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Original article

Sharp weighted bounds for the Hilbert transform of odd and even functions

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Abstract

Our aim is to establish sharp weighted bounds for the Hilbert transform of odd and even functions in terms of the mixed type characteristics of weights. These bounds involve A_p and A_∞ type characteristics. As a consequence, we obtain weighted bounds in terms of so-called Andersen–Muckenhoupt type characteristics.

Keywords: Hilbert transform; Sharp weighted bound; One-weight inequality

1. Introduction

In this paper, we investigate sharp weighted bounds, involving A_p and A_∞ characteristics of weights, for the Hilbert transform of odd and even functions. Following general results we derive these sharp weighted A_p bounds in terms of so-called Andersen-Muckenhoupt characteristics. Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{\mathcal{B}(X,Y)}$ which is defined in the standard way i.e. $\|T\|_{\mathcal{B}(X,Y)} = \sup_{\|f\|_X \le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_{\mathcal{B}(X)}$.

A non-negative locally integrable function (i.e. a weight function) w defined on $\mathbb{R}^n$ is said to satisfy the $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

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In 1972, B. Muckenhoupt [1] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy$$

is bounded in $L^p_w(\mathbb{R}^n)$. S. Buckley [2] investigated the sharp A_p bound for the operator M. In particular, he established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C \|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w \in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

Let H be the Hilbert transform given by

$$(Hf)(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R}.$$

In 1973 R. Hunt, B. Muckenhoupt and R. L. Wheeden [3] solved the one-weight problem for the Hilbert transform in terms of Muckenhoupt condition. In particular, they established the inequality

$$\|Hf\|_{L^p_w(\mathbb{R})} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})} \|f\|_{L^p_w(\mathbb{R})} \tag{1.2}$$

for some positive constant β and some constant c_p depending on p. S. Petermichl showed that the value of the exponent $\beta = \max\{1, p'/p\}$ in (1.2) is sharp. In particular, the following statement holds (see [4] for $p = 2$, [5] for $p \neq 2$):

Theorem A. *Let $1 < p < \infty$ and let w be a weight function on $\mathbb{R}$. Then there is a positive constant c_p depending only on p such that*

$$\|H\|_{\mathcal{B}(L^p_w)} \le c_p \|w\|^{\beta}_{A_p(\mathbb{R})}, \tag{1.3}$$

where $\beta = \max\left\{1, \frac{p'}{p}\right\}$. Moreover, the exponent in (1.3) is sharp.

We say that $w \in A_\infty(\mathbb{R}^n)$ if $w \in A_p(\mathbb{R})$ for some $p > 1$. In what follows we will use the symbol $\|\rho\|_{A_\infty}$ for the A_∞ characteristic of a weight function ρ:

$$\|\rho\|_{A_\infty} = \sup_I \frac{1}{\rho(I)} \int_I M(\rho \chi_I)(x) dx.$$

This characteristic appeared first in the papers by Fiji [6] and Wilson [7,8] and is lower than that the one introduced by Hruščev [9]:

$$[\rho]_{A_\infty} = \sup_I \left(\frac{1}{|I|} \int_I \rho(x) dx \right) \exp\left(\frac{1}{|I|} \int_I \log \rho^{-1}(x) dx \right).$$

In 2012, Hytönen, Perez and Rela [10] improved Buckley's result and obtained a sharp weighted bound involving A_∞ constant:

$$\|M\|_{\mathcal{B}(L^p_w)} \le c_n \left(\frac{1}{p-1} \|w\|_{A_p} \|\sigma\|_{A_\infty} \right)^{1/p}, \quad 1 < p < \infty, \quad \sigma = w^{1-p'}.$$

Later, in [11], it was proved that the sharp weighted bound involving the A_∞ characteristic for the Calderón–Zygmund operator provides an improved estimate than the one obtained by Hytönen in his celebrated paper [12] about the A_2 conjecture. We recall the result of [10] for the Hilbert transform H in the following theorem.

Theorem B. *Let H be the Hilbert transform and let $p \in (1,\infty)$. Then if $w \in A_p(\mathbb{R}_+)$, we have*

$$\|H\|_{\mathcal{B}(L^p_w)} \le \begin{cases} \|w\|_{A_p}^{2/p}\|\sigma\|_{A_\infty}^{2/p-1}, & \text{if } p \in (1,2], \\ \|w\|_{A_p}^{2/p}\|w\|_{A_\infty}^{1-2/p}, & \text{if } p \in [2,\infty), \end{cases} \tag{1.4}$$

where $\sigma := w^{1-p'}$.

It is known (see [11]) that

$$c_n\|\rho\|_{A_\infty} \le [\rho]_{A_\infty} \le \|\rho\|_{A_p}. \tag{1.5}$$

It can be checked that

$$[\sigma]_{A_\infty}^{p-1} \le \|\sigma\|_{A'_p}^{p-1} = \|w\|_{A_p}.$$

In the sequel we will use the following relation between weights $w : \mathbb{R} \to \mathbb{R}_+$ and $W : \mathbb{R}_+ \to \mathbb{R}_+$ (resp. between $\sigma : \mathbb{R} \to \mathbb{R}_+$ and $\Sigma : \mathbb{R}_+ \to \mathbb{R}_+$)

$$w(x) := \frac{W(\sqrt{|x|})}{2\sqrt{|x|}} \quad \Big(\text{resp. } \sigma(x) := \frac{\Sigma(\sqrt{|x|})}{2\sqrt{|x|}}\Big),$$

where $x \ne 0$.

Finally we mention that weighted sharp estimates for one-sided operators on the real line in terms of one-sided Muckenhoupt characteristics were established in [13] (see also [14] for related topics regarding multiple integral operators).

The relation $A \approx B$ means that there are positive constants c_1 and c_2 (in general these constants will depend only on the space exponents r or p) such that $c_1 B \le A \le c_2 B$.

For a weight function ρ and a measurable set $E \subset \mathbb{R}$, we denote

$$\rho(E) := \int_E \rho(x)dx.$$

Constants will be denoted by c or C (the same notation will be used even if they can differ from line to line).

2. Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}_+$ be odd. Then it is easy to check that Hf is even and given by $(Hf)(x) = (H_0 f)(x)$ for $x > 0$, where

$$(H_0 f)(x) = \frac{2}{\pi}\int_0^\infty \frac{tf(t)}{t^2 - x^2}dt, \quad x > 0.$$

If f is even, then Hf is odd and is given by $(Hf)(x) = (H_e f)(x)$ for $x > 0$, where

$$(H_e f)(x) = \frac{2}{\pi}\int_0^\infty \frac{xf(t)}{t^2 - x^2}dt.$$

Our aim is to investigate the sharp weighted bound of the type (1.4) for operators H_0 and H_e, and to derive sharp estimates of the type:

$$\|H_0 f\|_{L^p_W(\mathbb{R}_+)} \le c_p \|W\|_{A_p^0(\mathbb{R}_+)}^{\beta} \|f\|_{L^p_W(\mathbb{R}_+)}, \tag{2.1}$$

$$\|H_e f\|_{L^p_W(\mathbb{R}_+)} \le c_p \|W\|_{A_p^e(\mathbb{R}_+)}^{\gamma} \|f\|_{L^p_W(\mathbb{R}_+)} \tag{2.2}$$

where $1 < p < \infty$ and

$$\|W\|_{A_p^0(\mathbb{R}_+)} := \sup_{[a,b]\subset(0,\infty)} \left(\frac{1}{b^2-a^2}\int_a^b W(x)dx\right)\left(\frac{1}{b^2-a^2}\int_a^b x^{p'}W^{1-p'}(x)dx\right)^{p-1}$$

$$\|W\|_{A_p^e(\mathbb{R}_+)} := \sup_{[a,b]\subset(0,\infty)} \left(\frac{1}{b^2-a^2}\int_a^b x^p W(x)dx\right)\left(\frac{1}{b^2-a^2}\int_a^b W^{1-p'}(x)dx\right)^{p-1}.$$

K. Andersen [15] showed that if $1 < p < \infty$, then

(i) H_0 is bounded in $L^p_W(\mathbb{R}_+)$ if and only if $\|W\|_{A_p^0(\mathbb{R}_+)} < \infty$;

(ii) H_e is bounded in $L^p_W(\mathbb{R}_+)$ if and only if $\|W\|_{A_p^e(\mathbb{R}_+)} < \infty$.

The following lemma was proved in [15] but we give the proof because of the exponents of characteristics of weights.

Lemma 2.1. *Let $1 < r < \infty$ and let w be a non-negative measurable function on $(0, \infty)$. Then*

$$\|W\|_{A_r^0(\mathbb{R}_+)} \approx \|w\|_{A_r(\mathbb{R})}$$

with constants depending only on r.

Proof. First we show that

$$\|w\|_{A_r(\mathbb{R})} \le c_r \|W\|_{A_r^0(\mathbb{R}_+)}.$$

Let $[a, b] \subset (0, \infty)$. Then

$$\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} = \left(\int_a^b W(\sqrt{x})\frac{dx}{2\sqrt{x}}\right)\left(\int_a^b W^{1-r'}(\sqrt{x})\frac{dx}{(2\sqrt{x})^{1-r'}}\right)^{r-1}$$
$$= 2^r\left(\int_{\sqrt{a}}^{\sqrt{b}} W(x)dx\right)\left(\int_{\sqrt{a}}^{\sqrt{b}} x^{r'}W^{1-r'}(x)dx\right)^{r-1}.$$

If $\|W\|_{A_r^0(\mathbb{R}_+)} < \infty$, then the latter expression is bounded by

$$2^r\|W\|_{A_r^0(\mathbb{R}_+)}((\sqrt{b})^2 - (\sqrt{a})^2)^r = 2^r\|W\|_{A_r^0(\mathbb{R}_+)}(b-a)^r.$$

This follows from the definition of $\|W\|_{A_r^0(\mathbb{R}_+)}$.

Suppose now that $[a, b] \subset (-\infty, 0)$. Arguing as before, we see that

$$\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} = 2^r\left(\int_{\sqrt{-b}}^{\sqrt{-a}} W(x)dx\right)\left(\int_{\sqrt{-b}}^{\sqrt{-a}} x^{r'}W^{1-r'}(x)dx\right)^{r-1}$$
$$\le 2^r\|W\|_{A_r^0(\mathbb{R}_+)}(b-a)^r.$$

Now let $a < 0 < b$. Suppose that $c > 0$ is a number such that $[a, b] \subset [-c, c]$, and $[a, b]$ and $[-c, c]$ have at least one common endpoint. Then by using the above arguments we see that

$$\left(\int_a^b w(x)dx\right)\left(\int_a^b w^{1-r'}(x)dx\right)^{r-1} \le 2^r\left(\int_0^c w(x)dx\right)\left(\int_0^c w^{1-r'}(x)dx\right)^{r-1}$$
$$\le c_r\|W\|_{A_r^0(\mathbb{R}_+)}(b-a)^r$$

where c_r is a positive constant depending only on r. Finally,

$$\|w\|_{A_r(\mathbb{R})} \le c_r\|W\|_{A_r^0(\mathbb{R}_+)}.$$

Inequality $\|W\|_{A_r^0(\mathbb{R}_+)} \le c_r\|w\|_{A_r(\mathbb{R})}$ follows from the arguments similar to those used above. □

Now we introduce Wilson type A_∞ characteristic for weights defined on $\mathbb{R}_+$. The classes A^0_∞ and A^e_∞ are defined as follows:

$$A^0_\infty = \cup_{p>1} A^0_p; \quad A^e_\infty = \cup_{p>1} A^e_p.$$

Let $\|W\|_{A^0_\infty}$ be the A^0_∞ characteristic of a W on $\mathbb{R}_+$ defined as follows:

$$\|W\|_{A^0_\infty} = \sup_{(a,b)\subset\mathbb{R}_+} \frac{1}{W([a,b])} \int_a^b x\big(\overline{M}(W\chi_{(a,b)})\big)(x)dx,$$

where

$$\overline{M} f(x) = \sup_{(c,d)\ni x} \frac{1}{d^2-c^2} \int_c^d W(t)dt. \tag{2.3}$$

Here the supremum is taken over all interval $(c,d) \subset \mathbb{R}_+$ containing x.

The next statement will be useful to prove the main Theorem.

Lemma 2.2. *Let w be a weight on $\mathbb{R}$. Then the following relation holds:*

$$\|w\|_{A_\infty(\mathbb{R})} \approx \|W\|_{A^0_\infty(\mathbb{R}_+)} \tag{2.4}$$

with constants independent of w.

Proof. At first suppose that $I := (a,b) \subset \mathbb{R}_+$. Then it is easy to see that

$$\frac{1}{w(I)} \int_I M(w\chi_I)(x)dx \approx \frac{1}{W([\sqrt{a},\sqrt{b}])} \int_{\sqrt{a}}^{\sqrt{b}} x\overline{M}\big(W\chi_{[\sqrt{a},\sqrt{b}]}\big)(x)dx, \tag{2.5}$$

with constants independent of I and w, where $\overline{M}$ is defined by formula (2.3).

Next, we use the following observation: let $x \in (a,b)$,

$$M\big(w\chi_{(a,b)}\big)(x) \approx \overline{M}\big(W\chi_{(\sqrt{|a|},\sqrt{|b|})}\big)(\sqrt{x})$$

which can be obtained from the relation between w and W. In a similar manner, if $I := (a,b) \subset \mathbb{R}_-$, we have

$$\frac{1}{w(I)} \int_a^b M(w\chi_I)(x)dx \approx \frac{1}{W([\sqrt{-a},\sqrt{-b}])} \int_{\sqrt{-b}}^{\sqrt{-a}} x\overline{M}\big(W\chi_{(\sqrt{a},\sqrt{b})}\big)(x)dx. \tag{2.6}$$

Let now $0 \in I$. Then we represent $I = (a,0] \cup (0,b)$ to get

$$\begin{aligned}\frac{1}{w(I)} \int_I M(w\chi_I)(x)dx \le{}& \frac{1}{w(I)} \int_{(a,0)} M(w\chi_{(a,0)})(x)dx \\ &+ \frac{1}{w(I)} \int_{(a,0)} M(w\chi_{(0,b)})(x)dx + \frac{1}{w(I)} \int_{(0,b)} M(w\chi_{(a,0)})(x)dx \\ &+ \frac{1}{w(I)} \int_{(0,b)} M(w\chi_{(0,b)})(x)dx := S_1 + S_2 + S_3 + S_4.\end{aligned}$$

We have to estimate S_2 and S_3. Estimates for S_1 and S_4 can be derived in a similar manner by using the estimates

$$\frac{1}{w(I)} \int_a^0 M(w\chi_{[a,0]})(x)dx \le \frac{1}{w([a,0])} \int_a^0 M(w\chi_{[a,0]})(x)dx$$

and

$$\frac{1}{w(I)} \int_0^b M(w\chi_{[0,b]})(x)dx \le \frac{1}{w([0,b])} \int_0^b M(w\chi_{[0,b]})(x)dx.$$

Simple observations lead us to the estimates:

$$S_i \le C\frac{1}{W([0,\sqrt{A}])}\int_0^{\sqrt{A}} x\overline{M}\big(W\chi_{[0,\sqrt{A}]}\big)(x)dx \le C\|W\|_{A^0_\infty}, \quad i = 2, 3,$$

where $A := \max\{|a|, |b|\}$. Finally we have that

$$\|w\|_{A_\infty(\mathbb{R})} \le C\|W\|_{A^0_\infty}$$

with a constant C independent of w. The reverse estimate can be obtained in a similar manner. □

The next lemma is a consequence of (1.5), Lemmas 2.2 and 2.1.

Lemma 2.3. *Let* $1 < p < \infty$. *Then*

$$\|W\|_{A^0_\infty} \le C\|W\|_{A^0_p}.$$

In the sequel we assume that $\sigma = w^{1-p'}$. Taking into account the definition of Σ,we have that

$$\Sigma(u) = W^{1-p'}(u)(2u)^{p'}. \tag{2.7}$$

Theorem 2.1. *Let* $1 < p < \infty$. *Then* (i)

$$\|H_0\|_{\mathcal{B}(L^p_W)} \le \begin{cases} \|W\|^{2/p}_{A^0_p}(\|\Sigma\|_{A^0_\infty})^{2/p-1}, & \text{if } p \in (1,2],\\ \|W\|^{2/p}_{A^0_p}(\|W\|_{A^0_\infty})^{1-2/p}, & \text{if } p \in [2,\infty), \end{cases} \tag{2.8}$$

(ii)

$$\|H_e\|_{\mathcal{B}(L^p_W)} \le \begin{cases} \|W\|^{2/p}_{A^e_p}(\|W^{1-p'}\|_{A^0_\infty})^{1-2/p'}, & \text{if } p \in (1,2],\\ \|W\|^{2/p}_{A^e_p}(\|W_p\|_{A^0_\infty})^{2/p'-1}, & \text{if } p \in [2,\infty), \end{cases} \tag{2.9}$$

where W *and* Σ *are related by* (2.7) *and* $W_p(x) = W(x)(2x)^p$.

Proof. Let us prove (i). The proof for (ii) is a consequence of the dual arguments and will be discussed afterwards.

Let us denote $g(x) := f(\sqrt{x})$, $x > 0$, $g(x) = 0$ otherwise. Suppose that w and W are related as in Lemma 2.1, we have

$$\int_{-\infty}^{+\infty}|g(x)|^p w(x)dx = \int_0^\infty |f(\sqrt{x})|^p w(x)dx = \int_0^\infty |f(\sqrt{x})|^p\frac{W(\sqrt{x})}{2\sqrt{x}}dx = \int_0^\infty |f(u)|^p W(u)du.$$

Furthermore, for $x > 0$,

$$(Hg)(x) = \frac{1}{\pi}\int_0^\infty \frac{f(\sqrt{t})}{t-x}dt = \frac{1}{\pi}\int_0^\infty \frac{2tf(t)}{t^2-x}dt = (H_0f)(\sqrt{x}).$$

By definition, we have

$$\begin{aligned}\|H_0f\|^p_{L^p_W(\mathbb{R}_+)} &= \int_0^\infty |(H_0f)(x)|^p W(x)dx = \int_0^\infty |(H_0f)(\sqrt{u})|^p W(\sqrt{u})\frac{du}{2\sqrt{u}}\\ &= \int_0^\infty |(H_0f)(\sqrt{u})|^p w(u)du\\ &= \int_0^\infty |(Hg)(u)|^p w(u)du \le \|Hg\|^p_{L^p_w(\mathbb{R})}.\end{aligned}$$

Let $1 < p \le 2$. Then by Theorem B and Lemmas 2.1 and 2.2 we have that

$$\|H\|_{\mathcal{B}(L^p_w(\mathbb{R}))} \le \|w\|^{2/p}_{A_p(\mathbb{R})}\|\sigma\|^{2/p-1}_{A_\infty(\mathbb{R})} \approx \|W\|^{2/p}_{A^0_p(\mathbb{R}_+)}\|\Sigma\|^{2/p-1}_{A^0_\infty(\mathbb{R}_+)}$$

where $\sigma = w^{1-p'}$, $\sigma(x) = \Sigma(\sqrt{|x|})/(2\sqrt{|x|})$. Observe that W and Σ are related also by (2.7). The case $p \geq 2$ follows analogously. Thus we have (2.8).

To prove (2.9) we use the duality arguments. First observe that the Riesz identity for the classical Hilbert transform H and the appropriate substitution of the variable yields that

$$\int_{\mathbb{R}_+} (H_0 f)(x) g(x) dx = -\int_{\mathbb{R}_+} (H_e g)(x) f(x) dx.$$

Hence, it follows that the adjoint of H_o is H_e with the equation

$$\|H_e\|_{\mathcal{B}(L^p_w(\mathbb{R}_+))} = \|H_o\|_{\mathcal{B}(L^{p'}_\sigma(\mathbb{R}_+))}.$$

By applying case (i) and Lemmas 2.1 and 2.2 we have the desired result also for (ii). □

The next statement gives sharp weighted bound in terms of A_p characteristics.

Theorem 2.2. *Let $1 < p < \infty$ and let W be a weight function on $\mathbb{R}_+$. Then the following estimates hold*
(a)

$$\|H_0\|_{L^p_W(\mathbb{R}_+)} \leq c_p \|W\|^{\beta}_{A^0_p(\mathbb{R})}; \tag{2.10}$$

(b)

$$\|H_e\|_{L^p_W(\mathbb{R}_+)} \leq C_p \|W\|^{\beta}_{A^e_p(\mathbb{R}_+)} \tag{2.11}$$

with some positive constants c_p and C_p, respectively, depending only on p, where $\beta = \max\{1, \frac{p'}{p}\}$. Moreover the exponent β in (2.10) and (2.11) is best possible.

Proof. We prove (a). The estimate (b) follows from the duality arguments. Let $1 < p \leq 2$. To show the validity of (a) we use (2.8), Lemma 2.1 and relations

$$\|\Sigma\|_{A^0_\infty(\mathbb{R}_+)} \approx \|\sigma\|_{A_\infty(\mathbb{R})} \leq \|\sigma\|_{A_{p'}(\mathbb{R})} = \|w\|^{p'-1}_{A_p(\mathbb{R})} \approx \|W\|^{p'-1}_{A^0_p(\mathbb{R}_+)}.$$

The case $p > 2$ follows from the estimates:

$$\|W\|_{A^0_\infty(\mathbb{R}_+)} \approx \|w\|_{A_\infty(\mathbb{R})} \leq \|w\|_{A_p(\mathbb{R})} \approx \|W\|_{A^0_p(\mathbb{R})}.$$

Sharpness: First we will show the sharpness for $p = 2$. Let

$$g(x) = x^{\varepsilon-1}\chi_{(0,1)}, \quad w(x) = |x|^{1-\varepsilon}.$$

Then (see [4]) the following estimate holds:

$$\|g\|_{L^2(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|w\|_{A_2(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|Hg\|_{L^2_w(\mathbb{R})} \geq 4\varepsilon^{-3}.$$

Let now

$$f(x) = x^{2(\varepsilon-1)}\chi_{(0,1)}, \quad W(x) = |x|^{3-\varepsilon}.$$

Hence by using the same changing of variable we find that

$$\|f\|^2_{L^2_W(\mathbb{R})} \approx \frac{1}{\varepsilon}; \quad \|H_0 f\|^2_{L^2_W(\mathbb{R}_+)} \geq \varepsilon^{-3}.$$

Consequently, if the exponent $1-\varepsilon$ is the best possible for the A^0_2 characteristic in the one-weight inequality for some $\lambda > 0$, we have

$$4\varepsilon^{-3} \leq \|H_0 f\|_{L^2_W(\mathbb{R}_+)} \leq C\|W\|^{1-\lambda}_{A^0_2}\|f\|_{L^2_W(\mathbb{R})} \leq C\|W\|^{1-\varepsilon}_{A^0_2} \leq C\varepsilon^{\lambda-3}.$$

Let $1 < p < 2$. Suppose that $0 < \epsilon < 1$ and that $w(x) = |x|^{(1-\epsilon)(p-1)}$. Then it is easy to check that (see also [4])

$$\|w\|^{1/(p-1)}_{A_p} \approx \frac{1}{\epsilon}.$$

Observe also, that for the function defined by

$$f(x) = x^{\epsilon-1}\chi_{(0,1)}, \tag{2.12}$$

the relation $\|f\|_{L^p_w} \approx \frac{1}{\epsilon^{\frac{1}{p}}}$ holds. Let

$$g(x) = x^{2(\varepsilon-1)}, \quad W(x) = |x|^{2(1-\varepsilon)(p-1)}.$$

Then the following estimates can be checked easily by using the appropriate change of variables:

$$\begin{aligned}\|H_0 g\|_{L^p_w(\mathbb{R}_+)} &= 2^{-1/p}\|Hf\|_{L^p_w(\mathbb{R})} \geq 2^{-1/p}\frac{1}{\epsilon}\|f\|_{L^p_w(\mathbb{R})} \\ &\approx \|w\|^{p'/p}_{A_p}\|f\|_{L^p_w(\mathbb{R})} \approx \|W\|^{p'/p}_{A^0_p}\|g\|_{L^p_W(\mathbb{R}_+)}\end{aligned}$$

are fulfilled. Thus we have sharpness in (2.10) for $1 < p < 2$.

It remains to consider the case when $p > 2$. In the same manner as above, we can argue for the operator H_e and obtain the sharpness in (2.11) for $1 < p < \infty$. The duality arguments now imply the sharpness in (2.10) for $2 < p < \infty$. □

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