JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 1, Number 1 (2003), 35-43

On a measure of non-compactness for singular integrals

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(Communicated by V. Kokilashvili)

1991 Mathematics Subject Classification. 42B20

Keywords and phrases. Singular integrals, Riesz transforms, Hilbert transforms, Cauchy integral, weight.

Abstract. It is proved that there exists no weight pair (v, w) for which a singular integral operator is compact from the weighted Lebesgue space $L^p_w(R^n)$ to $L^p_v(R^n)$. Moreover, a measure of non-compactness for this operator is estimated from below. Analogous problems for Cauchy singular integrals defined on Jordan smooth curves are studied.

1. Introduction

In this paper we show that there exists no weight pair (v, w) for which the singular integral operator

$$Kf(x) = p \cdot v \cdot \int_{R^n} k(x-y)f(y)\,dy$$

is compact from the weighted Lebesgue space $L^p_w(R^n)$ to $L^p_v(R^n)$. A measure of non-compactness (essential norm)

$$\|K\|_{\mathcal{K}} \equiv \text{dist}\left\{K, \mathcal{K}(L^p_w(R^n), L^p_v(R^n))\right\}$$

for the operator K, where $\mathcal{K}(L^p_w(R^n), L^p_v(R^n))$ is a space of all compact operators acting from $L^p_w(R^n)$ to $L^p_v(R^n)$, is estimated from below. We also consider analogous problems for the Cauchy singular operator

$$S_\Gamma f(t) = p\cdot v\cdot \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)}{\tau - t}\,d\tau,\ \ t = t(s),\ 0\le s\le l\le\infty,$$

along a smooth Jordan curve Γ (of the complex plane) on which the arc-length is chosen as a parameter.

The essential norm $\|S_T\|_{\overline{\mathcal{K}}} \equiv \operatorname{dist}\{S_T, \mathcal{K}(L^p(T), L^p(T))\}$ for the operator S_T, where T is the unit circle, was calculated in [11-12] for $p = 2^n$ and $p = \frac{2^n}{2^n-1}$. In that paper a lower estimate for $\|S_T\|_{\overline{\mathcal{K}}}$ was also derived for all $p\in(1,\infty)$. An upper estimate for $\|S_T\|_{\overline{\mathcal{K}}}$, $1<p<\infty$, was obtained in [17]. In the case of weighted Lebesgue spaces with power weights the essential norm of a Cauchy singular integral over Lyapunov curves was calculated in [16]. The case of general Muckenhoupt weights was considered in [8], where it was shown that $\operatorname{dist}\{S_T, \mathcal{K}(L^2_w(T), L^2_w(T))\} = 1$ if and only if w has a vanishing mean oscillation.

2. Preliminaries

Let w be a locally integrable almost everywhere positive function (i.e. a weight) on Ω, where Ω is a domain in R^n. Denote by $L^p_w(\Omega)$ $(1<p<\infty)$ the weighted Lebesgue space which is a space of all measurable functions $f:\Omega\to R$ with finite norm

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p}.$$

If $w\equiv 1$, then we denote $L^p_w(\Omega)$ by $L^p(\Omega)$.

Definition 2.1. Let $1<p<\infty$. We say that the weight w belongs to $A_p(R^n)$ if

$$\sup\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w^{1-p'}(x)\,dx\right)^{p-1} < \infty,\ \ p' = \frac{p}{p-1},$$

where the supremum is taken over all balls B in R^n and $|B|$ is a measure of B.

We denote

$$g_B := \frac{1}{|B|}\int_B |g(x)|dx$$

for a measurable function g and a ball $B\subset R^n$.

In the sequel we shall assume that there exists a positive constant c such that

$$\|Kf\|_{L^2(R^n)} \le c\|f\|_{L^2(R^n)},\quad f\in C_0^\infty(R^n). \tag{2.1}$$

We shall also suppose that the kernel k satisfies the following two conditions:

(i) there exists a positive constsnt A such that the inequality

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^{\alpha} k(x)\Big| \leq A|x|^{-n-\alpha} \tag{2.2}$$

holds for all $x \in R^n$, $x \neq 0$, and $|\alpha| \leq 1$;

(ii) there exists a positive constant b and an unit vector u_0 such that

$$\big|k(x)\big| \geq b|x|^{-n} \tag{2.3}$$

when $x = \lambda \cdot u_0$ with $-\infty < \lambda < +\infty$.

It is easy to see that the Riesz transforms

$$R_j f(x) = \lim_{r\to 0} \gamma_n \int_{R^n\setminus B(x,r)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy, \quad j = 1, \dots, n,$$

where $x = (x_1, \dots, x_n) \in R^n$, $\gamma_n = \Gamma[(n+1)/2]/\pi^{(n+1)/2}$, satisfy conditions (2.1)–(2.3). If $n = 1$, then $R_1 f(x)$ is the Hilbert transform defined by

$$Hf(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy.$$

It is known that H is bounded in $L^p_w(R)$, $1 < p < \infty$, if and only if $w \in A_p(R)$ (see [13]). In [2] it was proved that the Calderón- Zygmund singular operator is bounded in $L^p_w(R^n)$, $1 < p < \infty$, if $w \in A_p(R^n)$. The necessity of the condition $w \in A_p(R^n)$ for the boundedness of R_j was established in [9], p. 417.

Theorem A ([18], Ch. 5, 4.2; 4.6). *If conditions* (2.1)$-$(2.2) *are satisfied and $w \in A_p(R^n)$, then the operator K is bounded in $L^p_w(R^n)$. Further, if* (2.1)–(2.3) *hold and K is bounded in $L^p_w(R^n)$, then $w \in A_p(R^n)$.*

Finally we note that optimal sufficient conditions on radial weight pairs (v, w) governing the boundedness of the Calderón-Zygmund operators from $L^p_w(R^n)$ to $L^p_v(R^n)$ were established in [4]. An analogous problem for singular integrals defined on measure spaces with a quasi-metric and doubling measure (SHT) was solved in [5] (see also [10], Ch. 9 and [6], Ch. 7).

Suppose that X and Y are Banach spaces. We denote by $\mathcal{K}(X,Y)$ the space of all compact linear operators acting from X to Y. Let $F_r(X,Y)$ be the space of all finite rank operators from X to Y.

The following statement is from [3] (Corollary V. 5. 4).

Lemma 2.1. *Let $1 \leq p < \infty$. Suppose that P is a bounded linear operator from X to Y, where $Y = L^p(\Omega)$. Then*

$$\operatorname{dist}\big\{P, \mathcal{K}(X,Y)\big\} = \operatorname{dist}\big\{P, F_r(X,Y)\big\}.$$

In the sequel by $B(x,r)$ will be denoted the ball with center x and radius r.

The next statement is similar to Lemma V.5.6 from [3].

Lemma 2.2. *Let $P \in F_r(X, L^p(\Omega))$, where $X = L^r_w(\Omega)$, $1 < r$, $p < \infty$. Then for any $a \in \Omega$ and $\varepsilon > 0$ there exist $R \in F_r(X, L^p(\Omega))$ and a positive number α such that the inequality*

$$\|(P-R)f\|_{L_p(\Omega)} \le \varepsilon \|f\|_X$$

holds and $\operatorname{supp} Rf \subset \Omega \backslash B(a,\alpha)$ *for all $f \in X$.*

Proof. From $P \in F_r(X,\Omega)$ it follows that there exist linearly independent functions $u_j \in L^p(\Omega)$ such that

$$Pf(x) = \sum_{j=1}^{N} \beta_j(f) u_j(x), \quad f \in X,$$

where β_j are linear functionals defined on X (i.e., $\beta_j \in X^*$).

On the other hand, there exists a positive constant c for which the inequality

$$\sum_{j=1}^{N} |\beta_j(f)| \le c\|f\|_X$$

holds.

Let us choose linearly independent functions $\Phi_j \in L^p(R^n)$ and real numbers α_j such that

$$\|u_j - \Phi_j\|_{L^p(\Omega)} < \varepsilon, \quad j \in \{1,2,\dots,N\},$$

and $\operatorname{supp} \Phi_j \subset \Omega \backslash B(a,\alpha_j)$. If

$$Rf(x) = \sum_{j=1}^{N} \beta_j(f) \Phi_j(x),$$

then it is evident that $R \in F_r(X, L^p(\Omega))$ and also

$$\|Pf - Rf\|_{L^p(\Omega)} \le \sum_{j=1}^{N} |\beta_j(f)| \|u_j - \Phi_j\|_{L^p(\Omega)} \le c\varepsilon \|f\|_X$$

for all $f \in X$. Let $\alpha = \min\{\alpha_j\}$. Then $\operatorname{supp} Rf \subset \Omega \backslash B(a,\alpha)$. □

3. Main Results

To prove the main results we need the following statement (see [18], Ch. 5, Section 4.6):

Lemma 3.1. *Let u_0 be the unit vector in R^n. Then by choosing $u = tu_0$, with t fixed sufficiently large we can guarantee that*

$$|k(r(u+v)) - k(ru)| \leq \frac{1}{2}|k(ru)|$$

whenever $r \in R \setminus \{0\}$ and $|v| \leq 2$.

Lemma 3.2. *Let $1 < p < \infty$ and condition (2.3) be satisfied. Then from the boundedness of K from $L^p_w(R^n)$ to $L^p_v(R^n)$ it follows that $w^{1-p'}$ is locally integrable.*

Proof. Suppose that $I(r) := \int_B w^{1-p'}(x)dx = \infty$ for some positive r, where $B = B(0,r)$. Then there exists $g \in L^p(B)$, $g \geq 0$, such that $\int_B w^{-1/p} g = \infty$. Let us assume that $f_r(y) = g(y)w^{-1/p}(y)\chi_B(y)$ and $B' = B(ru, r)$, where $u = tu_0$ (t is from Lemma 3.1 and u_0 is the unit vector taken so that (2.3) holds). Obviously, $x = ru + ux'$ for $x \in B'$ and $y = ry'$ for $y \in B$, where $|x'| < 1$ and $|y'| < 1$. Thus $x - y = r(u+v)$ with $|v| < 2$ and consequently Lemma 3.1 yields $|Kf_r(x)| \geq \frac{1}{2}(f_r)_B |k(ru)|$ for all $x \in B'$. Hence by (2.3) the following estimates hold:

$$\|Kf_r\|_{L^p_v(R^n)} \geq \|\chi_{B'}(x) K f_r(x)\|_{L^p_v(R^n)} \geq$$
$$\geq \frac{b}{2rt}(v_{B'})^{1/p} f_B = \infty.$$

On the other hand, $\|f_r\|_{L^p_w(R^n)} = \|g\|_{L^p(B)} < \infty$. Finally we conclude that $I(r) < \infty$ for all $r > 0$. □

Theorem 3.1. *Let $1 < p < \infty$. Suppose that conditions (2.1) – (2.3) are satisfied. Then there exists no weight pair (v, w) such that the singular integral operator K is compact from $L^p_w(R^n)$ to $L^p_v(R^n)$. Moreover, if K is bounded from $L^p_w(R^n)$ to $L^p_v(R^n)$, then the inequality*

$$\|K\|_{\mathcal{K}} \geq c \operatorname*{ess\,sup}_{a \in R^n} \left(\frac{v(a)}{w(a)} \right)^{1/p} \tag{3.1}$$

holds, where the positive constant c depends only on n, t and b (see Lemma 3.1 and (2.3)).

Let K be bounded from $L^p_w(R^n)$ to $L^p_v(R^n)$. Suppose that

$$K_v f(x) = v^{1/p} K f(x).$$

Then it is easy to verify that

$$\|K\|_{\mathcal{K}} = \operatorname{dist}\{K_v, \mathcal{K}(L^p_w(R^n), L^p(R^n))\}.$$

Let $\lambda > \|K\|_{\mathcal{K}}$. Then by Lemma 2.1 we have $\lambda > \alpha(K)$, where $\alpha(K) := \operatorname{dist}\{K_v, F_r\}$, $F_r := F_r(L^p_w(R^n), L^p(R^n))$. Consequently there exists $P \in F_r$ for which

$$\|K_v - P\| < \lambda.$$

Let $a \in R^n$. Using Lemma 2.2 we find that there exist a positive number β and $R \in F_r$ such that

$$\|P - R\| < \frac{\lambda - \|K_v - P\|}{2}$$

and $\operatorname{supp} Rf \subset R^n \setminus B(a, \beta)$ for all $f \in L^p_w(R^n)$. Hence

$$\|K_v - R\| \leq \lambda.$$

Thus the inequality

$$\|(K_v - R)f\|_{L^p(R^n)} \leq \lambda \|f\|_{L^p_w(R^n)} \tag{3.2}$$

holds for all $f \in L^p_w(R^n)$.

Let $B \equiv B(a, r)$, where $r < \beta$. Suppose that B' is the translation of B in the direction of u, i.e. $B' = B(a + ru, r)$, where $u = tu_0$, t is taken so that the conditions of Lemma 3.1 are satisfied and u_0 is the unit vector chosen so that (2.3) holds. Let f be any non-negative function supported in B. Consider $Tf(x)$ for $x \in B'$. We have

$$Kf(x) = \int_B k(x - y) f(y) dy$$

with $x = a + ru + rx'$, $|x'| < 1$. Since $y \in B$, we find that $y = a + ry'$ for $|y'| < 1$. Thus $x - y = r(u + r(y' - x')) = r(u + v)$ with $|v| < 2$. Further Lemma 3.1 and condition (2.3) yield

$$|Kf(x)| \geq \frac{1}{2} f_B |k(ru)| \geq c f_B \frac{1}{|B|}, \tag{3.3}$$

for all $x \in B'$, where $|B|$ denotes a measure of B and c is the positive constant depending only on n, b and t. Due to inequality (3.2) we obtain

$$\int_{B'} v(x) \left| \int_B k(x - y) f(y) dy \right|^p dx \leq \lambda^p \int_B (f(y))^p w(y) dy$$

for all non-negative f with $\operatorname{supp} f \subset B$. Let $f(x) = w^{1-p'}(x)\chi_B(x)$. Then using (3.3), we find that

$$c^p \left(\int_{B'} v(x) dx \right) f_B^p \frac{1}{|B|^p} \leq \lambda^p \int_B w^{1-p'}(y) dy.$$

Consequently by Lemma 3.2 we have

$$c^p v_{B'} ((w^{1-p'})_B)^{p-1} \leq \lambda^p. \tag{3.4}$$

Further, observe that the equality

$$\lim_{r \to 0} v_{B'} = v(a) \tag{3.5}$$

holds for almost all a. This follows from the obvious fact

$$|v_{B'} - v(a)| \leq \overline{c} \frac{1}{|\overline{B}|} \int_{\overline{B}} |v(x) - v(a)| dx \to 0$$

as $r \to 0$, where $\overline{B} = B(a, r(t+1))$ and $\overline{c}$ is a positive constant.

Inequalities (3.4) and (3.5) yield

$$c\left(\frac{v(a)}{w(a)}\right)^{1/p} \le \lambda$$

for almost all a; here the positive constant c depends only on a, n and t. As λ is an arbitrary number greater than $\|K\|_{\mathcal{K}}$, we conclude that (3.1) holds.

An analogous result for maximal functions was derived in [7].

We recall that Γ is called a smooth curve if $t'(s)$ is continuous (and in the case of its closedness , $t'(0) = t'(l)$). For smooth curves the boundedness of S_Γ in $L^p(\Gamma)$, $1 < p < \infty$, was derived in [1]. In [15] (see also [14], pp. 55-56) it was proved that S_Γ is bounded in $L^p_w(0,l)$ if and only if $w \in A_p(0,l)$.

Theorem 3.2. *Let $1 < p < \infty$. Suppose that Γ is a Jordan smooth curve. Then there exists no weight pair (v,w) such that the singular integral operator S_Γ is compact from $L^p_w(0,l)$ to $L^p_v(0,l)$. Moreover, if S_Γ is bounded from $L^p_w(0,l)$ to $L^p_v(0,l)$, then the inequality*

$$\|S_\Gamma\|_{\mathcal{K}} \equiv \operatorname{dist}\{S_\Gamma, \mathcal{K}(L^p_w(0,l), L^p_v(0,l))\} \ge \frac{1}{4\pi} \operatorname*{ess\,sup}_{a\in(0,l)} \left(\frac{v(a)}{w(a)}\right)^{1/p} \tag{3.6}$$

holds.

Proof. Let S_Γ be bounded from $L^p_w(0,l)$ to $L^p_v(0,l)$ and $a \in (0,l)$. Then, using Lemmas 2.1 and 2.2, there exist a positive number β and $R \in F_r(L^p_w(0,l), L^p_v(0,l))$ such that

$$\|(S_{\Gamma,v} - R)f\|_{L^p(0,l)} \le \lambda \|f\|_{L^p_w(0,l)}, \quad f \in L^p_w(0,l), \tag{3.7}$$

and $\operatorname{supp} Rf \subset (0,l) \setminus I(a,\beta)$, where

$$S_{\Gamma,v} f(x) = v^{1/p}(x) S_\Gamma f(x),$$

$I(a,\beta) = (a-\beta, a+\beta)$ and $\lambda > \|S_\Gamma\|_{\mathcal{K}}$. Further, let $I_1 := (a-r, a)$, $I_2 := (a, a+r)$, where $r < \beta$. Suppose that $\varphi(s) = f(t(s))$ is a non-negative function with $\operatorname{supp} \varphi \subset I_2$. Then (see [15], [14], p.56)

$$|S_\Gamma f(t(\sigma))| \ge \frac{1}{2\pi} \int_{I_2} \frac{\varphi(s)}{s-\sigma} ds \ge \frac{1}{4r\pi} \int_{I_2} \varphi(s)\, ds$$

for $\sigma \in I_1$ and sufficiently small r. Thus we have

$$|S_\Gamma f(t(\sigma))| \ge \left(\frac{1}{4r\pi} \int_{I_2} \varphi(s) ds\right) \chi_{I_1}(\sigma) \tag{3.8}$$

for any σ. Taking into account inequality (3.8) and the proof of Lemma 3.2 we have that $w^{1-p'}$ is locally integrable. Let $\varphi(s) = w^{1-p'}(s)\chi_{I_2}(s)$. Then by (3.7) we arrive at a conclusion that

$$\frac{1}{(4\pi)^p}\left(\frac{1}{r}\int_a^{a+r} w^{1-p'}(s)ds\right)^{p-1}\left(\frac{1}{r}\int_{a-r}^{a} v(s)ds\right) \leq \lambda^p$$

holds for all $a \in (0,l)$. The latter inequality yields (3.6). □

Remark. From the proof of Theorem 3.2 it easily follows that for the Hilbert transform H the constant c is equal to $\frac{1}{2\pi}$.

Acknowledgements. The research was partially supported by Grant No. 1.7 of the Georgian Academy of Sciences. I express my gratitude to Prof. V. Kokilashvili for drawing my attention to the problems considered in this paper and helpful remarks.

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(*Received : December 2002*)


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