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A note on two-weight inequalities for multiple Hardy-type operators

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Abstract. Necessary and sufficient conditions on a pair of weights guaranteeing two-weight estimates for the multiple Riemann-Liouville transforms are established provided that the weight on the right–hand side satisfies some additional conditions.

1. Introduction

In 1985 E. Sawyer [15] solved the two-weight problem for the two-dimensional Hardy transform

$$H_2 f(x,y) = \int_0^x \int_0^y f(t,\tau)\,dt\,d\tau, \quad x, y > 0.$$

Namely he proved the following statement:

Theorem A. *Let $1 < p \le q < \infty$. Then for the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ it is necessary and sufficient that the following three independent conditions are satisfied:*

(i)

$$A =: \sup_{a,b>0} (H_2' v(a,b))^{1/q} (H_2 \sigma(a,b))^{1/p'} < \infty, \tag{1.1}$$

where $\sigma =: w^{1-p'}$, $p' = \frac{p}{p-1}$;

(ii)

$$\int_0^a \int_0^b (H_2\sigma)^q v \leq A^q [H_2 \sigma(a,b)]^{p/q}$$

for all $a, b > 0$;

(iii)

$$\int_a^\infty \int_b^\infty (H_2' v)^{p'} \sigma \leq A^{p'} [H_2' v(a,b)]^{p'/q'}$$

for all $a, b > 0$, *where*

$$H_2' f(x,y) = \int_x^\infty \int_y^\infty f(t,\tau) dt d\tau, \quad x, y > 0.$$

In her doctoral thesis A. Wedestig [20] derived a two-weight criterion for the operator H_2 when the weight on the right–hand side is a product of two functions of separate variables. In particular, she proved

Theorem B. *Let* $1 < p \leq q < \infty$ *and let* $s_1, s_2 \in (1, p)$. *Suppose that the weight function* w *on* R^2_+ *has the form* $w(x,y) = w_1(x) w_2(y)$. *Then for the boundedness of the operator* H_2 *from* $L^p_w(R^2_+)$ *to* $L^q_v(R^2_+)$ *it is necessary and sufficient that*

$$\begin{aligned} A(s_1, s_2) \quad =: & \sup_{t_1, t_1 > 0} W_1(t_1)^{(s_1-1)/p} W_2(t_2)^{(s_2-1)/p} \\ & \times \left(\int_{t_1}^\infty \int_{t_2}^\infty v(x_1, x_2) W_1(x_1)^{\frac{q}{p}(p-s_2)} dx_1 dx_2 \right)^{1/q} < \infty, \end{aligned}$$

where $W_1(t_1) = \int_0^{t_1} w_1^{1-p'}(x_1) dx_1$ *and* $W_2(t_2) = \int_0^{t_2} w_2^{1-p'}(x_2) dx_2$.

Earlier some sufficient conditions for the validity of the two-weight inequality for H_2 were established in [16] and [19].

Necessary and sufficient conditions on the weight function v on R^2_+ governing the trace inequality

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty |R_{\alpha,\beta} f(x,y)|^q v(x,y) dx dy \right)^{1/q} \\ & \quad \leq c \left(\int_0^\infty \int_0^\infty |f(x,y)|^p dx dy \right)^{1/p}, \quad 1 < p \leq q < \infty, \end{aligned}$$

for the Riemann-Liouville operator with multiple kernels

$$R_{\alpha,\beta}f(x,y)=\int_0^x\int_0^y\frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau,$$

where $\alpha,\beta>1/p$, have been obtained in [8]. Analogous problem has been solved in [9] for $0<\alpha<1/p$ and $\beta>1/p$.

In this paper we establish boundedness criteria for the operator $R_{\alpha,\beta}$, $\alpha,\beta>1$, from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ when the weight w satisfies the one-dimensional doubling condition uniformly with respect to another variable. As a corollary we conclude that under this restriction the two-weight inequality for the operator H_2 holds if and only if the condition (1.1) is satisfied. When the weight function w has the form $w(x,y)=w_1(x)w_2(y)$ we show that also in this case a two-weight criterion for H_2 is (1.1) .

2. Preliminaries

Let ρ be an almost everywhere positive function on a subset E of R^n. We denote by $L^p_\rho(E)$, $1<p<\infty$, the set of all measurable functions $f:E\to R^1$ for which the norm

$$\|f\|_{L^p_\rho(E)}=\bigg(\int_E|f(x)|^p\rho(x)dx\bigg)^{1/p}$$

is finite.

Let us recall some well-known results for one-dimensional Hardy-type transforms.

A solution of the two-weight problem for the one-dimensional Hardy transform

$$Hf(x)=\int_0^x f(t)dt$$

has been given by B. Muchenhoupt [13] for $p=q$; by V. Kokilashvili [6], J. Bradley [2] and V. Maz'ya [11, Chapter 1] for $p\le q$. Namely the following statement holds.

Theorem C. *Let $1<p\le q<\infty$. Then the inequality*

$$\bigg(\int_0^\infty\bigg|\int_0^x f(y)dy\bigg|^q dx\bigg)^{1/q}\le c\bigg(\int_0^\infty|f(x)|^p dx\bigg)^{1/p}\tag{2.1}$$

with the positive constant c independent of f holds if and only if

$$A\equiv\sup_{t>0}\bigg(\int_t^\infty v(x)dx\bigg)^{1/q}\bigg(\int_0^t w^{1-p'}(x)dx\bigg)^{1/p'}<\infty.$$

Moreover, if c is the best constant in (2.1), *then there exists a positive constant b depending only on p and q such that the inequality $A \le c \le bA$ holds.*

Later on F. J. Martin–Reyes and E. Sawyer [10] and V. Stepanov [17] proved the next statement, which gives two-weight criteria for the Riemann-Liouville transform

$$R_\alpha f(x) = \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} dy,$$

where $\alpha > 1$.

Theorem D. *Let $1 < p \le q < \infty$, $\alpha > 1$. Then the operator R_α acts boundedly from $L^p_w(R_+)$ to $L^q_v(R_+)$ if and only if the following two conditions*

$$A_1 =: \sup_{t>0} \left(\int_t^\infty \frac{v(x)}{(x-t)^{(1-\alpha)q}} dx \right)^{1/q} \left(\int_0^t w^{1-p'}(x) dx \right)^{1/p'} < \infty;$$

$$A_2 =: \sup_{t>0} \left(\int_t^\infty v(x) dx \right)^{1/q} \left(\int_0^t \frac{w^{1-p'}(x)}{(t-x)^{(1-\alpha)p'}} dx \right)^{1/p'} < \infty$$

hold. Moreover, there exist positive constants c_1 and c_2 depending only on α, p and q such that $c_1 \max\{A_1, A_2\} \le \|R_\alpha\| \le c_2 \max\{A_1, A_2\}$.

Criteria for the boundedness of R_α from $L^p(R_+)$ to $L^q_v(R_+)$ when $1 < p \le q < \infty$ and $\alpha > 1/p$ have been obtained in [12] (see also [14]), while the similar result has been derived in [7], [3, Chapter 2], for $1 < p \le q < \infty$ and $0 < \alpha < 1/p$. When $1 < p < q < \infty$ a solution of the two-weight problem for potential operators has been given in [5].

The next statement concerning the discrete Hardy operator defined on Z perhaps is already known, but we give the proof of the theorem for the completeness (see also [1], [4] for two-weight criteria for the Hardy transform on Z_+):

Theorem E. *Let $1 < p \le q < \infty$ and let $\{a_n\}$, $\{b_n\}$ be positive sequences. The inequality*

$$\left(\sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{n} g_k \right|^q a_n^q \right)^{1/q} \le c \left(\sum_{n=-\infty}^{\infty} |g_n|^p b_n^p \right)^{1/p} \tag{2.2}$$

with the positive constant c independent of $\{g_k\}$ $(g_k \in l^p_{b^p_n}(Z))$, *holds if and only if*

$$B =: \sup_{n\in Z}\left(\sum_{k=n}^{\infty} a_k^q\right)^{1/q}\left(\sum_{k=-\infty}^{n} b_k^{-p'}\right)^{1/p'} < \infty.$$

Moreover, if c is the best constant in (2.2), *then*

$$B \le c \le Bq^{\frac{1}{q}}\left(\frac{q}{q-1}\right)^{(p-1)/p}.$$

Proof (*Sufficiency*). Let $\alpha_n = \left(\sum\limits_{k=-\infty}^{n} b_k^{-p'}\right)^{\frac{1}{qp'}}$. Due to Hölder's inequality we have

$$\begin{aligned}
&\left(\sum_{n=-\infty}^{\infty} a_n^q\left|\sum_{k=-\infty}^{n} f_k\right|^q\right)^{p/q}\\
&\le\left(\sum_{n=-\infty}^{\infty} a_n^q\left(\sum_{k=-\infty}^{n}|f_k\alpha_k b_k|^p\right)^{q/p}\left(\sum_{k=-\infty}^{n}(\alpha_k b_k)^{-p'}\right)^{\frac{q}{p'}}\right)^{p/q}\\
&=:\left(\sum_{n=-\infty}^{\infty} A_n\right)^{\frac{p}{q}}.
\end{aligned}$$

By Minkowsky's inequality $\left(\frac{q}{p}\ge 1\right)$ we obtain

$$\begin{aligned}
\left(\sum_{n=-\infty}^{\infty} A_n\right)^{p/q} &= \left(\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{n}|f_k\alpha_k b_k|^p a_n^p\left(\sum_{k=-\infty}^{n}(\alpha_k b_k)^{-p'}\right)^{\frac{p}{p'}}\right)^{\frac{q}{p}}\right)^{\frac{p}{q}}\\
&\le \sum_{k=-\infty}^{\infty}|f_k\alpha_k b_k|^p\left(\sum_{n=k}^{\infty} a_n^q\left(\sum_{k=-\infty}^{n}(\alpha_k b_k)^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{p}{q}}\\
&=:\sum_k D_k.
\end{aligned}$$

For the intrinsic sum we have

$$\sum_{n=k}^{\infty} a_n^q\left(\sum_{k=-\infty}^{n}(\alpha_k b_k)^{-p'}\right)^{\frac{q}{p'}} = \sum_{n=k}^{\infty} a_n^q\left(\sum_{k=-\infty}^{n}\left(\sum_{i=-\infty}^{k} b_k^{-p'}\right)^{-\frac{1}{q}} b_k^{-p'}\right)^{\frac{q}{p'}}.$$

Moreover, the next easily verifiable inequality

$$\sum_{k=-\infty}^{n} b_k^{-p'}\left(\sum_{i=-\infty}^{k} b_i^{-p'}\right)^{-\frac{1}{q}} \le q'\left(\sum_{k=-\infty}^{n} b_k^{-p'}\right)^{1/q'}$$

gives

$$\sum_{n=k}^{\infty} a_n^q\left(\sum_{i=-\infty}^{n} (\alpha_i b_i)^{-p'}\right)^{\frac{q}{p'}} = (q')^{\frac{q}{p'}}\sum_{n=k}^{\infty} a_n^q\left(\sum_{i=-\infty}^{n} b_i^{-p'}\right)^{\frac{q}{q'p'}}.$$

Further, the latter sum does not exceed

$$\lambda = B^{\frac{q}{q'}}(q')^{\frac{q}{p'}}\sum_{n=k}^{\infty} a_n^q\left(\sum_{i=n}^{\infty} a_i^q\right)^{-\frac{1}{q'}}.$$

The inequality

$$\sum_{n=k}^{\infty} a_n^q\left(\sum_{i=n}^{\infty} a_i^q\right)^{-\frac{1}{q'}} \le q\left(\sum_{i=k}^{\infty} a_i^q\right)^{1/q},$$

can also be easily verified. Therefore we obtain

$$\begin{aligned}
\lambda &\le qB^{\frac{q}{q'}}(q')^{\frac{q}{p'}}\left(\sum_{i=k}^{\infty} a_i^q\right)^{1/q} \\
&\le qB\cdot B^{\frac{q}{q'}}(q')^{\frac{q}{p'}}\left(\sum_{i=0}^{k} b_i^{-p'}\right)^{-\frac{1}{p'}} \\
&= qB^q(q')^{\frac{q}{p'}}\alpha_k^{-q}.
\end{aligned}$$

Finally we have

$$\sum_{k\in Z} D_k \le q^{\frac{p}{q}}B^p(q')^{\frac{p}{p'}}\sum_{k=-\infty}^{\infty}|f_k\alpha_k b_k|^p\alpha_k^{-p} = q^{\frac{p}{q}}B^p(q')^{\frac{p}{p'}}\sum_{k=-\infty}^{\infty}|f_k|^p\beta_k^p.$$

In order to prove necessity we take the sequence

$$g_k = \begin{cases}\beta_k^{-p'}, & k\le n,\\ 0, & k>n.\end{cases}$$

Then we have

$$\left(\sum_{i\in Z}^{\infty}\left(\sum_{k=-\infty}^{i} g_k\right)^q a_i^q\right)^{1/q} \geq \left(\sum_{i=n}^{\infty} a_i^q\right)^{1/q}\left(\sum_{k=-\infty}^{n}\beta_k^{-p'}\right).$$

On the other hand,

$$\left(\sum_{k=-\infty}^{\infty}|g_k|^p\beta_k^p\right)^{1/p} = \left(\sum_{k=-\infty}^{n}\beta_k^{-p'}\right)^{1/p}$$

and finally

$$B<\infty. \qquad \square$$

Analogously it follows

Theorem F. *Let* $1<p\le q<\infty$ *and let* m *be an integer. Suppose that* $\{a_n\}_{n=-\infty}^m$, $\{b_n\}_{n=-\infty}^m$ *are positive sequences. Then the two-weight inequality*

$$\left(\sum_{n=-\infty}^{m}\left|\sum_{k=-\infty}^{n} g_k\right|^q a_n^q\right)^{1/q} \le c\left(\sum_{n=-\infty}^{m}|g_n|^p b_n^p\right)^{1/p} \tag{2.3}$$

holds if and only if

$$B_m =: \sup_{-\infty<n\le m}\left(\sum_{k=n}^{m} a_k^q\right)^{1/q}\left(\sum_{k=-\infty}^{n} b_k^{-p'}\right)^{1/p'} <\infty.$$

Moreover, if c *is the best constant in* (2.3), *then*

$$B_m \le c \le B_m q^{\frac{1}{q}}\left(\frac{q}{q-1}\right)^{(p-1)/p}.$$

3. The Main Results

In order to formulate the main results of this paper we need the following definition:

Definition. *A nonnegative function* $\rho: R_+^2 \to R^1$ *is said to be a weight function with doubling condition uniformly with respect to* $x\in R_+$ *if there exists a positive constant* c *such that for arbitrary* $t>0$ *and almost all* $x>0$ *the inequality*

$$\int_0^{2t} \rho(x,y)dy \le c \int_0^t \rho(x,y)dy$$

holds. In this case we write $\rho \in DC(y)$*. Analogously we define the class* $DC(x)$*.*

Note that if the weight ρ is integrable on $[0,a]^2$, $a>0$, then $\rho \in DC(y)$ is equivalent to the condition: there exists a constant $c>0$ such that for all intervals of finite length $I \subset R_+$ and all $t>0$ the inequality

$$\int_I \int_0^{2t} \rho(x,y)dxdy \le c \int_I \int_0^t \rho(x,y)dxdy$$

holds.

Theorem 3.1. *Let* $1 < p \le q < \infty$ *and let* $\alpha, \beta \ge 1$*. Suppose that* $w^{1-p'} \in DC(y)$*. Then the operator* $R_{\alpha,\beta}$ *is bounded from* $L^p_w(R^2_+)$ *to* $L^q_v(R^2_+)$ *if and only if*

(i)

$$\begin{aligned} A_1 \quad &=: \quad \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}} dxdy \right)^{1/p'} \\ &\quad \times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty; \end{aligned}$$

(ii)

$$\begin{aligned} A_2 \quad &=: \quad \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dxdy \right)^{1/p'} \\ &\quad \times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(x-a)^{(1-\alpha)q} y^{(1-\beta)q}} dxdy \right)^{1/q} < \infty. \end{aligned}$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{A_1, A_2\}$*.*

Theorem 3.2. *Let* $1 < p \le q < \infty$ *and let* $\alpha, \beta \ge 1$*. Suppose that* $w^{1-p'} \in DC(x)$*. Then the operator* $R_{\alpha,\beta}$ *is bounded from* $L^p_w(R^2_+)$ *to* $L^q_v(R_+)$ *if and only if*

(i)

$$B_1 =: \sup_{a,b>0} \left(\int_0^a \int_0^b \frac{w^{1-p'}(x,y)}{(b-y)^{(1-\beta)q}} dxdy \right)^{1/p'} \times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty;$$

(ii)

$$B_2 =: \sup_{a,b>0} \left(\int_0^a \int_0^b w^{1-p'}(x,y) dxdy \right)^{1/p'} \times \left(\int_a^\infty \int_b^\infty \frac{v(x,y)}{(y-b)^{(1-\beta)q} x^{(1-\alpha)q}} dxdy \right)^{1/q} < \infty.$$

Moreover, $\|R_{\alpha,\beta}\| \approx \max\{B_1, B_2\}$.

Corollary 3.1. *Let* $1 < p \le q < \infty$. *Suppose that* $w^{1-p'} \in DC(x)$ *or* $w^{1-p'} \in DC(y)$. *Then the operator* H_2 *is bounded from* $L^p_w(R^2_+)$ *to* $L^q_v(R^2_+)$ *if and only if* (1.1) *holds.*

More general form of this corollary is the next statement:

Theorem 3.3. *Let* $1 < p \le q < \infty$. *Assume that the weight function* $w^{1-p'}$ *satisfies the condition*

$$\sup_{\substack{x>0 \\ k\in Z}} \left(\sum_{j=k}^\infty \left(\int_0^{2^j} w^{1-p'}(x,y)dy \right)^{1-p} \right) \left(\int_0^{2^{k+1}} w^{1-p'}(x,y)dx \right)^{p-1} < \infty.$$

Then the boundedness of H_2 *from* $L^p_w(R^2_+)$ *to* $L^q_v(R^2_+)$ *is equivalent to* (1.1).

The following theorem states that if the weight function w has the form $w(x,y) = w_1(x)w_2(y)$, then the boundedness of the operator H_2 from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ is equivalent to the first condition in the E. Sawyer's theorem.

Theorem 3.4. *Let* $1 < p \le q < \infty$ *and* $w(x,y) = w_1(x)w_2(y)$. *Then the operator* H_2 *is bounded from* $L^p_w(R^2_+)$ *to* $L^q_v(R^2_+)$ *if and only if*

$$D =: \sup_{a,b>0} \left(\int_0^a w_1^{1-p'}(x)dx \right)^{1/p'} \left(\int_0^b w_2^{1-p'}(y)dy \right)^{1/p'} \times \left(\int_a^\infty \int_b^\infty v(x,y)dxdy \right)^{1/q} < \infty.$$

4. Proof of the Main Results

In this section we present the proofs of the results formulated in the previous section.

Proof of Theorem 3.1. **Sufficiency**. First of all note that (see e.g., [18]) the condition $w^{1-p'} \in DC(y)$ implies the reverse doubling condition for $w^{1-p'}$ uniformly with respect to x, i.e., there exists the constants $\eta_1, \eta_2 > 1$ such that for all $t > 0$ and a.e. $x \in R_+$ the inequality

$$\int_0^{\eta_1 t} w^{1-p'}(x,y)dy \geq \eta_2 \int_0^t w^{1-p'}(x,y)dy \tag{4.1}$$

holds.

In the sequel we shall use the notation:

$$v_1(x,y) =: \frac{v(x,y)}{y^{(1-\beta)q}}; \;\; \widetilde{v}_{1,j}(x) =: \int_{\eta_1^j}^{\eta_1^{j+1}} v_1(x,y)dy;$$

$$F_j(t) =: \int_0^{\eta_1^{j+1}} f(t,\tau)d\tau; \;\; A =: \max\{A_1, A_2\}.$$

Let $f \geq 0$. Then taking into account the fact $\alpha \geq 1$ and using Theorem D we find that

$$\begin{aligned}
I \;\; &=: \;\; \int_0^\infty \int_0^\infty v(x,y)(R_{\alpha,\beta} f)^q(x,y)dxdy \\
&\leq \;\; \int_0^\infty \int_0^\infty v_1(x,y)\bigg(\int_0^x (x-t)^{\alpha-1}\bigg(\int_0^y f(t,\tau)d\tau\bigg)dt\bigg)^q dxdy \\
&\leq \;\; \sum_{j\in Z} \int_0^\infty \widetilde{v}_{1,j}(x)\bigg(\int_0^x (x-t)^{\alpha-1}F_j(t)dt\bigg)^q dx \\
&\leq \;\; cA^q \sum_{j\in Z}\bigg[\int_0^\infty \bigg(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy\bigg)^{1-p} F_j^p(x)dx\bigg]^{q/p} \\
&\leq \;\; cA^q \bigg[\int_0^\infty \sum_{j\in Z}\bigg(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy\bigg)^{1-p} F_j^p(x)dx\bigg]^{q/p}.
\end{aligned}$$

On the other hand, we have

$$\sup_{\substack{x>0\\k\in Z}} J(x,j) =: \sup_{\substack{x>0\\k\in Z}} \sum_{j=k}^\infty \bigg(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy\bigg)^{1-p} \tag{4.2}$$

$$\times \Big(\sum_{j=-\infty}^{k} \int_{\eta_1^j}^{\eta_1^{j+1}} w^{1-p'}(x,y)dy \Big)^{p-1} < \infty.$$

Indeed, (4.1) and the condition $w^{1-p'} \in DC(y)$ lead to the inequality:

$$\begin{aligned} J(x,j) &= \sum_{j=k}^{\infty} \Big(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy \Big)^{1-p} \Big(\int_0^{\eta_1^{k+1}} w^{1-p'}(x,y)dy \Big)^{p-1} \\ &\le \Big(\sum_{j=k}^{\infty} \eta_2^{(j-k)(1-p)} \Big) \Big(\int_0^{\eta_1^k} w^{1-p'}(x,y)dy \Big)^{1-p} \\ &\quad \times \Big(\int_0^{\eta_1^{k+1}} w^{1-p'}(x,y)dy \Big)^{p-1} \le c\,. \end{aligned}$$

Consequently, by virtue of Theorem E and Hölder's inequality we find that

$$\begin{aligned} &\Big[\int_0^\infty \sum_{j\in Z} \Big(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy \Big)^{1-p} F_j^p(x)dx \Big]^{q/p} \\ &\le c \Big[\int_0^\infty \sum_{j\in Z} \Big(\int_0^{\eta_1^j} w^{1-p'}(x,y)dy \Big)^{1-p} \Big(\sum_{k=-\infty}^{j} \int_{\eta_1^k}^{\eta_1^{k+1}} f(x,t)dt \Big)^p dx \Big]^{q/p} \\ &\le c \Big[\int_0^\infty \sum_{k\in Z} \Big(\int_{\eta_1^k}^{\eta_1^{k+1}} w^{1-p'}(x,\tau)d\tau \Big)^{1-p} \Big(\int_{\eta_1^k}^{\eta_1^{k+1}} f(x,t)dt \Big)^p dx \Big]^{q/p} \\ &\le c \Big[\int_0^\infty \sum_{j\in Z} \Big(\int_{\eta_1^j}^{\eta_1^{j+1}} w^{1-p'}(x,y)dy \Big)^{1-p} \\ &\quad \times \Big(\int_{\eta_1^j}^{\eta_1^{j+1}} w(x,t) f^p(x,t)dt \Big) \Big(\int_{\eta_1^j}^{\eta_1^{j+1}} w^{1-p'}(x,t)dt \Big)^{p-1} dx \Big]^{q/p} \\ &\le c \|f\|^q_{L^p(R^2_+)}\,. \end{aligned}$$

Necessity. Let $f \ge 0$ and let $a, b > 0$. It is easy to see that

$$\begin{aligned} I &\ge \int_0^\infty \int_0^\infty v(x,y) \Big(\int_0^x \int_0^{y/2} \frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} dt d\tau \Big)^q dx dy \\ &\ge c \Big(\int_a^\infty \int_b^\infty v_1(x,y) dx dy \Big) \Big(\int_0^a \int_0^{b/2} \frac{f(t,\tau)}{(a-t)^{1-\alpha}} dt d\tau \Big)^q. \end{aligned}$$

Using the latter inequality and the boundedness of $R_{\alpha,\beta}$ on the class of functions $f_{a,b}(x,y) = \chi_{(0,a)}(x)\chi_{(0,b/2)}(y) w^{1-p'}(x,y)(a-t)^{(\alpha-1)(q-1)}$,

$a, b > 0$, we find that

$$\left(\int_0^a\int_0^{b/2}\frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}}dxdy\right)^q\int_a^\infty\int_b^\infty v_1(x,y)dxdy$$
$$\leq I \leq c\left(\int_0^a\int_0^{b/2}\frac{w^{1-p'}(x,y)}{(a-x)^{(1-\alpha)q}}dxdy\right)^{q/p} < \infty.$$

Hence this inequality and the condition $w \in CD(y)$ give us the condition $A_1 < \infty$.

Taking into account the arguments used above and the fact that the operator $R_{\alpha,\beta}$ is bounded from $L^p_w(R^2_+)$ to $L^q_v(R^2_+)$ if and only if its dual operator

$$W_{\alpha,\beta}f(x,y) = \int_x^\infty\int_y^\infty(t-x)^{\alpha-1}(\tau-y)^{\beta-1}f(t,\tau)dtd\tau$$

is bounded from $L^{q'}_{v^{1-q'}}(R^2_+)$ to $L^{p'}_{w^{1-p'}}(R^2_+)$, we obtain that $A_2 < \infty$.

Proof of Theorem 3.2. The proof is similar to that of Theorem 3.1. □

Proof of Theorem 3.3. *Necessity* is obvious. In order to prove *sufficiency,* it is enough to take the sequence 2^k instead of η_1^k in the proof of Theorem 3.1. □

Proof of Theorem 3.4. First suppose that $S := \int_0^\infty w_2^{1-p'}(y)dy = \infty$. Let $\{x_k\}_{k=-\infty}^{+\infty}$ be a sequence of positive numbers for which the equality

$$2^k = \int_0^{x_k} w_2^{1-p'}(y)dy \tag{4.3}$$

holds for all $k \in Z$. It is clear that $\{x_k\}$ is increasing and $R_+ = \cup_{k\in Z}[x_k, x_{k+1})$. Besides, it is easy to verify that

$$2^k = \int_{x_k}^{x_{k+1}} w_2^{1-p'}(y)dy.$$

Let $f \geq 0$. We have

$$\begin{aligned} I &=: \int_0^\infty\int_0^\infty v(x,y)(H_2(x,y))^2dxdy \\ &= \sum_{k\in Z}\int_0^\infty\int_{x_k}^{x_{k+1}} v(x,y)\left(\int_0^x\int_0^y f(t,\tau)dtd\tau\right)^q dxdy \\ &\leq \sum_{k\in Z}\int_0^\infty\left(\int_{x_k}^{x_{k+1}} v(x,y)dy\right)\left(\int_0^x\left(\int_0^{x_{k+1}} f(t,\tau)d\tau\right)dt\right)^q dx \end{aligned}$$

$$= \sum_{k\in Z}\int_0^\infty V_k(x)\bigg(\int_0^x F_k(t)dt\bigg)^q dx,$$

where

$$V_k(x) =: \int_{x_k}^{x_{k+1}} v(x,y)dy; \quad F_k(t) =: \int_0^{x_{k+1}} f(t,\tau)d\tau.$$

Further, it is obvious that

$$D^q \ge \sup_{\substack{a>0\\ j\in Z}} \bigg(\int_a^\infty \int_{x_j}^{x_{j+1}} v(x,y)dxdy\bigg)\bigg(\int_0^a \int_0^{x_j} w^{1-p'}(x,y)dxdy\bigg)^{q/p'}.$$

Therefore by Theorem C we obtain

$$\begin{aligned} I &\le cD^q \sum_{j\in Z}\bigg[\int_0^\infty w_1(x)\bigg(\int_0^{x_j} w_2^{1-p'}(y)dy\bigg)^{1-p}(F_k(x))^p dx\bigg]^{q/p} \\ &\le cD^q\bigg[\int_0^\infty w_1(x)\sum_{j\in Z}\bigg(\int_0^{x_j} w_2^{1-p'}(y)dy\bigg)^{1-p} \\ &\quad \times\bigg(\sum_{k=-\infty}^{j}\int_{x_k}^{x_{k+1}} f(x,\tau)d\tau\bigg)^p dx\bigg]^{q/p}. \end{aligned}$$

On the other hand, (4.3) yields

$$\begin{aligned} &\sum_{k=n}^{+\infty}\bigg(\int_0^{x_k} w_2^{1-p'}(y)dy\bigg)^{1-p}\bigg(\sum_{k=-\infty}^{n}\int_{x_k}^{x_{k+1}} w_2^{1-p'}(y)dy\bigg)^{p-1} \\ &\quad= \sum_{k=n}^{+\infty}\bigg(\int_0^{x_k} w_2^{1-p'}(y)dy\bigg)^{1-p}\bigg(\int_0^{x_{n+1}} w_2^{1-p'}(y)dy\bigg)^{p-1} \\ &\quad= \bigg(\sum_{k=n}^{+\infty} 2^{k(1-p)}\bigg)2^{(n+1)(p-1)} \le c \end{aligned}$$

for all $n\in Z$. Hence by Theorem E and Hölder's inequality we have

$$\begin{aligned} I &\le cD^q\bigg[\int_0^\infty w_1(x)\sum_{j\in Z}\bigg(\int_{x_j}^{x_{j+1}} w_2^{1-p'}(y)dy\bigg)^{1-p} \\ &\quad \times\bigg(\int_{x_k}^{x_{k+1}} f(x,\tau)d\tau\bigg)^p dx\bigg]^{q/p} \\ &\le cD^q\bigg[\int_0^\infty w_1(x)\sum_{j\in Z}\bigg(\int_{x_k}^{x_{k+1}} w_2(\tau)f^p(x,\tau)d\tau\bigg)dx\bigg]^{q/p} = cD^q\|f\|^q_{L^p(R^2_+)}. \end{aligned}$$

If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{x_k\}_{k=-\infty}^{0}$ for which (4.3) holds for all $k \leq 0$. Arguing as in the case $S = \infty$ and using Theorem F instead of Theorem E, we finally obtain the desired result. □

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References

[1] G. Bennett, *Some elementary inequalities*, Quart. J. Math. Oxford, 38 (2) (1987), 401–425.

[2] J. S. Bradley, *Hardy inequality with mixed norms*, Canad. Math. Bull., 21 (1978), 405–408.

[3] D. E. Edmunds, V. Kokilashvili and A. Meskhi, *Bounded and compact integral operators*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[4] I. Gabisonija and A. Meskhi, Two-weighted inequalities for a discrete Hilbert transform, Proc. A. Razmadze Math. Inst., 116 (1998), 107–122.

[5] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 92 (1998), Longman, Harlow.

[6] V. M. Kokilashvili, *On Hardy's inequalities in weighted spaces* (Russian), Soobsch. Akad. Nauk Gruz. SSR, 96 (1979), 37–40.

[7] V. Kokilashvili and A. Meskhi, *On a trace inequality for one-sided potentials and applications to the solvability of nonlinear integral equations*, Georgian Math. J., 8 (3) (2001), 521–536.

[8] V. Kokilashvili and A. Meskhi, *On one-sided potentials with multiple kernels*, Integral Transf. Spec. Functions, accepted for publication.

[9] V. Kokilashvili and A. Meskhi, *On a trace inequality for one-sided potentials with multiple kernels*, Frac. Calc. Appl. Anal., 6 (4) (2003), 461–472.

[10] F. J. Martin-Reyes and E. Sawyer, *Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater*, Proc. Amer. Math. Soc., 106 (1989), 727–733.

[11] V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[12] A. Meskhi, *Solution of some weight problems for the Riemann-Liouville and Weyl operators*, Georgian Math. J., 5 (6) (1998), 565–574.

[13] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math., 44 (1972), 31–38.

[14] D. V. Prokhorov, *On the boundedness of a class of integral operators*, J. London Math. Soc., 61 (2) (2000), 617–628.

[15] E. Sawyer, *Weighted inequalities for the two-dimensional Hardy operator*, Studia Math., 82 (1985), 1–11.

[16] V. N. Sedov, *Weighted spaces, The imbedding theorem* (Russian), Diff. Uravneniya, 8 (1972), 1452-1462.

[17] V. D. Stepanov, *Weighted inequalities of Hardy type for higher derivatives, and their applications* (Russian), Soviet Math. Dokl., 38 (1989), 389–393.

[18] J. O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math., 1381 , Springer Verlag, Berlin, 1989.

[19] F. A. Sysoeva, *Generalization of a certain Hardy inequality* (Russian), Izv. Vusch. Uchev. Zaved. Matematika, 49 (6) (1965), 140–143.

[20] A. Wedestig, *Weighted inequalities of Hardy-type and their limiting inequalities*, Doctoral Thesis, 2003.

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