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ON TWO-WEIGHT INEQUALITIES FOR MULTIPLE HARDY–TYPE OPERATORS

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In this note necessary and sufficient conditions on a pair of weights guaranteeing two-weight inequalities for the multiple Hardy-type transform

$$R_{\alpha,\beta}f(x,y)=\int\limits_0^x\int\limits_0^y \frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau \quad \alpha,\beta\geq 1,$$

are presented provided that the weight on the right–hand side satisfies some additional conditions.

Let ρ be an almost everywhere positive function on a subset E of R^n. We denote by $L^p_\rho(E)$, $1<p<\infty$, the set of all measurable functions $f: E\to R^1$ for which the norm

$$\|f\|_{L^p_\rho(E)}=\Bigg(\int\limits_E |f(x)|^p\rho(x)dx\Bigg)^{1/p}$$

is finite.

A solution of the two-weight problem for the two-dimensional Hardy operator

$$H_2 f(x,y)=\int\limits_0^x\int\limits_0^y f(t,\tau)dtd\tau, \quad x,y>0.$$

has been found in 1985 by E. Sawyer [13]. Namely he proved

Theorem A. *Let $1<p\leq q<\infty$. Then for the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ it is necessary and sufficient that the following three independent conditions are satisfied:*

(i)

$$A=:\sup_{a,b>0}(H_2'v(a,b))^{1/q}(H_2\sigma(a,b))^{1/p'}<\infty, \qquad (*)$$

where $\sigma=:w^{1-p'}$, $p'=\frac{p}{p-1}$;

(ii)

$$\int\limits_0^a\int\limits_0^b (H_2\sigma)^q v\leq A^q[H_2\sigma(a,b)]^{p/q}$$

for all $a,b>0$;

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(iii)

$$\int_a^\infty \int_b^\infty (H_2' v)^{p'} \sigma \le A^{p'} [H_2' v(a,b)]^{p'/q'}$$

for all $a, b > 0$, *where*

$$H_2' f(x,y) = \int_x^\infty \int_y^\infty f(t,\tau) dt d\tau, \quad x, y > 0.$$

In her doctoral thesis A. Wedestig [17] derived a two-weight criterion for the operator H_2 when the weight on the right–hand side is a product of two functions of separate variables. In particular, she proved

Theorem B. *Let* $1 < p \le q < \infty$ *and let* $s_1, s_2 \in (1,p)$. *Suppose that the weight function* w *on* R_+^2 *has the form* $w(x,y) = w_1(x) w_2(y)$. *Then for the boundedness of the operator* H_2 *from* $L_w^p(R_+^2)$ *to* $L_v^q(R_+^2)$ *it is necessary and sufficient that*

$$A(s_1,s_2) =: \sup_{t_1,t_1>0} W_1(t_1)^{(s_1-1)/p} W_2(t_2)^{(s_2-1)/p} \times$$

$$\times \Bigg(\int_{t_1}^\infty \int_{t_2}^\infty v(x_1,x_2) W_1(x_1)^{\frac{q}{p}(p-s_2)} dx_1 dx_2 \Bigg)^{1/q} < \infty,$$

where $W_1(t_1) = \int_0^{t_1} w_1^{1-p'}(x_1) dx_1$ *and* $W_2(t_2) = \int_0^{t_2} w_2^{1-p'}(x_2) dx_2$.

Earlier some sufficient conditions for the validity of the two-weight inequality for H_2 were established in [14] and [16].

Necessary and sufficient conditions on the weight function v on R_+^2 governing the trace inequality

$$\Bigg(\int_0^\infty \int_0^\infty |R_{\alpha,\beta} f(x,y)|^q v(x,y) dx dy \Bigg)^{1/q} \le c \Bigg(\int_0^\infty \int_0^\infty |f(x,y)|^p dx dy \Bigg)^{1/p},$$

$$1 < p \le q < \infty,$$

for the Riemann-Liouville operator with multiple kernels $R_{\alpha,\beta}$ $\alpha, \beta > 1/p$, have been obtained in [6]. Analogous problem has been solved in [7] for $0 < \alpha < 1/p$ and $\beta > 1/p$

A solution of the two-weight problem for the one-dimensional Hardy transform

$$Hf(x) = \int_0^x f(t) dt$$

has been given by B. Muchenhoupt [11] for $1 < p = q < \infty$; by V. Kokilashvili [4], J. Bradley [1] and V. Maz'ya [9] (Ch.1) for $1 < p \le q < \infty$:

Later on F. J. Martin–Reyes and E. Sawyer [8] and V. Stepanov [15] established two-weight criteria for the Riemann-Liouville transform

$$R_\alpha f(x) = \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} dy$$

for $\alpha > 1$.

Criteria for the boundedness of R_α from $L^p(R_+)$ to $L_v^q(R_+)$ when $1 < p \le q < \infty$ and $\alpha > 1/p$ have been found in [10] (see also [12]), while the similar result has been derived in [5] for $1 < p \le q < \infty$ and $0 < \alpha < 1/p$ (see also [2], Ch. 2). When $1 < p < q < \infty$ a solution of the two-weight problem for potential operators has been given in [3].

Definition. A nonnegative function $\rho : R^2_+ \to R^1$ is said to be a weight function with doubling condition uniformly with respect to $x \in R_+$ if there exists a positive constant c such that for arbitrary $t > 0$ and almost all $x > 0$ the inequality

$$\int_0^{2t} \rho(x,y)dy \le c \int_0^t \rho(x,y)dy$$

holds. In this case we write $\rho \in DC(y)$. Analogously we define the class $DC(x)$.

The main results of the present note are the following statements:

Theorem 1. *Let $1 < p \le q < \infty$ and let $\alpha, \beta \ge 1$. Suppose that $w^{1-p'} \in DC(y)$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if*

$$\text{(i)} \qquad A_1 =: \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty;$$

$$\text{(ii)} \qquad A_2 =: \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dxdy \right)^{1/p'} \times$$

$$\times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q} y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty.$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{A_1, A_2\}$.

Theorem 2. Let $1 < p \le q < \infty$ and let $\alpha, \beta \ge 1$. Suppose that $w^{1-p'} \in DC(x)$. Then the operator $R_{\alpha,\beta}$ is bounded from $L^p_w(R^2_+)$ to $L^q_v(R_+)$ if and only if

$$\text{(i)} \qquad B_1 =: \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(b-y)^{(1-\beta)q}} dxdy \right)^{1/p'} \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty;$$

$$\text{(ii)} \qquad B_2 =: \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dxdy \right)^{1/p'} \times$$

$$\times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(y-b)^{(1-\beta)q} x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty.$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{B_1, B_2\}$.

Corollary. Let $1 < p \le q < \infty$. Suppose that $w^{1-p'} \in DC(x)$ or $w^{1-p'} \in DC(y)$. Then the operator H_2 is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if the condition $(*)$ of Theorem A holds.

More general form of this corollary is the next statement:

Theorem 3. *Let $1 < p \le q < \infty$. Assume that the weight function $w^{1-p'}$ satisfies the condition*

$$\sup_{\substack{x>0 \\ k \in Z}} \left(\sum_{j=k}^\infty \left(\int_0^{2^j} w^{1-p'}(x,y)dy \right)^{1-p} \right) \left(\int_0^{2^{k+1}} w^{1-p'}(x,y)dx \right)^{p-1} < \infty.$$

Then the boundedness of H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ is equivalent to the condition $(*)$ of Theorem A.

The following theorem states that if the weight function w has the form $w(x,y) = w_1(x)w_2(y)$, then the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ is equivalent to the first condition in the Sawyer's theorem.

Theorem 4. *Let $1 < p \le q < \infty$ and $w(x,y) = w_1(x)w_2(y)$. Then the operator H_2 is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if*

$$D =: \sup_{a,b>0} \bigg(\int_0^a w_1^{1-p'}(x)dx \bigg)^{1/p'} \bigg(\int_0^b w_2^{1-p'}(y)dy \bigg)^{1/p'} \times$$
$$\times \bigg(\int_a^\infty \int_b^\infty v(x,y)\, dxdy \bigg)^{1/q} < \infty.$$

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