha_2}$ is bounded from $L^p(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}_+)$ if and only if*

$$
\begin{aligned}
\widetilde{I}_1 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b\frac{w^{1-p'}(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty\frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}<\infty,\\
\widetilde{I}_2 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty\frac{v(x_1,x_2)}{(x_1-a)^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}<\infty.
\end{aligned}
\tag{4.7}
$$

Moreover, $\|(\mathcal{R}\mathcal{H})_{\alpha_1,\alpha_2}\| \approx \max\{\widetilde{I}_1,\widetilde{I}_2\}$.

(ii) *The operator* $(\mathcal{W}\mathcal{H})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v\mathbb{R}_+)$ *if and only if*

$$\widetilde{J}_1 := \sup_{a,b>0}\left(\int_0^a\int_b^\infty \frac{v(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_0^b w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty,$$
$$\widetilde{J}_2 := \sup_{a,b>0}\left(\int_0^a\int_b^\infty v(x_1,x_2)x_2^{(1-\alpha_2)q}dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_0^b \frac{w^{1-p'}(x_1,x_2)}{(x_1-a)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.8}$$

Moreover, $\|(\mathcal{W}\mathcal{H})_{\alpha_1,\alpha_2}\| \approx \max\{\widetilde{J}_1,\widetilde{J}_2\}$.

(iii) *The operator* $(\mathcal{R}\mathcal{H}')_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+)$ *if and only if*

$$\widetilde{J}_1' := \sup_{a,b>0}\left(\int_a^\infty\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_0^a\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_2^{(1-\alpha_2)p'}(a-x_1)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'} < \infty,$$
$$\widetilde{J}_2' := \sup_{a,b>0}\left(\int_a^\infty\int_0^b \frac{v(x_1,x_2)}{(x_1-a)^{(1-\alpha_1)q}}dx_1dx_2\right)^{1/q}\left(\int_0^a\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_2^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.9}$$

Moreover, $\|(\mathcal{R}\mathcal{H}')_{\alpha_1,\alpha_2}\| \approx \max\{\widetilde{J}_1',\widetilde{J}_2'\}$.

(iv) *The operator* $(\mathcal{W}\mathcal{H}')_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+)$ *if and only if*

$$\widetilde{I}_1' := \sup_{a,b>0}\left(\int_0^a\int_0^b \frac{v(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)q}}dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_2^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'} < \infty,$$
$$\widetilde{I}_2' := \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_2^{(1-\alpha_2)p'}(x_1-a)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.10}$$

Moreover, $\|(\mathcal{W}\mathcal{H}')_{\alpha_1,\alpha_2}\| \approx \max\{\widetilde{I}_1',\widetilde{I}_2'\}$.

Proof. Let $w(x_1,x_2) = w_1(x_1)w_2(x_2)$. The proof of the case $v(x_1,x_2) = v_1(x_1)v_2(x_2)$ is followed by duality arguments. We prove, for example, part (i). Proofs of other parts are similar and, therefore, are omitted. We follow the proof of Theorem 3.4 of [25] (see also the proof of Theorem 1.1.6 in [13]).

Sufficiency. First suppose that $S := \int_0^\infty w_2^{1-p'}(x_2)dx_2 = \infty$. Let $\{a_k\}_{k=-\infty}^{+\infty}$ be a sequence of positive numbers for which the equality

$$2^k = \int_0^{a_k} w_2^{1-p'}(x_2)dx_2 \tag{4.11}$$

holds for all $k \in \mathbb{Z}$. It is clear that $\{a_k\}$ is increasing and $\mathbb{R}_+ = \cup_{k\in\mathbb{Z}}[a_k, a_{k+1})$. Moreover, it is easy to verify that

$$2^k = \int_{a_k}^{a_{k+1}} w_2^{1-p'}(x_2)dx_2. \tag{4.12}$$

Let $f \geq 0$. We have that

$$\begin{aligned}
&\|(\mathcal{RH})_{\alpha_1,\alpha_2} f\|^q_{L^q(v,\mathbb{R}^2_+)} \\
&\quad = \int_{\mathbb{R}^2_+} v(x_1,x_2)\left((\mathcal{RH})_{\alpha_1,\alpha_2} f\right)^q(x_1,x_2)dx_1dx_2 \\
&\quad \leq \sum_{k\in\mathbb{Z}} \int_0^\infty \int_{a_k}^{a_{k+1}} \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}} \left(\int_0^{x_1}\int_0^{x_2} \frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}}dt_1dt_2\right)^q dx_1dx_2 \\
&\quad \leq \sum_{k\in\mathbb{Z}} \int_0^\infty \left(\int_{a_k}^{a_{k+1}} \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}}dx_2\right)\left(\int_0^{x_1}(x_1-t_1)^{\alpha_1-1}\left(\int_0^{a_{k+1}} f(t_1,t_2)dt_2\right)dt_1\right)^q dx_1 \\
&\quad = \sum_{k\in\mathbb{Z}} \int_0^\infty V_k(x_1)\left(\int_0^{x_1}(x_1-t_1)^{(\alpha_1-1)}F_k(t_1)dt_1\right)^q dx_1,
\end{aligned} \tag{4.13}$$

where

$$V_k(x_1) := \int_{a_k}^{a_{k+1}} \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}}dx_2, \qquad F_k(t_1) := \int_0^{a_{k+1}} f(t_1,t_2)dt_2. \tag{4.14}$$

It is obvious that

$$\begin{aligned}
\widetilde{I}_1^q &\geq \sup_{\substack{a>0\\ j\in\mathbb{Z}}} \left(\int_a^\infty \int_{a_j}^{a_{j+1}} \frac{v(x_1,x_2)}{(x_1-a)^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\right)\left(\int_0^a\int_0^{a_j} \frac{w^{1-p'}(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{q/p'}, \\
\widetilde{I}_2^q &\geq \sup_{\substack{a>0\\ j\in\mathbb{Z}}} \left(\int_a^\infty \int_{a_j}^{a_{j+1}} \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}}dx_1dx_2\right)\left(\int_0^a\int_0^{a_j} \frac{w^{1-p'}(x_1,x_2)}{(a-x_1)^{(1-\alpha_1)p'}}dx_1dx_2\right)^{q/p'}.
\end{aligned} \tag{4.15}$$

Hence, by using the two-weight criteria for the one-dimensional Riemann-Liouville operator without singularity (see [24]), we find that

$$\begin{aligned}
&\left\|(\mathcal{R}\mathcal{H})_{\alpha_1,\alpha_2} f\right\|^q_{L^q(v,\mathbb{R}^2_+)} \\
&\quad \le c\widetilde{I}^q \sum_{j\in\mathbb{Z}} \left[\int_0^\infty w_1(x_1)\left(\int_0^{a_j} w_2^{1-p'}(x_2)dx_2\right)^{1-p} (F_j(x_1))^p dx_1\right]^{q/p} \\
&\quad \le c\widetilde{I}^q \left[\int_0^\infty w_1(x_1)\sum_{j\in Z}\left(\int_0^{a_j} w_2^{1-p'}(x_2)dx_2\right)^{1-p}\left(\sum_{k=-\infty}^{j}\int_{a_k}^{a_{k+1}} f(x_1,t_2)dt_2\right)^p dx_1\right]^{q/p},
\end{aligned} \tag{4.16}$$

where $\widetilde{I} = \max\{\widetilde{I}_1, \widetilde{I}_2\}$.

On the other hand, (4.11) yields

$$\begin{aligned}
&\sum_{k=n}^{+\infty}\left(\int_0^{a_k} w_2^{1-p'}(x_2)dx_2\right)^{1-p}\left(\sum_{k=-\infty}^{n}\int_{a_k}^{a_{k+1}} w_2^{1-p'}(x_2)dx_2\right)^{p-1} \\
&\quad = \sum_{k=n}^{+\infty}\left(\int_0^{a_k} w_2^{1-p'}(x_2)dx_2\right)^{1-p}\left(\int_0^{a_{n+1}} w_2^{1-p'}(x_2)dx_2\right)^{p-1} = \left(\sum_{k=n}^{+\infty}2^{k(1-p)}\right)2^{(n+1)(p-1)} \le c
\end{aligned} \tag{4.17}$$

for all $n \in \mathbb{Z}$. Hence by Hardy's inequality in discrete case (see, for example, [25, 26]) and Hölder's inequality we have that

$$\begin{aligned}
&\left\|(\mathcal{R}\mathcal{H})_{\alpha_1,\alpha_2} f\right\|^q_{L^q(v,\mathbb{R}^2_+)} \\
&\quad \le c\widetilde{I}^q \left[\int_0^\infty w_1(x_1)\sum_{j\in\mathbb{Z}}\left(\int_{a_j}^{a_{j+1}} w_2^{1-p'}(x_2)dx_2\right)^{1-p}\left(\int_{a_j}^{a_{j+1}} f(x_1,t_2)dt_2\right)^p dx_1\right]^{q/p} \\
&\quad \le c\widetilde{I}^q \left[\int_0^\infty w_1(x_1)\sum_{j\in\mathbb{Z}}\left(\int_{a_j}^{a_{j+1}} w_2(t_2)f^p(x_1,t_2)dt_2\right)dx_1\right]^{q/p} = c\widetilde{I}^q\|f\|^q_{L^p(w,\mathbb{R}^2_+)}.
\end{aligned} \tag{4.18}$$

If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{a_k\}_{k=-\infty}^{0}$ for which (4.11) holds for all $k \in \mathbb{Z}_-$. Arguing as in the case of $S = \infty$, we finally obtain the desired result.

Necessity follows by choosing the appropriate test functions. The details are omitted.

To prove, for example, (iii), we choose the sequence $\{x_k\}$ so that $\int_{x_k}^\infty w_2^{1-p'}(x)dx = 2^k$ (notice that x_k is decreasing) and argue as in the proof of (i). □

Proposition 4.4. *Let $1 < p \le q < \infty$, and let $\alpha_1, \alpha_2 \ge 1$. Suppose that either $w(x_1,x_2) = w_1(x_1)$ $w_2(x_2)$ or $v(x_1,x_2) = v_1(x_1)v_2(x_2)$ for some one-dimensional weights: w_1, w_2, v_1, and v_2.*

(i) *The operator* $(\mathcal{H}\mathcal{R})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+^2)$ *if and only if*

$$I_1 := \sup_{a,b>0}\left(\int_0^a\int_0^b \frac{w^{1-p'}(x_1,x_2)}{(b-x_2)^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}}dx_1dx_2\right)^{1/q} < \infty,$$

$$I_2 := \sup_{a,b>0}\left(\int_0^a\int_0^b w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/p'}\left(\int_a^\infty\int_b^\infty \frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}(x_2-b)^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q} < \infty. \tag{4.19}$$

Moreover, $\|(\mathcal{H}\mathcal{R})_{\alpha_1,\alpha_2}\| \approx \max\{I_1, I_2\}$.

(ii) *The operator* $(\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+)$ *if and only if*

$$J_1 := \sup_{a,b>0}\left(\int_a^\infty\int_0^b \frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}(b-x_2)^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}\left(\int_0^a\int_b^\infty w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{1/p'} < \infty,$$

$$J_2 := \sup_{a,b>0}\left(\int_a^\infty\int_0^b v(x_1,x_2)x_1^{(\alpha_1-1)q}dx_1dx_2\right)^{1/q}\left(\int_0^a\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{(x_2-b)^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.20}$$

Moreover, $\|(\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2}\| \approx \max\{J_1, J_2\}$.

(iii) *The operator* $(\mathcal{H}'\mathcal{R})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+)$ *if and only if*

$$J_1' := \sup_{a,b>0}\left(\int_0^a\int_b^\infty v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_0^b \frac{w^{1-p'}(x_1,x_2)}{x_1^{(1-\alpha_1)p'}(b-x_2)^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'} < \infty,$$

$$J_2' := \sup_{a,b>0}\left(\int_0^a\int_b^\infty \frac{v(x_1,x_2)}{(x_2-b)^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_0^b \frac{w^{1-p'}(x_1,x_2)}{x_1^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.21}$$

Moreover, $\|(\mathcal{H}'\mathcal{R})_{\alpha_1,\alpha_2}\| \approx \max\{J_1', J_2'\}$.

(iv) *The operator* $(\mathcal{H}'\mathcal{W})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p(w,\mathbb{R}_+^2)$ *to* $L^q(v,\mathbb{R}_+)$ *if and only if*

$$I_1' := \sup_{a,b>0}\left(\int_0^a\int_0^b \frac{v(x_1,x_2)}{(b-x_2)^{(1-\alpha_2)q}}dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_1^{(1-\alpha_1)p'}}dx_1dx_2\right)^{1/p'} < \infty,$$

$$I_2' := \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_b^\infty \frac{w^{1-p'}(x_1,x_2)}{x_1^{(1-\alpha_1)p'}(x_2-b)^{(1-\alpha_2)p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.22}$$

Moreover, $\|(\mathcal{H}'\mathcal{W})_{\alpha_1,\alpha_2}\| \approx \max\{I_1', I_2'\}$.

Proof of this proposition is similar to Proposition 4.3 by changing the order of variables.

Theorem 4.5. *Let $1 < p \le q < \infty$, and let $0 < \alpha_1, \alpha_2 \le 1$. Suppose that the weight function w on $\mathbb{R}_+^2$ is of product type, that is, $w(x_1, x_2) = w_1(x_1)w_2(x_2)$. Suppose also that $W_1(\infty) = W_2(\infty) = \infty$.*

(i) *If $v \in DC(y)$, then $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}_+^2)$ to $L^q(v, \mathbb{R}_+^2)$ if and only if*

$$\begin{aligned} A_1 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)(a-x_1)^{\alpha_1 q}dx_1dx_2\right)^{1/q} \\ &\quad\times\left(\int_0^a w_1(x_1)dx_1\right)^{-1/p}\left(\int_b^\infty W_2^{-p'}(x_2)w_2(x_2)x_2^{\alpha_2 p'}dx_2\right)^{1/p'} < \infty, \end{aligned} \tag{4.23}$$

$$\begin{aligned} A_2 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q} \\ &\quad\times\left(\int_a^\infty\int_b^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)(x_1-a)^{\alpha_1 p'}x_2^{\alpha_2 p'}dx_1dx_2\right)^{1/p'} < \infty. \end{aligned} \tag{4.24}$$

(ii) *If $v \in DC(x)$, then $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}_+^2)$ to $L^q(v, \mathbb{R}_+^2)$ if and only if*

$$\begin{aligned} B_1 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)(b-x_2)^{\alpha_2 q}dx_1dx_2\right)^{1/q} \\ &\quad\times\left(\int_a^\infty W_1^{-p'}(x_1)w_1(x_1)x_1^{\alpha_1 p'}dx_1\right)^{1/p'}\left(\int_0^b w_2(x_2)dx_2\right)^{-1/p} < \infty, \\ B_2 &:= \sup_{a,b>0}\left(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q} \\ &\quad\times\left(\int_a^\infty\int_b^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)(x_2-b)^{\alpha_2 p'}x_1^{\alpha_1 p'}dx_1dx_2\right)^{1/p'} < \infty. \end{aligned} \tag{4.25}$$

(iii) *If $v \in DC(x) \cap DC(y)$, then $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}_+^2)$ to $L^q(v, \mathbb{R}_+^2)$ if and only if*

$$\begin{aligned} C_1 &:= \sup_{a,b>0}\left(\int_a^\infty\int_b^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)x_2^{\alpha_2 p'}x_1^{\alpha_1 p'}dx_1dx_2\right)^{1/p'} \\ &\quad\times\left(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty. \end{aligned} \tag{4.26}$$

Proof. By using Proposition A we see that the operator $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{\text{dec}}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if the inequality

$$\begin{aligned}\Bigg(\int_{\mathbb{R}^2_+}\Bigg(\int_0^{x_1}\int_0^{x_2}\left[\int_0^{\tau_1}\int_0^{\tau_2}\frac{g(t_1,t_2)dt_1dt_2}{(\tau_1-t_1)^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}\right]d\tau_1d\tau_2\Bigg)^{p'}\\ \times W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\Bigg)^{1/p'}\le c\left(\int_{\mathbb{R}^2_+}g^{q'}v^{1-q'}\right)^{1/q'}\end{aligned}\tag{4.27}$$

holds for all $g\ge 0$. Further, it is easy to see that

$$\begin{aligned}&\int_0^{x_1}\int_0^{x_2}\left[\int_0^{\tau_1}\int_0^{\tau_2}\frac{g(t_1,t_2)dt_1dt_2}{(\tau_1-t_1)^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}\right]d\tau_1d\tau_2\\ &\quad=\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)\left[\int_{t_1}^{x_1}\int_{t_2}^{x_2}\frac{d\tau_1d\tau_2}{(\tau_1-t_1)^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}\right]dt_1dt_2\\ &\quad=c_{\alpha_1,\alpha_2}\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)(x_1-t_1)^{\alpha_1}(x_2-t_2)^{\alpha_2}dt_1dt_2.\end{aligned}\tag{4.28}$$

Hence $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{\text{dec}}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if $\mathcal{R}_{\alpha_1+1,\alpha_2+1}$ is bounded from $L^{q'}(v^{1-q'},\mathbb{R}^2_+)$ to $L^{p'}(W^{-p'}w,\mathbb{R}^2_+)$.

By using Theorem D, (i) and (ii) follow immediately.

To prove (iii) we show that if $v\in DC(x)\cap DC(y)$, then (4.26) implies (4.23) and (4.24). Let $a,b>0$. Then $a\in[2^{m_0},2^{m_0+1})$ for some $m_0\in\mathbb{Z}$. By using the doubling condition with respect to the first variable uniformly to the second one and Remark 4.2, we see that

$$\begin{aligned}&\left(\int_0^a\int_0^b v(x_1,x_2)(a-x_1)^{\alpha_1 q}dx_1dx_2\right)^{p'/q}\left(\int_0^a w_1(x_1)dx_1\right)^{-p'/p}\\ &\quad=c\left(\int_0^a\int_0^b v(x_1,x_2)(a-x_1)^{\alpha_1 q}dx_1dx_2\right)^{p'/q}\left(\int_a^\infty W_1^{-p'}(x_1)w_1(x_1)dx_1\right)\\ &\quad\le c\sum_{k=m_0}^{\infty}\left(\int_{2^k}^{2^{k+1}}W_1^{-p'}(x_1)w_1(x_1)dx_1\right)2^{(m_0+1)\alpha_1 p'}\left(\int_0^{2^{m_0}}\int_0^b v(x_1,x_2)dx_1dx_2\right)^{p'/q}\\ &\quad\le c\sum_{k=m_0}^{\infty}\left(\int_{2^k}^{2^{k+1}}x_1^{\alpha_1 p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\right)c_1^{(m_0-k)(p'/q)}\left(\int_0^{2^k}\int_0^b v(x_1,x_2)dx_1dx_2\right)^{p'/q}\\ &\quad\le cC_1^{p'}\left(\int_b^\infty W_2^{-p'}(x_2)w_2(x_2)x_2^{\alpha_2 p'}dx_2\right)^{-1}.\end{aligned}\tag{4.29}$$

Hence, $A_1\le C_1$. In a similar manner we can show that $A_2\le C_1$.

For necessity, let us see, for example, that (4.23) implies (4.26). For $a \in [2^{m_0}, 2^{m_0+1})$, by using the doubling condition for v with respect to the first variable and Remark 4.2, we have

$$\begin{aligned}
&\left(\int_0^a \int_0^b v(x_1,x_2)dx_1dx_2\right)^{p'/q}\left(\int_a^\infty W_1^{-p'}(x_1)w(x_1)x_1^{\alpha_1 p'}dx_1\right)\\
&\quad\le c\sum_{k=m_0}^\infty \left(\int_{2^k}^{2^{k+1}} W_1^{-p'}(x_1)w(x_1)dx_1\right)2^{k\alpha_1 p'}\left(\int_0^{2^{m_0+1}}\int_0^b v(x_1,x_2)dx_1dx_2\right)^{p'/q}\\
&\quad\le c\sum_{k=m_0}^\infty \left(\int_{2^k}^{2^{k+1}} W_1^{-p'}(x_1)w(x_1)dx_1\right)c_1^{(m_0-k+2)(p'/q)}\left(\int_0^{2^{k-1}}\int_0^b \left(2^k-x_1\right)^{\alpha_1 q} v(x_1,x_2)dx_1dx_2\right)^{p'/q}\\
&\quad\le cA_1^{p'}\left(\int_b^\infty W_2^{-p'}(x_2)w_2(x_2)x_2^{\alpha_2 p'}dx_2\right)^{-1}.
\end{aligned}\tag{4.30}$$

Hence, taking the supremum with respect to a and b, we find that $C_1 \le cA_1$. □

The following statements give analogous statement for the mixed-type operator $(\mathcal{RW})_{\alpha_1,\alpha_2}$ and $(\mathcal{WR})_{\alpha_1,\alpha_2}$.

Theorem 4.6. *Let $1 < p \le q < \infty$, and let $0 < \alpha_1, \alpha_2 \le 1$. Suppose that the weight function w on $\mathbb{R}_+^2$ is of product type, that is, $w(x_1,x_2) = w_1(x_1)w_2(x_2)$. Suppose also that $W_1(\infty) = W_2(\infty) = \infty$.*

(i) *The operator $(\mathcal{RW})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}_+^2)$ to $L^q(v,\mathbb{R}_+^2)$ if and only if*

$$\sup_{a,b>0}\left(\int_0^a\int_0^b \frac{x_1^{\alpha_1 q}v(x_1,x_2)}{(b-x_2)^{-\alpha_2 q}}dx_1dx_2\right)^{1/q}\left(\int_0^a\int_0^b w_1(x_1)w_2(x_2)dx_1dx_2\right)^{-1/p} < \infty, \tag{4.31}$$

$$\begin{aligned}
&\sup_{a,b>0}\left(\int_0^a\int_0^b x_1^{\alpha_1 q}v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_0^a w_1(x_1)dx_1\right)^{-1/p}\\
&\quad\times\left(\int_b^\infty W_2^{-p'}(x_2)w_2(x_2)(x_2-b)^{\alpha_2 p'}dx_2\right)^{1/p'} < \infty,
\end{aligned}\tag{4.32}$$

$$\begin{aligned}
&\sup_{a,b>0}\left(\int_a^\infty\int_0^b \frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}(b-x_2)^{-\alpha_2 q}}dx_1dx_2\right)^{1/q}\left(\int_0^a x_1^{p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\right)^{1/p'}\\
&\quad\times\left(\int_0^b w_2(x_2)dx_2\right)^{-1/p} < \infty,
\end{aligned}\tag{4.33}$$

$$\sup_{a,b>0}\left(\int_a^\infty\int_0^b x_1^{(\alpha_1-1)q}v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_0^a\int_b^\infty \frac{W^{-p'}(x_1,x_2)w(x_1,x_2)x_1^{p'}}{(x_2-b)^{-\alpha_2 p'}}dx_1dx_2\right)^{1/p'} < \infty. \tag{4.34}$$

(ii) *The operator* $(\mathcal{W}\mathcal{R})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$ *if and only if*

$$\sup_{a,b>0}\left(\int_0^a\int_0^b \frac{x_2^{\alpha_2 q}v(x_1,x_2)}{(a-x_1)^{-\alpha_1 q}}dx_1dx_2\right)^{1/q}\left(\int_0^a\int_0^b w_1(x_1)w_2(x_2)dx_1dx_2\right)^{-1/p}<\infty, \tag{4.35}$$

$$\begin{aligned}\sup_{a,b>0}&\left(\int_0^a\int_0^b x_2^{\alpha_2 q}v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_0^b w_2(x_2)dx_2\right)^{-1/p}\\&\times\left(\int_a^\infty W_1^{-p'}(x_1)w_1(x_1)(x_1-a)^{\alpha_1 p'}dx_1\right)^{1/p'}<\infty,\end{aligned} \tag{4.36}$$

$$\begin{aligned}\sup_{a,b>0}&\left(\int_0^a\int_b^\infty \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}(a-x_1)^{-\alpha_1 q}}dx_1dx_2\right)^{1/q}\left(\int_0^a w_1(x_1)dx_1\right)^{-1/p}\\&\times\left(\int_0^b x_2^{p'}W_2^{-p'}(x_2)w_2(x_2)dx_2\right)^{1/p'}<\infty,\end{aligned} \tag{4.37}$$

$$\sup_{a,b>0}\left(\int_0^a\int_b^\infty x_2^{(\alpha_2-1)q}v(x_1,x_2)dx_1dx_2\right)^{1/q}\left(\int_a^\infty\int_0^b \frac{W^{-p'}(x_1,x_2)w(x_1,x_2)x_2^{p'}}{(x_1-a)^{-\alpha_1 p'}}dx_1dx_2\right)^{1/p'}<\infty. \tag{4.38}$$

Proof. We prove *part* (i). The proof of *part* (ii) is similar by changing the order of variables.

First we show that the two-sided pointwise relation $(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}f \approx (\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2}f$, $f\downarrow$, holds. Indeed, by using the fact that f is nonincreasing in the first variable, we find that

$$\begin{aligned}&(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}f(x_1,x_2)\\&\quad=\int_0^{x_1/2}\int_{x_2}^\infty(\cdots)+\int_{x_1/2}^{x_1}\int_{x_2}^\infty(\cdots)\\&\quad\le c'_{\alpha_1}x_1^{\alpha_1-1}\int_0^{x_1/2}\int_{x_2}^\infty\frac{f(t_1,t_2)}{(t_2-x_2)^{1-\alpha_2}}dt_1dt_2+c''_{\alpha_1}x_1^{\alpha_1-1}\int_0^{x_1/2}\int_{x_2}^\infty\frac{f(t_1,t_2)}{(t_2-x_2)^{1-\alpha_2}}dt_1dt_2\\&\quad\le c_{\alpha_1,\alpha_2}(\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2}f(x_1,x_2).\end{aligned} \tag{4.39}$$

The inequality

$$(\mathcal{H}\mathcal{W})_{\alpha_1,\alpha_2}f(x_1,x_2)\le(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}f(x_1,x_2) \tag{4.40}$$

is obvious because $x_1-t_1\le x_1$ for $0<t_1\le x_1$.

Further, it is easy to check that

$$\begin{aligned}
&\int_0^{x_1}\int_0^{x_2}\left(\int_{\tau_1}^{\infty}\int_0^{\tau_2}\frac{g(t_1,t_2)}{t_1^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}dt_1dt_2\right)d\tau_1d\tau_2\\
&\quad=\int_0^{x_1}\int_0^{x_2}\left(\int_{\tau_1}^{x_1}\int_0^{\tau_2}\frac{g(t_1,t_2)}{t_1^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}dt_1dt_2\right)d\tau_1d\tau_2\\
&\qquad+\int_0^{x_1}\int_0^{x_2}\left(\int_{x_1}^{\infty}\int_0^{\tau_2}\frac{g(t_1,t_2)}{t_1^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}dt_1dt_2\right)d\tau_1d\tau_2\\
&\quad=\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)t_1^{\alpha_1-1}\left(\int_0^{t_1}\int_{t_2}^{x_2}(\tau_2-t_2)^{\alpha_2-1}d\tau_1d\tau_2\right)dt_1dt_2\\
&\qquad+\int_{x_1}^{\infty}\int_0^{x_2}g(t_1,t_2)t_1^{\alpha_1-1}\left(\int_0^{x_1}\int_{t_2}^{x_2}(\tau_2-t_2)^{\alpha_2-1}d\tau_1d\tau_2\right)dt_1dt_2\\
&\quad=c\int_0^{x_1}\int_0^{x_2}g(t_1,t_2)t_1^{\alpha_1}(x_2-t_2)^{\alpha_2}dt_1dt_2\\
&\qquad+cx_1\int_{x_1}^{\infty}\int_0^{x_2}g(t_1,t_2)t_1^{\alpha_1-1}(x_2-t_2)^{\alpha_2}dt_1dt_2.
\end{aligned}\tag{4.41}$$

Hence, since the boundedness of $(\mathscr{H}\mathcal{W})_{\alpha_1,\alpha_2}$ from $L^p_{\mathrm{dec}}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ is equivalent to the inequality (see also [4])

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2_+}\left(\int_0^{x_1}\int_0^{x_2}\left[\int_{\tau_1}^{\infty}\int_0^{\tau_2}\frac{g(t_1,t_2)dt_1dt_2}{t_1^{1-\alpha_1}(\tau_2-t_2)^{1-\alpha_2}}\right]d\tau_1d\tau_2\right)^{p'}W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{1/p'}\\
&\quad\le c\left(\int_{\mathbb{R}^2_+}g^{q'}v^{1-q'}\right)^{1/q'},
\end{aligned}\tag{4.42}$$

we can conclude that Proposition 4.4 yields the desired result. □

Proposition 4.7. *Let the conditions of Theorem 4.6 be satisfied. Then*

(i) *if $v\in DC(x)$, then $(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if* (4.33) *and* (4.34) *hold;*

(ii) *if $v\in DC(y)$, then $(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if* (4.32) *and* (4.34) *are satisfied;*

(iii) *if $v\in DC(x)\cap DC(y)$, then $(\mathcal{R}\mathcal{W})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if* (4.34) *holds.*

Proof. (i) Taking into account the arguments used in Theorem 4.5, we can prove that (4.34) implies (4.32) and (4.33) implies (4.31).

(ii) It can be checked that (4.32) implies (4.31) and (4.34) implies (4.33). To show that, for example, (4.32) implies (4.31), we take $a, b > 0$. Then $b \in [2^{m_0}, 2^{m_0+1})$ for some integer m_0. By using the doubling condition for v with respect to the second variable, we have

$$\begin{aligned}
&\left(\int_0^a \int_0^b x_1^{\alpha_1 q} v(x_1, x_2)(b - x_2)^{\alpha_2 q} dx_1 dx_2\right)^{p'/q} \left(\int_0^b w_2(x_2) dx_2\right)^{-p'/q} \\
&\quad \le c\left(\int_0^a \int_0^{2^{m_0+1}} x_1^{\alpha_1 q} v(x_1, x_2) dx_1 dx_2\right)^{p'/q} \left(\int_{2^{m_0}}^\infty W_2^{-p'}(x_2) w_2(x_2) dx_2\right) 2^{(m_0+1)\alpha_2 p'} \\
&\quad \le c \sum_{k \ge m_0} \left(\int_{2^k}^{2^{k+1}} W_2^{-p'}(x_2) w_2(x_2) dx_2\right) \left(\int_0^a \int_0^{2^{k-1}} x_1^{\alpha_1 q} v(x_1, x_2) dx_1 dx_2\right)^{p'/q} \\
&\qquad \times c_1^{(m_0-k)p'/q} 2^{(m_0+1)\alpha_2 p'} \\
&\quad \le c \sum_{k \ge m_0} \left(\int_{2^k}^{2^{k+1}} W_2^{-p'}(x_2) w_2(x_2) \left(x_2 - 2^{k-1}\right)^{\alpha_2 p'} dx_2\right) \left(\int_0^a \int_0^{2^{k-1}} x_1^{\alpha_1 q} v(x_1, x_2) dx_1 dx_2\right)^{p'/q} \\
&\qquad \times c_1^{(m_0-k)p'/q} \\
&\quad \le c\left(\int_0^a w_1(x_1) dx_1\right)^{1/p}.
\end{aligned} \tag{4.43}$$

By a similar manner it follows that (4.34) implies (4.33). The proof of (iii) is similar, and we omit it. □

The proof of the next statement is similar to that of Proposition 4.7.

Proposition 4.8. *Let the conditions of Theorem 4.6 be satisfied. Then*

(i) *if $v \in DC(x)$, then $(\mathcal{WR})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if* (4.36) *and* (4.38) *hold;*

(ii) *if $v \in DC(y)$, then $(\mathcal{WR})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if* (4.37) *and* (4.38) *are satisfied;*

(iii) *if $v \in DC(x) \cap DC(y)$, then $(\mathcal{WR})_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if* (4.38) *holds.*

Now we are ready to discuss the operators $\mathcal{I}_{\alpha_1,\alpha_2}$ on the cone of nonincreasing functions.

Theorem 4.9. *Let $1 < p \le q < \infty$, and let $0 < \alpha_1, \alpha_2 < 1$. Suppose that the weight v belongs to the class $DC(y)$. Let $w(x_1, x_2) = w_1(x_1) w_2(x_2)$ for some one-dimensional weight functions w_1 and w_2 and $W_1(\infty) = W_2(\infty) = \infty$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if conditions* (2.10), (2.11), (4.23), (4.24), (4.32), (4.34), (4.37), *and* (4.38) *are satisfied.*

Theorem 4.10. *Let $1 < p \leq q < \infty$, and let $0 < \alpha_1,\ \alpha_2 < 1$. Suppose that the weight v belongs to the class $DC(x)$. Let $w(x_1, x_2) = w_1(x_1)w_2(x_2)$ for some one-dimensional weight functions w_1 and w_2 and $W_1(\infty) = W_2(\infty) = \infty$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if conditions* (2.10), (2.12), (4.25), (4.33), (4.34), (4.36), *and* (4.38) *are satisfied.*

Theorem 4.11. *Let $1 < p \leq q < \infty$, and let $0 < \alpha_1,\ \alpha_2 < 1$. Suppose that the weight $v \in DC(x) \cap DC(y)$. Let $w(x_1, x_2) = w_1(x_1)w_2(x_2)$ for some one-dimensional weight functions w_1 and w_2 and $W_1(\infty) = W_2(\infty) = \infty$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if conditions* (2.10), (4.26), (4.34), *and* (4.38) *are satisfied.*

Proofs of these statements follow immediately from the pointwise estimate

$$\mathcal{I}_{\alpha_1,\alpha_2} f = \mathcal{R}_{\alpha_1,\alpha_2} f + \mathcal{W}_{\alpha_1,\alpha_2} f + \mathcal{R}\mathcal{W}_{\alpha_1,\alpha_2} f + \mathcal{W}\mathcal{R}_{\alpha_1,\alpha_2} f. \tag{4.44}$$

Corollary B, Theorem 4.5, and Propositions 4.7 and 4.8.

The next statement shows that the two-weight inequality for $\mathcal{I}_{\alpha_1,\alpha_2}$ can be characterized by one condition when $w \approx 1$.

Corollary 4.12. *Let $1 < p \leq q < \infty$, and let $0 < \alpha_1, \alpha_2 < 1/p$. Suppose that $v \in DC(x) \cup DC(y)$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(1, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if*

$$D := \sup_{a,b>0} a^{(\alpha_1-(1/p))} b^{(\alpha_2-(1/p))} \left(\int_0^a \int_0^b v(t,\tau)\,dt\,d\tau \right)^{1/q} < \infty. \tag{4.45}$$

Proof. *Necessity* can be derived by substituting the test function $f_{a,b}(x) = \chi_{(0,a)\times(0,b)}(x)$ in the two-weight inequality for $\mathcal{I}_{\alpha_1,\alpha_2}$.

Sufficiency follows by using Theorems 4.9 and 4.10 and the arguments of the proof of Corollary 3.5 with respect to each variable. Details are omitted. □

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