Potential-Type Operators in $L^{p(x)}$ Spaces

D. E. Edmunds and A. Meskhi

Abstract. In this paper we derive weight inequalities for one-sided and Riesz potentials in $L^{p(x)}$ spaces under the condition that p satisfies a weak Lipschitz condition. Compactness of these operators in $L^{p(x)}$ spaces is also established.

Keywords: *Lebesgue spaces with variable exponent, Riemann–Liouville and Weyl operators, Riesz potentials*

AMS subject classification: Primary 46E30, secondary 47B34

0. Introduction

Our aim is to establish some weight inequalities for one-sided and Riesz potentials in Lebesgue spaces $L^{p(x)}$ with variable exponent. Diening (see [1]) proved that if p satisfies a weak Lipschitz condition, then the Hardy-Littlewood maximal operator is bounded in $L^{p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$. Sobolev-type theorems for Riesz potentials were derived by Samko (see [10 - 11]). For further properties of Lebesgue and Sobolev spaces with variable exponent see the papers [2- 3, 6]. Finally, we mention that necessary and sufficient integral conditions on the weight function v governing the boundedness/compactness of the Riemann–Liouville operator R_α from the classical Lebesgue space $L^p(\mathbb{R}_+)$ to the weighted space $L^p_v(\mathbb{R}_+)$ $(1 < p < \infty, \frac{1}{p} < \alpha < 1)$ were derived in [7] (for $p = 2$ see [8]).

The paper is organized as follows:

In Section 1 we establish some properties of $L^{p(x)}$ spaces. In Section 2 we derive weight inequalities for one-sided and Riesz potentials in these spaces. Section 3 is devoted to the compactness problems of the operators mentioned above in Lebesgue spaces with variable exponent. Constants (often different constants in the same series of inequalities) will generally be denoted by c.

D. E. Edmunds: Univ. of Sussex, Centre for Math. Anal. & Appl., Brighton BN1 9QH, Sussex, United Kingdom; d.e.edmunds@sussex.ac.uk

A. Meskhi: A. Razmadze Math. Inst. of the Georgian Acad. Sci., 1 M. Aleksidze St., 380093 Tbilisi, Georgia; meskhi@rmi.acnet.ge

1. Preliminaries

Let Ω be a domain in $\mathbb{R}^n$ and let $p : \Omega \to (1, \infty)$ be a measurable function. Throughout the paper we shall assume that

$$P \equiv \operatorname*{ess\,sup}_{x \in \Omega} p(x) < \infty \qquad \text{and} \qquad p_0 \equiv \operatorname*{ess\,inf}_{x \in \Omega} p(x) > 1.$$

Definition 1.1. By $L^{p(x)}$ we denote the set of all measurable functions f defined on Ω such that

$$I_p(f) := \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

It is known (see, e.g., [6, 9, 12]) that the functional

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : I_p\left(\tfrac{f}{\lambda}\right) \le 1 \right\}$$

is a norm on $L^{p(x)}(\Omega)$.

Proposition 1.1 (see, e.g., [6, 10]).

(i) *The inequalities*

$$\|f\|_{p(\cdot)}^P \le I_p(f) \le \|f\|_{p(\cdot)}^{p_0}, \quad \|f\|_{p(\cdot)} \le 1$$

$$\|f\|_{p(\cdot)}^{p_0} \le I_p(f) \le \|f\|_{p(\cdot)}^P, \quad \|f\|_{p(\cdot)} \ge 1$$

hold.

(ii) *If E is a measurable set in Ω and χ_E is its characteristic function, then*

$$|E|^{\frac{1}{p_0}} \le \|\chi_E\|_{p(\cdot)} \le |E|^{\frac{1}{P}}, \quad |E| \ge 1$$

$$|E|^{\frac{1}{p_0}} \ge \|\chi_E\|_{p(\cdot)} \ge |E|^{\frac{1}{P}}, \quad |E| < 1.$$

(iii) *The generalization of Hölder's inequality*

$$\left| \int_\Omega f(x) \varphi(x)\, dx \right| \le k \|f\|_{p(\cdot)} \|\varphi\|_{p'(\cdot)}$$

holds, where $p'(x) = \frac{p(x)}{p(x)-1}$ and the constant $k > 0$ depends only on p.

Furthermore, if we introduce another norm $\|\cdot\|_{p(\cdot)}^*$ by

$$\|f\|_{p(\cdot)}^* = \sup_{\|\varphi\|_{p'(\cdot)} \le 1} \left| \int_\Omega f(x) \varphi(x) \right|,$$

then this norm is equivalent to $\|\cdot\|_{p(\cdot)}$.

Definition 1.2. A function g is said to belong to W-Lip (Ω) (or to satisfy a weak Lipschitz condition) if $g \in C(\Omega)$ and there exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $0 < |x - y| < \frac{1}{2}$ the inequality $|g(x) - g(y)| \le \frac{A}{-\log|x-y|}$ holds.

The next lemma follows immediately:

Lemma 1.1. *The function $p : (0,1) \to \mathbb{R}$ belongs to W-Lip $(0,1)$ if and only if $p' \in W$-Lip$(0,1)$, where $p'(x) = \frac{p(x)}{p(x)-1}$.*

Lemma 1.2. *Let the function $\alpha : (0,1) \to (0,1]$ belong to W-Lip $(0,1)$. Then there exists a constant $c > 0$ such that for all $x, y \in (0,1)$ the inequality $(x-y)^{\alpha(x)-1} \leq c\,(x-y)^{\alpha(y)-1}$ holds.*

Proof. Let $a(x,y) \equiv (x-y)^{\alpha(x)-\alpha(y)}$. Then

$$|\log a(x,y)| = |\alpha(x) - \alpha(y)| \log \frac{1}{x-y} \leq c \log \frac{1}{x-y} \cdot \log^{-1} \frac{1}{x-y} = c.$$

Consequently, $0 < c_2 \leq a(x,y) \leq c_1 < \infty$. Finally,

$$(x-y)^{\alpha(x)-1} = (x-y)^{\alpha(x)-\alpha(y)+\alpha(y)-1} \leq c\,(x-y)^{\alpha(y)-1}$$

and the lemma is proved ∎

The following result is a special case of one proved recently by Diening (see [1]).

Theorem A. *Let $p : (0,1) \to (1,\infty)$ be uniformly continuous and let $p \in W$-Lip $(0,1)$. Then the maximal operator M defined by*

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} \int_{[x-h,x+h]\cap[0,1]} |f(y)|\, dy \qquad (1.1)$$

is bounded in $L^{p(x)}(0,1)$.

2. Boundedness

In this section we establish some weight inequalities for the Riemann–Liouville operator R_α defined by

$$(R_\alpha f)(x) = \int_0^x (x-y)^{\alpha(x)-1} f(y)\, dy \qquad (0 < \alpha(x) \leq 1, x > 0)$$

in the spaces $L^{p(x)}(0,1)$ and $L^{p(x)}(\mathbb{R}_+)$. The appropriate problems for the Weyl operators W_α and $\mathcal{W}_\alpha$ defined by

$$(W_\alpha f)(x) = \int_x^\infty (y-x)^{\alpha(x)-1} f(y)\, dy \qquad \left(0 < \alpha(x) \leq 1, x > 0\right)$$

$$(\mathcal{W}_\alpha f)(x) = \int_x^1 (y-x)^{\alpha(x)-1} f(y)\, dy \qquad \left(0 < \alpha(x) \leq 1, x \in (0,1)\right)$$

respectively are studied.

To prove the promised inequalities we shall need some auxiliary results.

Lemma 2.1. *Let* $\alpha : (0,1) \rightarrow (0,1]$ *be measurable. Then there exists a constant* $c > 0$ *such that for all* $x \in (0,1)$ *and* $0 \leq f \in L(a,b)$ $(0 < a < b < 1)$ *the inequality*

$$\varphi(x) x^{-\alpha(x)} R_\alpha f(x) \leq c M_- f(x)$$

holds, where

$$M_- f(x) = \sup_{0 < h \leq x} \frac{1}{h} \int_{x-h}^{x} |f(y)| \, dy$$

and

$$\varphi(x) \equiv 2^{\alpha(x)} - 1. \tag{2.1}$$

Proof. We have

$$
\begin{aligned}
(R_\alpha f)(x) &= \int_0^x \frac{f(y)}{(x-y)^{1-\alpha(x)}} \, dy \\
&= \sum_{k=0}^{+\infty} \int_{x-\frac{x}{2^k}}^{x-\frac{x}{2^{k+1}}} f(y)(x-y)^{\alpha(x)-1} \, dy \\
&\leq \sum_{k=0}^{+\infty} \left(\frac{x}{2^{k+1}} \right)^{\alpha(x)-1} \int_{x-\frac{x}{2^k}}^{x-\frac{x}{2^{k+1}}} f(y) \, dy \\
&\leq \sum_{k=0}^{+\infty} \left(\frac{x}{2^{k+1}} \right)^{\alpha(x)-1} \int_{x-\frac{x}{2^k}}^{x} f(y) \, dy \\
&\leq c\, x^{\alpha(x)} M_- f(x) \sum_{k=1}^{+\infty} 2^{-\alpha(x)k} \\
&= c\, x^{\alpha(x)} M_- f(x) (\varphi(x))^{-1}
\end{aligned}
$$

and the lemma is proved ∎

Now we can derive a Hardy-type theorem in the spaces $L^{p(\cdot)}(0,1)$ (for the classical Lebesgue case see, e.g., [4: Section 329]).

Theorem 2.1. *Let* p *be a uniformly continuous function on* $(0,1)$ *which belongs to* W-Lip $(0,1)$. *Then there exists a constant* $c > 0$ *such that for all* $f \in L^{p(x)}(0,1)$ *the inequality*

$$\left\| x^{-\alpha(x)} \varphi(x)(R_\alpha f)(x) \right\|_{L^{p(x)}(0,1)} \leq c \left\| f(x) \right\|_{L^{p(x)}(0,1)}$$

holds, where φ *is defined by* (2.1).

Proof. The proof follows from Lemma 2.1, Theorem A and inequality $M_- f(x) \leq M f(x)$ ∎

Corollary 2.1. *Let* p *satisfy the conditions of Theorem* 2.1. *Then the operator* R_α *is bounded in* $L^{p(x)}(0,1)$.

Now we are able to investigate the boundedness of the Riesz potential I_α defined by

$$(I_\alpha f)(x) = \int_0^1 |x-y|^{\alpha(x)-1} f(y) \, dy \qquad (0 < \alpha(x) < 1)$$

in $L^{p(x)}(0,1)$ spaces.

Theorem 2.2. *Let p satisfy the conditions of Theorem 2.1 and assume that $\alpha \in W$-Lip $(0,1)$. Then the operator I_α is bounded in $L^{p(x)}(0,1)$.*

Proof. Taking into account the simple equality

$$I_\alpha f = R_\alpha f + \mathcal{W}_\alpha f, \qquad \text{where} \ \ (\mathcal{W}_\alpha f)(x) = \int_x^1 (y-x)^{\alpha(x)-1} f(y)\, dy$$

it is sufficient to establish the boundedness of $\mathcal{W}_\alpha$ in $L^{p(x)}(0,1)$. For this we observe that $\mathcal{W}_\alpha$ acts boundedly in $L^{p(x)}(0,1)$ if and only if R_α is bounded in $L^{p'(x)}(0,1)$, where $p'(x) = \frac{p(x)}{p(x)-1}$. Indeed, let $\|g\|_{p'(\cdot)} \leq 1$ and $\|f\|_{p(\cdot)} \leq 1$. Using the equivalence of the norms $\|\cdot\|^*_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$, Tonelli's theorem and Lemma 1.2, we find that

$$\|\mathcal{W}_\alpha(\cdot)f\|_{p(\cdot)} \leq c \sup_{\|g\|_{p'(x)} \leq 1} \left| \int_0^1 g(x)(\mathcal{W}_\alpha f)(x)\, dx \right|$$

$$\leq c \sup_{\|g\|_{p'(x)} \leq 1} \left(\int_0^1 |g(x)|(\mathcal{W}_\alpha |f|)(x)\, dx \right)$$

$$\leq c \sup_{\|g\|_{p'(x)} \leq 1} \int_0^1 |f(y)| \left(\int_0^y (y-x)^{\alpha(y)-1}|g(x)|\, dx \right) dy$$

$$\leq c \sup_{\|g\|_{p'(x)} \leq 1} \|f\|_{p(\cdot)} \|(R_\alpha|g|)\|_{p'(\cdot)}$$

$$\leq c.$$

Analogously, it follows that if $\mathcal{W}_\alpha$ is bounded in $L^{p(\cdot)}(0,1)$, then $\|(R_\alpha f)(\cdot)\|_{p'(\cdot)} \leq c$ for all $\|f(\cdot)\|_{p'(\cdot)} \leq 1$ ∎

Theorem 2.3. *Let $\alpha : \mathbb{R}_+ \to (0,1]$ be a non-decreasing function and let $p : \mathbb{R}_+ \to (1,\infty)$ be a measurable function. Assume also that $\overline{p}(x) \equiv p(\frac{x}{1-x})$ belongs to W-Lip$(0,1)$, where $x \in (0,1)$. Then there exists a constant $c > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}_+)$ the inequality*

$$\left\| v(x)(R_\alpha(f \cdot w))(x) \right\|_{p(x)} \leq c \, \|f(x)\|_{p(x)}$$

holds, where

$$w(x) = (1+x)^{\frac{2}{p(x)} - \alpha(x) - 1},$$

$$v(x) = (1+x)^{\frac{2}{p'(x)} + 1} x^{-\alpha(x)} \varphi(\tfrac{x}{x+1})$$

and φ is defined by (2.1).

Proof. Let $\|g\|_{p'(\cdot)} \leq 1$ and $\|f\|_{p(\cdot)} \leq 1$. Then using the change of variable $x = \frac{t}{1-t}$ $(t \in (0,1))$ and the notation $\overline{\psi}(t) \equiv \psi(\frac{t}{t-1})$ for measurable $\psi : (0,1) \to \mathbb{R}$,

we obtain

$$\int_{\mathbb{R}_+} v(x) \left| \int_0^x (x-y)^{\alpha(x)-1} f(y) w(y)\, dy \right| g(x)\, dx$$

$$= \int_0^1 \overline{v}(t) \left| \int_0^{\frac{t}{1-t}} \left(\frac{t}{1-t} - y \right)^{\overline{\alpha}(t)-1} f(y) w(y)\, dy \right| \overline{g}(t) \frac{dt}{(1-t)^2}$$

$$= \int_0^1 \overline{v}(t) \left(\int_0^t \frac{(t-\tau)^{\overline{\alpha}(t)-1}}{[(1-t)(1-\tau)]^{\overline{\alpha}(t)-1}} |\overline{f}(\tau)| \overline{w}(\tau) \frac{d\tau}{(1-\tau)^2} \right) \overline{g}(t) \frac{dt}{(1-t)^2}$$

$$\leq \int_0^1 \overline{v}(t) \left(\int_0^t (t-\tau)^{\overline{\alpha}(t)-1} (1-\tau)^{-\overline{\alpha}(t)-1} |\overline{f}(\tau)| \overline{w}(\tau)\, d\tau \right) (1-t)^{-\alpha(t)-1} \overline{g}(t)\, dt$$

$$\leq \int_0^1 \overline{v}(t) \left(\int_0^t (t-\tau)^{\overline{\alpha}(t)-1} (1-\tau)^{-\overline{\alpha}(\tau)-1} |\overline{f}(\tau)| \overline{w}(\tau)\, d\tau \right) (1-t)^{-\alpha(t)-1} \overline{g}(t)\, dt$$

$$\leq \left\| \overline{g}(t)(1-t)^{-\frac{2}{\overline{p}'(t)}} \right\|_{\overline{p}'(t)} \left\| \varphi(t) t^{-\alpha(t)} (R_\alpha \widetilde{f})(t) \right\|_{\overline{p}(t)},$$

where $\widetilde{f}(x) = |\overline{f}(x)|(1-x)^{-\frac{2}{\overline{p}'(x)}}$. Further, by Theorem 2.1 and Proposition 1.1 we have

$$\left\| \varphi(t) t^{-\alpha(t)} (R_\alpha \widetilde{f})(t) \right\|_{\overline{p}(t)} \leq c \left\| \widetilde{f}(t) \right\|_{\overline{p}(t)}$$

$$\leq c \left(\int_0^1 |\overline{f}(t)|^{\overline{p}(t)} (1-t)^{-2} dt \right)^{\frac{1}{P}}$$

$$= c \left(\int_{\mathbb{R}_+} |f(t)|^{p(t)}\, dt \right)^{\frac{1}{P}}$$

$$\leq c \left\| f(t) \right\|_{p(t)}^{\frac{p_0}{P}}$$

$$\leq c.$$

Analogously, using the change of variable $t = \frac{\tau}{1-\tau}$, we find $\left\| \overline{g}(t)(1-t)^{-\frac{2}{\overline{p}'(t)}} \right\|_{\overline{p}'(t)} \leq 1$. Taking into account the inequalities derived above, we conclude that

$$\left\| v(t)(R_\alpha(fw))(t) \right\|_{p(t)} \leq c, \quad \left\| f(\cdot) \right\|_{p(\cdot)} \leq 1$$

and the statement is proved ∎

Taking into account duality arguments and the equivalence of the norms $\| \cdot \|_{p(\cdot)}$ and $\| \cdot \|_{p(\cdot)}^*$, we easily obtain the following

Theorem 2.4. *Let $p : \mathbb{R}_+ \to (1, \infty)$ be such that $\overline{p}(x) \equiv p(\frac{x}{1-x})$ belongs to W-Lip$(0,1)$ $(x \in (0,1))$. Assume also that α is a non-decreasing function defined on $[0,1]$ and that $\alpha \in W$-Lip$(0,1)$. Then there exists a constant $c > 0$ such that for all $f \in L^{p(x)}(\mathbb{R}_+)$ the inequality*

$$\left\| v(x)(W_\alpha(f \cdot w))(x) \right\|_{p(x)} \leq c \left\| f(x) \right\|_{p(x)}$$

holds, where

$$v(x) = (1 + x)^{\frac{2}{p'(x)} - \alpha(x) - 1},$$

$$w(x) = (1 + x)^{\frac{2}{p(x)} + 1} x^{-\alpha(x)} \varphi\left(\frac{x}{x+1}\right)$$

and φ is defined by (2.1).

3. Compactness

In this section we study the compactness properties of the operators R_α, W_α and I_α in the spaces $L^{p(x)}(0, 1)$.

Lemma 3.1. *Let $1 < p_0 \leq p(x) \leq P < \infty$. Then*

$$\operatorname{supp} L^{p(x)}(\Omega) := \left\{ \cup \operatorname{supp} f : f \in L^{p(\cdot)}(\Omega) \right\} = \Omega,$$

where $\operatorname{supp} f := \{ x \in \Omega : f(x) \neq 0 \}$.

Proof. Let us represent $\Omega = \sum_n \Omega_n$, where Ω_n are subsets of Ω with finite measure. By Proposition 1.1 we conclude that the functions $f_n = \chi_{\Omega_n}$ belong to $L^{p(\cdot)}(\Omega)$ for every n ∎

The next lemma is taken from [6].

Lemma 3.2. *Let $1 < p_0 \leq p(x) \leq P < \infty$. Then the norm $\|\cdot\|_{p(\cdot)}$ is monotonic, i.e. if $|f(x)| \leq |g(x)|$ a.e. and $f, g \in L^{p(\cdot)}$, then $\|f(\cdot)\|_{p(\cdot)} \leq \|g(\cdot)\|_{p(\cdot)}$.*

Lemma 3.3 (see, e.g., [6]). *Let the function p satisfy the conditions of Lemma 3.2. Then $L^{p(\cdot)}$ is a Banach space.*

Lemma 3.4. *Let the function p satisfy the conditions of Lemma 3.2. Then the space $L^{p(\cdot)}(\Omega)$ has the property that if (g_n) is a sequence of functions such that $g_n \downarrow 0$ a.e., then $\|g_n(\cdot)\|_{p(\cdot)} \downarrow 0$.*

Proof. If $g_n \downarrow 0$ a.e., then $g_n(x)^{p(x)} \downarrow 0$ a.e. Hence using the Lebesgue monotone convergence theorem, we conclude that $I_p(g_n) \downarrow 0$ as $n \to \infty$. Consequently, $\|g_n(\cdot)\|_{p(\cdot)} \downarrow 0$ ∎

From this lemma we easily obtain

Lemma 3.5. *Let p satisfy the conditions of Lemma 3.2. Then from the fact $0 \leq g_n(x) \uparrow g(x)$ a.e., where $g \in L^{p(\cdot)}$, it follows that $\|g_n(\cdot)\|_{p(\cdot)} \to \|g(\cdot)\|_{p(\cdot)}$.*

In the sequel we shall denote by $L^{p(\cdot)}(\Omega)[L^{q(\cdot)}(\Omega)]$ the space of all measurable functions $k(s, t)$ on $\Omega \times \Omega$ satisfying the following conditions:

(i) The function $s \to k(s, t)$ belongs to $L^{q(s)}(\Omega)$ for a.a. $t \in \Omega$.

(ii) The function $\|k(\cdot, t)\|_{q(\cdot)}$ belongs to $L^{p(t)}(\Omega)$.

From Lemmas 3.1 - 3.5 we can obtain the following lemma (see [5: Chapter XI/ Section 4, Lemma 2]).

Lemma 3.6. *Let p and q be given measurable functions on Ω such that*

$$1 < p_0 \le p(x) \le P < \infty$$
$$1 < q_0 \le q(x) \le Q < \infty$$
$$(x \in \Omega)$$

where $p_0 = \operatorname{ess\,inf}_{x \in \Omega} p(x), P = \operatorname{ess\,sup}_{x \in \Omega} p(x), q_0 = \operatorname{ess\,inf}_{x \in \Omega} q(x), Q = \operatorname{ess\,sup}_{x \in \Omega} q(x)$. Then the space $L^{(p \cdot)}(\Omega)[L^{q(\cdot)}(\Omega)]$ contains an everywhere dense subset H_1 consisting of all functions of the form $\sum_{i=1}^{i_0} \chi_{B_i}(s)x_i(t)$ $(s, t \in \Omega)$ where the sets B_i are point-wise disjoint, $\chi_{B_i} \in L^{q(\cdot)}(\Omega)$ and $x_i \in L^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Now we are ready to prove

Theorem 3.1. *Let p and q satisfy the condition of Lemma 3.6. Then from the condition $M \equiv \big\| \|k(x, y)\|_{p'(y)} \big\|_{q(x)} < \infty$ the compactness of the operator K defined by*

$$(Kf)(x) = \int_\Omega k(x, y)f(y)\,dy \qquad (x \in \Omega)$$

from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ follows.

Proof. By Lemma 3.6 we know that the set of all functions

$$k_m(s, t) = \sum_{i=1}^m \eta_i(s)\lambda_i(t) \qquad (s, t \in \Omega)$$

is dense in $L^{q(\cdot)}[L^{p'(\cdot)}]$, where $\eta_i = \chi_{B_i}$ (B_i are point-wise disjoint sets) belongs to $L^{q(\cdot)}(\Omega)$ and $\lambda_i \in L^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. First we show the boundedness of K. By Hölder's inequality we have

$$|(Kf)(x)| \le k\|f(y)\|_{p(y)}\|k(x, y)\|_{p'(y)}.$$

Hence

$$\|(Kf)(\cdot)\|_{q(\cdot)} \le k\|f(\cdot)\|_{p(\cdot)} \big\| \|k(x, y)\|_{p'(y)} \big\|_{q(x)} \le kM\|f(\cdot)\|_{p(\cdot)}.$$

Moreover, $\|K\| \le kM$.

Now we prove the compactness of K. Let

$$(K_n\varphi)(x) = \int_\Omega k_n(x, y)\varphi(y)\,dy.$$

Note that

$$(K_n\varphi)(x) = \sum_{i=1}^n \eta_i(x) \int_\Omega \lambda_i(y)\varphi(y)\,dy \equiv \sum_{i=1}^n \eta_i(x)b_i,$$

where $b_i = \int_\Omega \lambda_i(y)\varphi(y)\,dy$. This means that K_n is a finite rank operator, and so it is compact. We have

$$\|K - K_n\| = \sup_{\|f\|_{p(\cdot)} \le 1} \|(K - K_n)f\|_{q(\cdot)}$$

$$\le c \sup_{\|f(\cdot)\|_{p(\cdot)} \le 1} \sup_{\|g(\cdot)\|_{q'(\cdot)} \le 1} \left| \int_\Omega [(K - K_n)f(x)]g(x)\,dx \right|.$$

Further, we find that

$$\left| \int_\Omega \left[(K - K_n) f(x) \right] g(x) \, dx \right|$$

$$\leq k \| g(\cdot) \|_{q'(\cdot)} \| (K - K_n) f(\cdot) \|_{q(\cdot)}$$

$$\leq k^2 \| g(\cdot) \|_{q'(\cdot)} \left\| \| k(x, y) - k_n(x, y) \|_{p'(y)} \right\|_{q(x)} \| f(\cdot) \|_{p(\cdot)}.$$

Let $\varepsilon > 0$. Then we can choose $n \in \mathbb{N}$ such that $\| K - K_n \| < \varepsilon$. Thus the operator K is compact as it is the limit of finite rank operators ∎

Theorem 3.2. *Let* $1 < p_0 \leq p(x) \leq P < \infty$ *and* $1 < q \leq q(x) \leq Q < \infty$. *Assume that* $0 < \alpha(x) \leq 1$ *for* $x \in (0, 1)$ *and that* $p \in W\text{-Lip}(0, 1)$. *Assume also that* $\alpha(x)p(x) \geq 1$ *for all* $x \in (0, 1)$ *and* $S \equiv \left\| \left[(\alpha(x) - 1)p'(x) + 1 \right]^{-1} \right\|_{q(x)} < \infty$. *Then the operator* R_α *is compact from* $L^{p(\cdot)}(0, 1)$ *to* $L^{q(\cdot)}(0, 1)$.

Proof. By Theorem 3.1 it is sufficient to show that

$$M \equiv \left\| \| (x - y)^{\alpha(x) - 1} \chi_{(0,x)}(y) \|_{p'(y)} \right\|_{q(x)} < \infty.$$

By Lemma 1.1 we have that $p' \in W\text{-Lip}(0, 1)$. Hence

$$\int_0^x (x - y)^{(\alpha(x) - 1)p'(y)} dy \leq c \int_0^x (x - y)^{(\alpha(x) - 1)p'(x)} dy$$

$$= c \frac{x^{(\alpha(x) - 1)p'(x) + 1}}{(\alpha(x) - 1)p'(x) + 1}$$

$$\leq \frac{c}{(\alpha(x) - 1)p'(x) + 1}$$

for every $x \in (0, 1)$. Consequently, by the condition $S < \infty$ and Proposition 1.1 we finally see that $M < \infty$ ∎

From this statement we obtain

Theorem 3.3. *Let* $1 < p_0 \leq p(x) \leq P < \infty$ *and* $1 < q_0 \leq q(x) \leq Q < \infty$. *Assume that* $p \in W\text{-Lip}(0, 1)$ *and* $\inf(\alpha(x)p(x)) > 1$. *Then the operator* R_α *is compact from* $L^{p(\cdot)}(0, 1)$ *to* $L^{q(\cdot)}(0, 1)$.

Proof. By Theorem 3.2 it suffices to show that that there exist a constant $\sigma > 0$ such that for all $x \in (0, 1)$ the inequality $\left[(\alpha(x) - 1)p'(x) + 1 \right]^{-1} \leq \sigma$ holds. Indeed, let $\lambda = \inf(\alpha(x)p(x))$ and let us choose σ such that $\alpha(x)p(x) > \sigma(P - 1) + 1$. Hence $(\alpha(x)p(x) - 1)p'(x) + 1 > \sigma$ for all $x \in (0, 1)$ ∎

Theorem 3.4. *Let* p, q *and* α *satisfy the conditions of Theorem* 3.2. *Then the operator* I_α *is compact from* $L^{p(\cdot)}(0, 1)$ *to* $L^{q(\cdot)}(0, 1)$.

Proof. It is obvious that $I_\alpha f = R_\alpha f + W_\alpha f$. By Theorem 3.2 R_α is compact. To show that W_α is also compact we observe that, by the condition $p' \in W\text{-Lip}(0, 1)$,

$$\int_x^1 (y - x)^{(\alpha(x) - 1)p'(y)} dy \leq c \int_x^1 (y - x)^{(\alpha(x) - 1)p'(x)} dy$$

$$\leq c \frac{(1 - x)^{(\alpha(x) - 1)p'(x) + 1}}{(\alpha(x) - 1)p'(x) + 1}.$$

Hence

$$\big\| \, \| (y - x)^{\alpha(x)-1} \chi_{(x,1)}(y) \|_{p'(y)} \big\|_{q(x)} < \infty$$

and the statement is proved ∎

Now from this statement we have

Theorem 3.5. *Let α, p and q satisfy the conditions of Theorem* 3.3. *Then the operator I_α is compact from $L^{p(\cdot)}(0,1)$ to $L^{q(\cdot)}(0,1)$.*

References

[1] Diening, L.: *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$.* C.R. Acad. Sci. Paris (to appear).

[2] Edmunds, D. E. and J. Rákosník: *Sobolev embeddings with variable exponent.* Studia Math. 143 (2000), 267 – 293.

[3] Edmunds, D. E. and J. Rákosník: *Sobolev embeddings with variable exponent* II. Math. Nachr. (to appear).

[4] Hardy, G. H., Littlewood, J. E. and G. Pólya: *Inequalities.* Cambridge: University Press 1934.

[5] Kantorovich, L. P. and G. P. Akilov: *Functional Analysis.* Oxford: Pergamon 1982.

[6] Kovácik, O. and J. Rákosník: *On spaces $L^{p(x)}$ and $W^{k,p(x)}$.* Czech. Math. J. 41 (1991), 592 – 618.

[7] Meskhi, A: *Solution of some weight problems for the Riemann-Liouville and Weyl operators.* Georgian Math. J. 5 (1998), 565 – 574.

[8] Newman, J. and M. Solomyak: *Two-sided estimates on singular values for a class of integral operators on the semi-axis.* Int. Equ. Oper. Theory 20 (1994), 335 – 349.

[9] Samko, S: *Diffentiation and integration of variable order and the spaces $L^{p(\cdot)}$.* Contemp. Math. 212 (1998), 203 – 219.

[10] Samko, S: *Convolution type operators in $L^{p(x)}$.* Int. Transf. & Spec. Funct. 7 (1998), 123 – 144.

[11] Samko, S: *Convolution and potential type operators in $L^{p(x)}(R^n)$.* Int. Transf. & Spec. Funct. 7 (1998), 261 – 284.

[12] Sharapudinov, I. I: *On a topology of the space $L^{p(x)}([0,1])$.* Mat. Zam. 26 (1979), 613 – 632.