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Interpolation on variable Morrey spaces defined on quasi-metric measure spaces

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In this paper, we show the validity of a Riesz–Thorin type interpolation theorem for linear operators acting from variable exponent Lebesgue spaces into variable exponent Morrey space in the framework of quasi-metric measure spaces.

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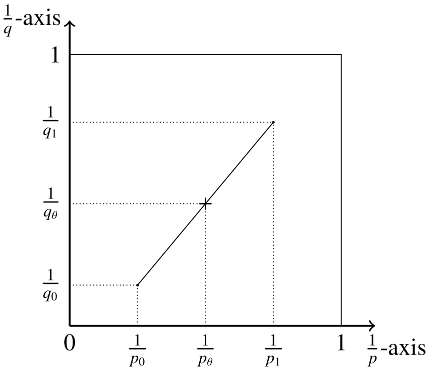

Fig. 1. Riesz square.

1. Introduction

The classical Riesz–Thorin interpolation theorem is a well-known result in harmonic analysis, cf. [13,30] where, loosely speaking, we obtain boundedness results of certain type of operators using the information on the endpoints. The geometric interpretation of this fact is that if the operator is of strong type (p_0, q_0) and of strong type (p_1, q_1), then it is of strong type (p_θ, q_θ) where the reciprocal of (p_θ, q_θ) belongs to the line joining the reciprocals of (p_0, q_0) and (p_1, q_1), viz.

$$\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}\right) = (1-\theta)\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right);$$

the schematic of this fact is called the *Riesz square*, as in Fig. 1.

A generalization of this theorem when the target space is a Morrey–Campanato space was given by S. Campanato and M. Murthy in [6]. It should be noted that it is not possible to prove Riesz–Thorin interpolation theorem when the domain space is a Morrey type space; for example, E. Stein and A. Zygmund [31] constructed an example of a bounded linear operator on H^α and L^2 but not on L^q, $q > 2$ and BMO. Further results on such a failure may be found in the papers by A. Ruiz and L. Vega [26] and O. Blasco et al. [5], where there were given examples of operators bounded from L^{p_i,λ_i} to L^{q_i}, which are not bounded in the intermediate spaces. The Riesz–Thorin interpolation theorem in the framework of variable exponent Lebesgue spaces was first proved using the abstract complex interpolation method of Calderón (cf. [8, Ch. 7]), and more recently by P. Nguyen in his PhD thesis [22] via Thorin's idea. For interpolation results for positive operators in variable exponent Lebesgue spaces we refer to [7] (see also [9] for related results).

Morrey spaces first appeared in 1938 in the work of C. Morrey [21] in relation to some problems in partial differential equations. During last decade, Morrey spaces were widely studied because of their proposed applications in various allied fields of sciences (see e.g. [11]). For more details regarding Morrey spaces we refer to the survey papers [24,25].

Function spaces with variable exponent are a very active area of research nowadays (see e.g. [18,19]) and one of the reasons is the wide variety of applications of such spaces, e.g., in the modeling of electro-rheological fluids [27] as well as thermo-rheological fluids [3], in the study of image processing [1,32] and in differential equations with non-standard growth. Lebesgue spaces with variable exponent in the framework of quasi-metric measure space have also been studied by several authors, for which we refer to [14,12].

In this paper, we prove a variant of Riesz–Thorin interpolation theorem when the domain space is the variable exponent Lebesgue space $L^{p(\cdot)}(X)$ and the target space is a variable exponent Morrey space $L^{q(\cdot),\lambda(\cdot)}(Y)$ where X and Y are a quasi-metric measure space (QMMS), where we adapt techniques presented in [6,22].

Throughout the paper constants (often different constants in the same series of inequalities) will mainly be denoted by c or C; by the symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$; the relation $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 a \le b \le c_2 a$.

2. Preliminaries

Let (X, d, μ) be a QMMS with a complete measure μ such that the space of compactly supported continuous functions are dense in $L^1_\mu(X)$. A quasi-metric d is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(a) $d(x, y) = 0$ for all $x \in X$.
(b) $d(x, y) > 0$ for all $x \neq y$, $x, y \in X$.
(c) There is a constant $a_0 > 0$ such that $d(x, y) = a_0 d(y, x)$ for all $x, y \in X$.
(d) There is a constant $a_1 > 0$ such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

Let $d_X = \operatorname{diam}(X) = \sup\{d(x, y) : x, y \in X\}$. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ be the ball with center x and radius $r > 0$. We will assume that $0 < \mu(B(x, r)) < \infty$ for every $x \in X$ and $r > 0$. It is obvious that the conditions $d_X < \infty$ and the assumption that all balls have finite measure imply $\mu(X) < \infty$. For further literature on the subject of quasi-metric measure spaces we refer e.g. to the recent book [4].

2.1. Variable exponent spaces

Let E be a measurable set in (X, μ) with positive measure. We denote:

$$p^-(E) := \inf_E p, \qquad p^+(E) := \sup_E p$$

for a measurable function p on E. Suppose that $1 \le p^-(E) \le p^+(E) < \infty$. We say that a measurable function f on E belongs to $L^{p(\cdot)}(E)$ (or to $L^{p(x)}(E)$) if

$$S_{p(\cdot),E}(f)=\int\limits_{E}|f(x)|^{p(x)}\,\mathrm{d}\mu(x)<\infty.$$

It is a Banach space with respect to the norm (see e.g. [15,20,28,29])

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\left\{\eta>0:S_{p(\cdot),E}\left(\frac{f}{\eta}\right)\le 1\right\}.$$

For the following propositions we refer to [20,28,29].

Proposition A. *Let E be a measurable subset of X. Suppose that $1 \le p^-(E) \le p^+(E) < \infty$. Then*

(i)

$$\|f\|^{p^+(E)}_{L^{p(\cdot)}(E)}\le S_{p(\cdot),E}(f)\le\|f\|^{p^-(E)}_{L^{p(\cdot)}(E)},\quad \|f\|_{L^{p(\cdot)}(E)}\le 1;$$
$$\|f\|^{p^-(E)}_{L^{p(\cdot)}(E)}\le S_{p(\cdot),E}(f)\le\|f\|^{p^+(E)}_{L^{p(\cdot)}(E)},\quad \|f\|_{L^{p(\cdot)}(E)}\ge 1;$$

(ii) *Hölder's inequality*

$$\Big|\int\limits_{E}f(x)g(x)\,\mathrm{d}\mu(x)\Big|\le\Big(\frac{1}{p^-(E)}+\frac{1}{(p^+(E))'}\Big)\ \|f\|_{L^{p(\cdot)}(E)}\ \|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f\in L^{p(\cdot)}(E)$ and $g\in L^{p'(\cdot)}(E)$.

Proposition B. *Let $1\le r(x)\le p(x)$ and let E be a subset of X with $\mu(E)<\infty$. Then the following inequality*

$$\|f\|_{L^{r(\cdot)}(E)}\le(\mu(E)+1)\|f\|_{L^{p(\cdot)}(E)}$$

holds.

The following lemma can be found in [8, p. 27].

Lemma A. *Let E be a measurable subset of X. Suppose that $1\le p^-(E)\le p^+(E)<\infty$. Then*

$$\|f\|_{L^{p(\cdot)}(E)}\le S_{p(\cdot),E}(f)+1$$

holds.

Definition 2.1. We say that a μ-measurable function $p : X \to [1,\infty)$ belongs to the class $\mathcal{P}_\mu^{\log}(X)$ if the inequality

$$|p(x) - p(y)| \le \frac{-A}{\ln \mu B(x, d(x,y))}$$

holds for all $x, y \in X$ such that $\mu B(x, d(x,y)) \le 1/2$.

For the next lemma we refer to [24,18].

Lemma B. *Let (X, d, μ) be a QMMS with finite measure, and let $p \in \mathcal{P}_\mu^{\log}(X)$. Then*

$$\left\|\chi_{B(x,r)}\right\|_{L^{p(\cdot)}} \le \mu(B(x,r))^{\frac{1}{p(x)}}.$$

The Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$ were introduced by several authors more or less simultaneously: by A. Almeida, J. Hasanov and S. Samko [2], V. Kokilashvili and A. Meskhi [16,17], T. Ohno [23] and X. Fan [10]. Let $1 \le p(\cdot) < p^+(X) < \infty$ and $0 \le \lambda(\cdot) \le 1$ be μ-measurable functions. We say that a function $f \in L^{p(\cdot)}(X)$ belongs to $L^{p(\cdot),\lambda(\cdot)}(X)$ if

$$I_{p(\cdot),\lambda(\cdot)}(f) = \sup_{x\in X, r>0} \frac{1}{\mu(B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \left|f(y)\right|^{p(y)} \mathrm{d}\mu(y) < \infty.$$

The norm on variable exponent Morrey spaces can be introduced in the following ways (see e.g. [2,16,17,24]):

$$\|f\|_1 = \inf\left\{\eta > 0 : I_{p(\cdot),\lambda(\cdot)}\left(f/\eta\right) \le 1\right\},$$

and

$$\|f\|_2 = \sup_{x\in X, R>0} \left\|\left(\mu B(x,r)\right)^{\frac{-\lambda(x)}{p(\cdot)}} f\chi_{B(x,r)}\right\|_{L^{p(\cdot)}(E)},$$

and

$$\|f\|_3 = \sup_{x\in X, r>0} \left(\mu B(x,r)\right)^{\frac{-\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r))}.$$

It can be verified that $\|f\|_1 = \|f\|_2$. Further, if $p \in \mathcal{P}_\mu^{\log}(X)$ then $\|f\|_1$ and $\|f\|_2$ are equivalent to the norm $\|f\|_3$ (see e.g. [24]). Therefore it is possible to introduce the norm in several ways which are equivalent provided that the exponent satisfies some condition. We define the norm on variable exponent Morrey space as:

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} = \|f\|_3.$$

It is easy to see that if $\lambda = 0$, then $L^{p(\cdot),0}(X) = L^{p(\cdot)}(X)$. When $p(x) \equiv const$ and $\lambda(x) \equiv const$ then $L^{p(\cdot),\lambda(\cdot)}(X)$ coincide with the classical Morrey space $L^{p,\lambda}(X)$.

The next lemma gives the embedding of variable Morrey spaces into variable Lebesgue space in case $d_X < \infty$. Here we present the proof of this lemma for the sake of completeness.

Lemma C. *Let (X, d, μ) be a QMMS with $\mu(X) < \infty$. Suppose that $1 \le p(\cdot) < p^+(X) < \infty$ and $0 \le \lambda(\cdot) \le 1$. Then, $L^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X)$ and moreover for every $f \in L^{p(\cdot),\lambda(\cdot)}(X)$, $x \in X$ and $r > 0$ we have*

$$\|f\|_{L^{p(\cdot)}(B(x,r))} \le \mu(B(x,r))^{\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}.$$

Further the following inequality

$$\int\limits_X f(y)g(y)\,\mathrm{d}\mu(y) \le c\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \tag{2.1}$$

holds.

Proof. Suppose that $f \in L^{p(\cdot),\lambda(\cdot)}(X)$. Let $x \in X$ and $r > 0$, then

$$\begin{aligned}\|f\|_{L^{p(\cdot)}(B(x,r))} &= \mu(B(x,r))^{\frac{\lambda(x)}{p(x)}} \frac{1}{\mu(B(x,r))^{\frac{\lambda(x)}{p(x)}}} \|f\|_{L^{p(\cdot)}(B(x,r))} \\ &\le \mu(B(x,r))^{\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}.\end{aligned}$$

Since p is bounded, hence taking supremum with respect to $x \in X$ and $r > 0$ we have the following estimate

$$\begin{aligned}\|f\|_{L^{p(\cdot)}(X)} &\le \max\{1, \mu(X)\}^{(\frac{\lambda}{p})^+(X)} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \\ &\le c_{p,\lambda,\mu} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}.\end{aligned}$$

Consequently, via Hölder's inequality, for $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$ there is a positive constant c such that

$$\int\limits_X f(y)g(y)\,\mathrm{d}\mu(y) \le c\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \tag{2.2}$$

holds. □

3. Complex interpolation in variable exponent Morrey spaces

In this section, we prove the main result of this paper. We begin this section with an auxiliary lemma.

Lemma 3.1. *Let (X_k, d_k, μ_k) be QMMS for $k = 1, 2$ and $\mu_2(X_2) < \infty$. Let p_1 and p_2 be bounded exponents and $0 \le \lambda(\cdot) \le 1$ and let $T : L^{p_1(\cdot)}(X_1) \to L^{p_2(\cdot),\lambda(\cdot)}(X_2)$ be linear and continuous. For $k = 1, 2$, suppose that there are positive real number B_k and M_k, measurable sets A_k with $\mu_1(A_1) < \infty$ and measurable functions $m_k, b_k : X_k \to \mathbb{R}$ such that $-M_k \le m_k(x_k) \le M_k$ and $0 \le b_k(x_k) \le B_k$, for almost every $x_k \in X_k$. For $z \in \mathbb{S}$, define*

$$F(z) := \int\limits_{X_2} T[a_1^{m_1(\cdot)z+b_1(\cdot)}\chi_{A_1}(\cdot)](x_2)a_2^{m_2(x_2)z+b_2(x_2)}\chi_{A_2}(x_2)\,\mathrm{d}\mu_2(x_2),$$

where a_k, $k = 1, 2$ are positive real numbers. Then F is continuous and bounded on the strip $\mathbb{S} = \{z : 0 \le \mathrm{Re}(z) \le 1\}$ and analytic on $\mathrm{int}(\mathbb{S})$.

Proof. Let $x_k \in X_k$ and $z \in \mathbb{C}$. Denote

$$\alpha_k(x_k, z) := a_k^{m_k(x_k)z+b_k(x_k)}\chi_{A_k}(x_k)$$

and

$$Q_k(x_k, z, w) := \frac{\alpha_k(x_k, z) - \alpha_k(x_k, w)}{z - w} - \alpha_k(x_k, z)m_k(x_k)\log a_k.$$

Therefore, we may represent F as

$$F(z) := \int\limits_{X_2} T[\alpha_1(\cdot, z)](x_2)\alpha_2(x_2, z)\,\mathrm{d}\mu_2(x_2).$$

Notice that for almost everywhere $x_k \in X_k$ the following point-wise estimate holds

$$\begin{aligned}|\alpha_k(x_k, z)| &= \left|a_k^{m_k(x_k)\,\mathrm{Re}(z)+b_k(x_k)}a_k^{im_k(x_k)\,\mathrm{Im}(z)}\chi_{A_k}(x_k)\right| \\ &= a_k^{m_k(x_k)\,\mathrm{Re}(z)+b_k(x_k)}\chi_{A_k}(x_k) \\ &\le D_k\chi_{A_k}(x_k), \qquad (3.1)\end{aligned}$$

where $D_k := \max\limits_{t\in[-M_k, M_k+B_k]} a_k^t$. Further, by virtue of (3.1) we have the following estimate,

$$\begin{aligned}|Q_k(x_k, z, w)| &= |\alpha_k(x_k, z)|\left|\frac{a_k^{m_k(x_k)(w-z)} - 1}{w - z} - m_k(x_k)\log a_k\right| \\ &= |\alpha_k(x_k, z)|\left|\frac{\mathrm{e}^{[m_k(x_k)\,\log a_k](w-z)} - 1}{w - z} - m_k(x_k)\log a_k\right| \\ &= |\alpha_k(x_k, z)|\left|\frac{\left[\sum_{j=0}^{\infty}\frac{[m_k(x_k)\,\log a_k(w-z)]^j}{j!}\right] - 1}{w - z} - m_k(x_k)\log a_k\right|\end{aligned}$$

$$
\begin{aligned}
&= |\alpha_k(x_k,z)|\left|\sum_{j=2}^{\infty}\frac{[m_k(x_k)\log a_k]^j(w-z)^{j-1}}{j!}\right| \\
&\le D_k\chi_{A_k}(x_k)\sum_{j=2}^{\infty}\frac{(M_k|\log a_k|)^j|w-z|^{j-1}}{j!} \\
&\le D_k{M_k}^2|\log a_k|^2|z-w|\chi_{A_k}(x_k)\sum_{j=0}^{\infty}\frac{[M_k|\log a_k||w-z|]^j}{j!} \\
&= D_k{M_k}^2|\log a_k|^2|z-w|\mathrm{e}^{M_k|\log a_k||w-z|}\chi_{A_k}(x_k) \qquad (3.2)
\end{aligned}
$$

for almost every $x_k \in X_k$. Now for sufficiently small $|z-w|$, we have that $D_k{M_k}^2|\log a_k|^2|z-w|\mathrm{e}^{M_k|\log a_k||w-z|}$ is no greater than 1. Thus,

$$
\begin{aligned}
S_{p'_k(\cdot),X_k}(Q_k(\cdot,z,w)) := \int_{X_k}|Q_k(x_k,z,w)|^{p'_k(x_k)}\,\mathrm{d}\mu_k(x_k) \\
\le D_kM_k^2|\log a_k|^2|z-w|\mathrm{e}^{M_k|\log a_k||w-z|}\mu_k(A_k) \\
\to 0, \qquad (3.3)
\end{aligned}
$$

as $w\to z$. Since $(p'_k)^+<\infty$, it follows that

$$\lim_{w\to z}\|Q_k(\cdot,z,w)\|_{L^{p'_k(\cdot)}(X_k)}=0. \qquad (3.4)$$

Similarly,

$$\lim_{w\to z}\|Q_k(\cdot,z,w)\|_{L^{p_k(\cdot)}(X_k)}=0. \qquad (3.5)$$

Taking into account (3.4) and the following inequality

$$
\begin{aligned}
&\|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p'_k(\cdot)}(X_k)} \\
&\quad\le |z-w|\Big(\|Q_k(\cdot,z,w)\|_{L^{p'_k(\cdot)}(X_k)}+\|\alpha_k(\cdot,z)m_k(\cdot)\log a_k\|_{L^{p'_k(\cdot)}(X_k)}\Big),
\end{aligned}
$$

we have

$$\lim_{w\to z}\|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p'_k(\cdot)}(X_k)}=0. \qquad (3.6)$$

Analogously we have

$$\lim_{w\to z}\|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p_k(\cdot)}(X_k)}=0. \qquad (3.7)$$

Now we prove the analyticity of F:

$$\begin{aligned}
&\frac{F(z)-F(w)}{z-w}\\
&=\frac{\int\limits_{X_2} T[\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)-\int\limits_{X_2} T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,w)\,\mathrm{d}\mu_2(x_2)}{z-w}\\
&=\int\limits_{X_2}\frac{T[\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)-T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,z)}{z-w}\,\mathrm{d}\mu_2(x_2)\\
&\qquad+\int\limits_{X_2}\frac{T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,z)-T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,w)}{z-w}\,\mathrm{d}\mu_2(x_2)\\
&=\int\limits_{X_2} T\left[\frac{\alpha_1(\cdot,z)-\alpha_1(\cdot,w)}{z-w}\right](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)\\
&\qquad\qquad+\int\limits_{X_2} T[\alpha_1(\cdot,w)](x_2)\left[\frac{\alpha_2(x_2,z)-\alpha_2(x_2,w)}{z-w}\right]\mathrm{d}\mu_2(x_2)\\
&=: I+J.
\end{aligned}$$

Denote

$$I' := \int\limits_{X_2} T[\alpha_1(\cdot,z)m_1(\cdot)\log a_1](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)$$

and

$$J' := \int\limits_{X_2} T[\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2\,\mathrm{d}\mu_2(x_2).$$

We show that $I \to I'$ and $J \to J'$ as $w \to z$. Firstly, we show $I \to I'$; using linearity of the operator T along with the Lemma C we obtain

$$\begin{aligned}
|I-I'| &= \Bigg|\int\limits_{X_2} T\left[\frac{\alpha_1(\cdot,z)-\alpha_1(\cdot,w)}{z-w}\right](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)\\
&\qquad\qquad-\int\limits_{X_2} T[\alpha_1(\cdot,z)m_1(\cdot)\log a_1](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)\Bigg|\\
&= \Bigg|\int\limits_{X_2} T[Q_1(\cdot,z,w)](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2)\Bigg|\\
&\le c\|T[Q_1(\cdot,z,w)](\cdot)\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_2(\cdot,z)\|_{L^{p_2'(\cdot)}(X_2)}\\
&\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|Q_1(\cdot,z,w)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)\|_{L^{p_2'(\cdot)}(X_2)}
\end{aligned}$$

where c is the constant appearing in Lemma C. The desired result follows from Lemma A, (3.1) and (3.5). To show $J \to J'$ again split $J - J'$ as two integrals:

$$\begin{aligned} J - J' &= \int\limits_{X_2} T[\alpha_1(\cdot, w)](x_2)\left[\frac{\alpha_2(x_2,z)-\alpha_2(x_2,w)}{z-w}\right] \mathrm{d}\mu_2(x_2) \\ &\qquad - \int\limits_{X_2} T[\alpha_1(\cdot, z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2 \,\mathrm{d}\mu_2(x_2) \\ &= \int\limits_{X_2} T[\alpha_1(\cdot, w)](x_2)\left[\frac{\alpha_2(x_2,z)-\alpha_2(x_2,w)}{z-w} - \alpha(x_2,z)m_2(x_2)\log a_2\right] \mathrm{d}\mu_2(x_2) \\ &\qquad + \int\limits_{X_2} T[\alpha_1(\cdot, w)-\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2 \,\mathrm{d}\mu_2(x_2) \\ &= \int\limits_{X_2} T[\alpha_1(\cdot, w)](x_2)Q_2(x_2,z,w)\,\mathrm{d}\mu_2(x_2) \\ &\qquad + \int\limits_{X_2} T[\alpha_1(\cdot, w)-\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2 \,\mathrm{d}\mu_2(x_2) \\ &=: S_1 + S_2. \end{aligned}$$

By means of Lemma C, the boundedness of T and (3.1) we have the following estimate,

$$\begin{aligned} |S_1| &\le c\|T[\alpha_1(\cdot,w)](\cdot)\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)} \\ &\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_1(\cdot,w)\|_{L^{p_1(\cdot)}(X_1)}\|Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)} \\ &\le cD_1\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\chi_{A_1}(\cdot)\|_{L^{p_1(\cdot)}(X_1)}\|Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)}. \end{aligned} \tag{3.8}$$

Analogously,

$$\begin{aligned} |S_2| &\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda}(X_2)} \\ &\quad \times \|\alpha_1(\cdot,w)-\alpha_1(\cdot,z)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)m_2(\cdot)\log a_2\|_{L^{p_2'(\cdot)}(X_2)} \end{aligned} \tag{3.9}$$

Now, expression on the right-hand side in (3.8) tends to 0 as $w \to z$ by virtue of Lemma A and (3.4), while the expression on the right-hand side in (3.9) tends to 0 as $w \to z$ by virtue of Lemma A and (3.6). Therefore, $J \to J'$, as $w \to z$. Hence, F is analytic on $\mathrm{int}(\mathbb{S})$; and $F' = I' + J'$. Now we show that F is continuous in the entire strip $\mathbb{S}$. In fact we use the same technique as above:

$$
\begin{aligned}
&|F(z)-F(w)| \\
&= \Big| \int\limits_{X_2} T[\alpha_1(\cdot,z)-\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,z)\,\mathrm{d}\mu_2(x_2) \\
&\qquad\qquad + \int\limits_{X_2} T[\alpha_1(\cdot,w)](x_2)[\alpha_2(x_2,z)-\alpha_2(x_2,w)]\,\mathrm{d}\mu_2(x_2)\Big| \\
&\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_1(\cdot,z)-\alpha_1(\cdot,w)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)\|_{L^{p_2'(\cdot)}(X_2)} \\
&\ + cD_1\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\chi_{A_1}(\cdot)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)-\alpha_2(\cdot,w)\|_{L^{p_2'(\cdot)}(X_2)}.
\end{aligned}
$$

As $w \to z$ in $\mathbb{S}$, both terms in the above sum tends to 0 by virtue of Lemma A, (3.6) and (3.7), proving the continuity of F in $\mathbb{S}$. Finally, F is bounded in $\mathbb{S}$. Indeed, by the boundedness of T and invoking Lemma C, Lemma A and estimate (3.1) we have:

$$
\begin{aligned}
|F(z)| &\le c\|T[\alpha_1(\cdot,z)]\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_2(\cdot,z)\|_{L^{p_2'(\cdot)}(X_2)} \\
&\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_1(\cdot,z)\|_{L^{p_1(\cdot)}}\|\alpha_2(\cdot,z)\|_{L^{p_2'(\cdot)}(X_2)} \\
&\le cD_1D_2\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\chi_{A_1}(\cdot)\|_{L^{p_1(\cdot)}(X_1)}\|\chi_{A_2}(\cdot)\|_{L^{p_2'(\cdot)}(X_2)} \\
&< \infty
\end{aligned}
$$

which ends the proof. □

Finally, we prove the Riesz–Thorin theorem in the setting of variable Morrey spaces defined on quasi-metric measure spaces.

Theorem 3.1. *Let (X,μ) and (Y,ν) be σ-finite, complete QMMS. For $k=0,1$, assume that $1 \le p_k(\cdot), q_k(\cdot) < q_k^+(Y) < \infty$ and $0 \le \lambda_k \le 1$. Suppose that we have a linear operator $T : L^{p_k(\cdot)}(X) \to L^{q_k(\cdot),\lambda_k(\cdot)}(Y)$ such that for all $f \in L^{p_k(\cdot)}(X)$*

$$
\|Tf\|_{L^{q_k(\cdot),\lambda_k(\cdot)}(Y)} \le M_k\|f\|_{L^{p_k(\cdot)}(X)} \tag{3.10}
$$

holds. For $z \in \mathbb{S} := \{z : 0 < \mathrm{Re}(z) < 1\}$, define p_z, q_z and λ_z by

$$
\frac{1}{p_z(x)} = \frac{1-z}{p_0(x)} + \frac{z}{p_1(x)}, \qquad \frac{1}{q_z(x)} = \frac{1-z}{q_0(x)} + \frac{z}{q_1(x)},
$$

and

$$
\frac{\lambda_z(x)}{q_z(x)} = (1-z)\frac{\lambda_0(x)}{q_0(x)} + z\frac{\lambda_1(x)}{q_1(x)}.
$$

Then, given any $\theta \in (0,1)$, the inequality

$$\|Tf\|_{L^{q_\theta(\cdot),\lambda_\theta(\cdot)}(Y)} \leq cM_0^{1-\theta}M_1^\theta\|f\|_{L^{p_\theta(\cdot)}(X)}$$

holds for every $f \in L^{p_\theta(\cdot)}(X)$.

Proof. Since T is linear, we may assume that $f \neq 0$, otherwise the inequality holds for $f = 0$. By the homogeneity of norm and the scaling argument we may assume $\|f\|_{L^{p_\theta(\cdot)}(X)} \leq 1$ and show that

$$\|Tf\|_{L^{q_\theta(\cdot),\lambda_\theta(\cdot)}(Y)} \leq cM_0^{1-\theta}M_1^\theta. \tag{3.11}$$

We will show (3.11) for simple functions in X and since simple function are dense in $L^{p(\cdot)}(X)$ we will have the estimate for all $f \in L^{p_\theta(\cdot)}(X)$.

Let us assume f, g are simple and complex valued function defined on X and Y, respectively, by

$$f(x) = \sum_{j=1}^{m} a_j \mathrm{e}^{\mathrm{i}\alpha_j}\chi_{A_j}(x),$$

$$g(x) = \sum_{k=1}^{n} b_k \mathrm{e}^{\mathrm{i}\beta_k}\chi_{B_k}(x),$$

where the $a_j, b_k > 0$ and $\alpha_j, \beta_k \in \mathbb{R}$, $\mu(A_j), \mu(B_k) < \infty$, and the $\{A_j\}$ and $\{B_k\}$ are, respectively, pairwise disjoint. Now define

$$f_z(x) = \sum_{j=1}^{m} a_j^{\frac{p_\theta(x)}{p_z(x)}} \mathrm{e}^{\mathrm{i}\alpha_j}\chi_{A_j}(x)$$

$$g_z(y) = \sum_{k=1}^{n} b_k^{\frac{q'_\theta(y)}{q'_z(y)}} \mathrm{e}^{\mathrm{i}\beta_k}\chi_{B_k}(y).$$

Finally, for every $y \in Y$, $r > 0$ and $z \in \mathbb{C}$, we put

$$F(y,r,z) := \int\limits_{B(y,r)} T(f_z(s))g_z(s)\,\mathrm{d}\nu(s).$$

Firstly note that for every $\theta \in (0,1)$, $p_\theta(y) \in [1,\infty)$. Further for almost every $x \in X$, $p_\theta(x) = \frac{p_0(x)p_1(x)}{(1-\theta)p_1(x)+\theta p_0(x)} \leq p_0^+p_1^+ < \infty$ and hence $p_\theta(x) \in [1, p_0^+p_1^+]$. Moreover,

$$-1 < \frac{1}{p_1^-} - 1 \leq \frac{1}{p_1(x)} - \frac{1}{p_0(x)} \leq 1 - \frac{1}{p_0^-} < 1,$$

for almost every $x \in X$. Let $\Phi_1(x) := p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right]$ and $\Phi_0(x) := \frac{p_\theta(x)}{p_0(x)}$. Hence, regarding

$$\frac{p_\theta(x)}{p_z(x)} = \Phi_1(x)z + \Phi_0(x)$$

as a linear polynomial in z and that Φ_1 maps X to the interval $[-p_0^+ p_1^+, p_0^+ p_1^+]$ while the map Φ_0 maps X to the interval $[0, p_0^+ p_1^+]$. Analogously for $\frac{q'_\theta(x)}{q'_z(x)}$, we have similar estimates. Since we can write F as

$$F(y,r,z) = \sum_{j=1}^{m}\sum_{k=1}^{n} \int_{B(y,r)} T\big[a_j^{\frac{p_\theta(\cdot)}{p_z(\cdot)}} \chi_{A_j}(\cdot)\big](s) b_k^{\frac{q'_\theta(s)}{q'_z(s)}} \chi_{B_k}(s)\,\mathrm{d}\nu(s),$$

hence for almost every $y \in Y$, Lemma 3.1 ensures that F is analytic on int($\mathbb{S}$) and continuous and bounded on $\mathbb{S}$.

Since A_j are pairwise disjoint and $a_j > 0$, we have for $z = \mathrm{i}t$, with $t \in \mathbb{R}$

$$\begin{aligned} S_{p_0(\cdot),B(y,r)}(f_z) &= \int_{B(y,r)} \Big|\sum_{j=1}^{m} a_j^{\frac{p_\theta(x)}{p_z(x)}} \mathrm{e}^{\mathrm{i}\alpha_j} \chi_{A_j}(x)\Big|^{p_0(x)} \mathrm{d}\mu(x) \\ &= \int_{B(y,r)} \Big|\sum_{j=1}^{m} a_j^{p_\theta(x)[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}]\mathrm{i}t + \frac{p_\theta(x)}{p_0(x)}} \mathrm{e}^{\mathrm{i}\alpha_j} \chi_{A_j}(x)\Big|^{p_0(x)} \mathrm{d}\mu(x) \\ &= \int_{B(y,r)} \sum_{j=1}^{m} \Big|a_j^{p_\theta(x)[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}]\mathrm{i}t + \frac{p_\theta(x)}{p_0(x)}} \mathrm{e}^{\mathrm{i}\alpha_j} \chi_{A_j}(x)\Big|^{p_0(x)} \mathrm{d}\mu(x) \\ &= \int_{B(y,r)} \sum_{j=1}^{m} a_j^{p_\theta(x)} \chi_{A_j}(x)\,\mathrm{d}\mu(x) \\ &= \int_{B(y,r)} \Big|\sum_{j=1}^{m} a_j \mathrm{e}^{\mathrm{i}\alpha_j} \chi_{A_j}(x)\Big|^{p_\theta(x)} \mathrm{d}\mu(x) \\ &= S_{p_\theta(\cdot),B(y,r)}(f) \\ &\le 1 \end{aligned}$$

since $\|f\|_{L^{p_\theta(\cdot)}(X)} \le 1$. Hence $\|f_z\|_{L^{p_0(\cdot)}(B(y,r))} \le 1$. A similar argument shows that $\|g_z\|_{L^{q'_0(\cdot)}(B(y,r))} \le 1$ for $z = \mathrm{i}t$. Now by Hölder's inequality, Lemma C and (3.10) we have

$$\begin{aligned} |F(y,r,\mathrm{i}t)| &\le \Big|\int_{B(y,r)} T(f_z(s)) g_z(s)\,\mathrm{d}\nu(s)\Big| \\ &\le c\|Tf_z\|_{L^{q_0(\cdot)}(B(y,r))} \|g_z\|_{L^{q'_0(\cdot)}(B(y,r))} \\ &\le c\|Tf_z\|_{L^{q_0(\cdot)}(B(y,r))} \end{aligned}$$

$$\le c\nu(B(y,r))^{\frac{\lambda_0(y)}{q_0(y)}}\|Tf_z\|_{L^{q_0(\cdot),\lambda_0(\cdot)}(Y)}$$
$$\le c\nu(B(y,r))^{\frac{\lambda_0(y)}{q_0(y)}}M_0\|f_z\|_{L^{p_0(\cdot)}(X)}$$
$$\le c\nu(B(y,r))^{\frac{\lambda_0(y)}{q_0(y)}}M_0.$$

An analogous argument with $\mathrm{Re}(z) = 1$ and with the exponents p_1 and q_1 yields

$$|F(y, r, 1 + \mathrm{i}t)| \le c\nu(B(y,r))^{\frac{\lambda_1(y)}{q_1(y)}}M_1.$$

Finally, using Hadamard's three lines lemma gives

$$|F(y, r, \theta)| \le c\nu(B(y,r))^{\frac{\lambda_\theta(y)}{q_\theta(y)}}M_0^{1-\theta}M_1^{\theta}.$$

Also,

$$\sup_{\|g\|_{L^{q'_\theta}B(y,r)}\le 1}|F(y, r, \theta)| = \|Tf\|_{L^{q_\theta(\cdot)}(B(y,r))}.$$

Hence for almost every $y \in Y$ and $r > 0$ we have

$$\nu(B(y,r))^{\frac{-\lambda_\theta(y)}{q_\theta(y)}}\|Tf\|_{L^{q_\theta(\cdot)}(B(y,r))} \le cM_0^{1-\theta}M_1^{\theta},$$

which implies that

$$\|Tf\|_{L^{q_\theta(\cdot),\lambda_\theta(\cdot)}(Y)} \le cM_0^{1-\theta}M_1^{\theta}.$$

This completes the proof. □

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