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Research Article

Weighted Estimates of a Measure of Noncompactness for Maximal and Potential Operators

Muhammad Asif[1] and Alexander Meskhi[2]

[1] *Abdus Salam School of Mathematical Sciences, GC University, c-II, M. M. Alam Road, Gulberg III, Lahore 54660, Pakistan*
[2] *A. Razmadze Mathematical Institute, Georgian Academy of Sciences, 1, M. aleksidze Street, 0193 Tbilisi, Georgia*

Correspondence should be addressed to Alexander Meskhi, alex72meskhi@yahoo.com

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A measure of noncompactness (essential norm) for maximal functions and potential operators defined on homogeneous groups is estimated in terms of weights. Similar problem for partial sums of the Fourier series is studied. In some cases, we conclude that there is no weight pair for which these operators acting between two weighted Lebesgue spaces are compact.

1. Introduction

In the papers [1–3], the measure of noncompactness (essential norm) of maximal functions, singular integrals, and identity operators acting in weighted Lebesgue spaces defined on $\mathbb{R}^n$ with different weights was estimated from below. In this paper, we investigate the same problem for maximal functions and potentials defined on homogeneous groups. Analogous estimates for the partial sums of Fourier series are also derived. For truncated potentials, we have two-sided estimates of the essential norm.

A result analogous to that of [2] has been obtained in [4, 5] for the Hardy-Littlewood maximal operator with more general differentiation basis on symmetric spaces. The essential norm for Hardy-type transforms and one-sided potentials in weighted Lebesgue spaces has been estimated in [6–9] (see also [10]). For two-sided estimates of the essential norm for the Cauchy integrals see [11–14]. The same problem in the one-weighted setting has been studied in [15, 16].

The one-weight problem for the Hardy-Littlewood maximal functions was solved by Muckenhoupt [17] (for maximal functions defined on the spaces of homogeneous type see, e.g., [18]) and for fractional maximal functions and Riesz potentials by Muckenhoupt and Wheeden [19]. Two-weight criteria for the Hardy-Littlewood maximal functions have been obtained in [20]. Necessary and sufficient conditions guaranteeing the boundedness of the Riesz potentials from one weighted Lebesgue space into another one were derived by Sawyer [21, 22] and Gabidzashvili and Kokilashvili [23] (see also [24]). However, conditions derived in [23] aremore transparent than those of [21]. For the solution of the two-weight problem for operators with positive kernels on spaces of homogeneous type see [25] (see also [10, 26] for related topics).

Earlier, the trace inequality for the Riesz potentials (boundedness of Riesz potentials from L^p to L^q_v) was established in [27, 28]. The two-weight criteria for fractional maximal functions were obtained in [22, 29, 30] (see also [25] for more general case).

Necessary and sufficient conditions guaranteeing the compactness of the Riesz potentials have been derived in [31] (see also [10, Section 5.2]). The one-weight problem for the Hilbert transform and partial sums of the Fourier series was solved in [32].

The paper is organized as follows. In Section 2, we give basic concepts and prove some lemmas. Section 3 is divided into 4 parts. Section 3.1 concerns maximal functions; potential operators are discussed in Sections 3.2 and 3.3. Section 3.4 is devoted to the partial sums of Fourier series.

Constants (often different constants in the same series of inequalities) will generally be denoted by c or C.

2. Preliminaries

A homogeneous group is a simply connected nilpotent Lie group G on a Lie algebra g with the one-parameter group of transformations $\delta_t = \exp(A\log t)$, $t > 0$, where A is a diagonalized linear operator in G with positive eigenvalues. In the homogeneous group G, the mappings $\exp o\, \delta_t\, o \exp^{-1}$, $t > 0$, are automorphisms in G, which will be again denoted by δ_t. The number $Q = \operatorname{tr} A$ is the homogeneous dimension of G. The symbol e will stand for the neutral element in G.

It is possible to equip G with a homogeneous norm $r : G \to [0,\infty)$ which is continuous on G, smooth on $G \setminus \{e\}$, and satisfies the conditions

(i) $r(x) = r(x^{-1})$ for every $x \in G$;

(ii) $r(\delta_t x) = tr(x)$ for every $x \in G$ and $t > 0$;

(iii) $r(x) = 0$ if and only if $x = e$;

(iv) there exists $c_o > 0$ such that

$$r(xy) \le c_o(r(x) + r(y)), \quad x, y \in G. \tag{2.1}$$

In the sequel, we denote by $B(a,\rho)$ and $\overline{B}(a,\rho)$ open and closed balls, respectively, with the center a and radius ρ, that is,

$$B(a,\rho) := \{y \in G;\ r(ay^{-1}) < \rho\}, \qquad \overline{B}(a,\rho) := \{y \in G;\ r(ay^{-1}) \le \rho\}. \tag{2.2}$$

It can be observed that $\delta_\rho B(e,1) = B(e,\rho)$.

Let us fix a Haar measure $|\cdot|$ in G such that $|B(e,1)| = 1$. Then, $|\delta_t E| = t^Q|E|$. In particular, $|B(x,t)| = t^Q$ for $x \in G$, $t > 0$.

Examples of homogeneous groups are the Euclidean n-dimensional space $\mathbb{R}^n$, the Heisenberg group, upper triangular groups, and so forth. For the definition and basic properties of the homogeneous group, we refer to [33, page 12] and [25].

Proposition A. *Let G be a homogeneous group and let $S = \{x \in G : r(x) = 1\}$. There is a (unique) Radon measure σ on S such that for all $u \in L^1(G)$,*

$$\int_G u(x)dx = \int_0^\infty \int_S u(\delta_t \overline{y}) t^{Q-1} d\sigma(\overline{y}) dt. \tag{2.3}$$

For the details see, for example, [33, page 14].

We call a weight a locally integrable almost everywhere positive function on G. Denote by $L^p_w(G)$ $(1 < p < \infty)$ the weighted Lebesgue space, which is the space of all measurable functions $f : G \to \mathbb{C}$ with the norm

$$\|f\|_{L^p_w(G)} = \left(\int_G |f(x)|^p w(x)dx\right)^{1/p} < \infty. \tag{2.4}$$

If $w \equiv 1$, then we denote $L^p_1(G)$ by $L^p(G)$.

Let $X = L^p_w(G)(1 < p < \infty)$ and denote by X^* the space of all bounded linear functionals on X. We say that a real-valued functional F on X is sublinear if

(i) $F(f + g) \leq F(f) + F(g)$ for all nonnegative $f, g \in X$;

(ii) $F(\alpha f) = |\alpha| F(f)$ for all $f \in X$ and $\alpha \in \mathbb{C}$.

Let T be a sublinear operator $T : X \to L^q(G)$, then, the norm of the operator T is defined as follows:

$$\|T\| = \sup\left\{\|Tf\|_{L^q(G)} : \|f\|_X \leq 1\right\}. \tag{2.5}$$

Moreover, T is order preserving if $Tf(x) \geq Tg(x)$ almost everywhere for all nonnegative f and g with $f(x) \geq g(x)$ almost everywhere. Further, if T is sublinear and order preserving, then obviously it is nonnegative, that is, $Tf(x) \geq 0$ almost everywhere if $f(x) \geq 0$.

The measure of noncompactness for an operator T which is bounded, order preserving, and sublinear from X into a Banach space Y will be denoted by $\|T\|_{\kappa(X,Y)}$ (or simply $\|T\|_\kappa$) and is defined as

$$\|T\|_{\kappa(X,Y)} = \operatorname{dist}\{T, \mathcal{K}(X,Y)\} \equiv \inf\left\{\|T - K\| : K \in \mathcal{K}(X,Y)\right\}, \tag{2.6}$$

where $\mathcal{K}(X,Y)$ is the class of all compact sublinear operators from X to Y. If $X = Y$, then we use the symbol $\mathcal{K}(X)$ for $\mathcal{K}(X,Y)$.

Let X and Y be Banach spaces and let T be a continuous linear operator from X to Y. The entropy numbers of the operator T are defined as follows:

$$e_k(T) = \inf\left\{\varepsilon > 0 : T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} (b_i + \varepsilon U_Y) \text{ for some } b_1, \ldots, b_{2^{k-1}} \in Y\right\}, \tag{2.7}$$

where U_X and U_Y are the closed unit balls in X and Y, respectively. It is well known (see, e.g., [34, page 8]) that the measure of noncompactness of T is greater than or equal to $\lim_{n\to\infty} e_n(T)$.

In the sequel, we assume that X is a Banach space which is a certain subset of all Haar-measurable functions on G. We denote by $S(X)$ the class of all bounded sublinear functionals defined on X, that is,

$$S(X) = \left\{F : X \to \mathbb{R},\ F\text{-sublinear and } \|F\| = \sup_{\|x\|\leq 1} |F(x)| < \infty\right\}. \tag{2.8}$$

Let M be the set of all bounded functionals F defined on X with the following property:

$$0 \le Ff \le Fg, \tag{2.9}$$

for any $f, g \in X$ with $0 \le f(x) \le g(x)$ almost every. We also need the following classes of operators acting from X to $L^p(G)$:

$$\begin{aligned} F_L(X, L^p(G)) &:= \Big\{ T : Tf(x) = \sum_{j=1}^{m} \alpha_j(f) u_j,\ m \in \mathbb{N},\ u_j \ge 0,\ u_j \in L^p(G), \\ &\qquad u_j \text{ are linearly independent and } \alpha_j \in X^* \bigcap M \Big\}, \\ F_S(X, L^p(G)) &:= \Big\{ T : Tf(x) = \sum_{j=1}^{m} \beta_j(f) u_j,\ m \in \mathbb{N},\ u_j \ge 0,\ u_j \in L^p(G), \\ &\qquad u_j \text{ are linearly independent and } \beta_j \in S(X) \bigcap M \Big\}. \end{aligned} \tag{2.10}$$

If $X = L^p(G)$, we will denote these classes by $F_L(L^p(G))$ and $F_S(L^p(G))$, respectively. It is clear that if $P \in F_L(X, L^p(G))$ (resp., $P \in F_S(X, L^p(G))$), then P is compact linear (*resp., compact sublinear*) from X to $L^p(G)$.

We will use the symbol $\alpha(T)$ for the distance between the operator $T : X \to L^p(G)$ and the class $F_S(X, L^p(G))$, that is,

$$\alpha(T) := \operatorname{dist}\{T, F_S(X, L^p(G))\}. \tag{2.11}$$

For any bounded subset A of $L^p(G)$ $(1 < p < \infty)$, let

$$\begin{aligned} \Phi(A) &:= \inf\{\delta > 0 : A \text{ can be covered by finitely many open balls in } L^p(G) \text{ of radius } \delta\}, \\ \Psi(A) &:= \inf_{P \in F_L(L^p(G))} \sup\{\|f - Pf\|_{L^p(G)} : f \in A\}. \end{aligned} \tag{2.12}$$

We will need a statement similar to Theorem V.5.1 of Chapter V of [35] (for Euclidean spaces see [2]).

Theorem A. *For any bounded subset $K \subset L^p(G)$ $(1 \le p < \infty)$, the inequality*

$$2\Phi(K) \ge \Psi(K) \tag{2.13}$$

holds.

Proof. Let $\varepsilon > \Phi(K)$. Then, there are $g_1, g_2, \dots, g_N \in L^p(G)$ such that for all $f \in K$ and some $i \in \{1, 2, \dots, N\}$,

$$\|f - g_i\|_{L^p(G)} < \varepsilon. \tag{2.14}$$

Further, given $\delta > 0$, let $\overline{B}$ be the closed ball in G with center e such that for all $i \in \{1, 2, \ldots, N\}$,

$$\left(\int_{G\setminus\overline{B}} |g_i(x)|^p dx\right)^{1/p} < \frac{1}{2}\delta. \tag{2.15}$$

It is known (see [33, page 8]) that every closed ball in G is a compact set. Let us cover $\overline{B}$ by open balls with radius h. Since $\overline{B}$ is compact, we can choose a finite subcover $\{B_1, B_2, \ldots, B_n\}$. Further, let us assume that $\{E_1, E_2, \ldots, E_n\}$ is a family of pairwise disjoint sets of positive measure such that $\overline{B} = \bigcup_{i=1}^n E_i$ and $E_i \subset B_i$ (we can assume that $E_1 = B_1 \cap \overline{B}$, $E_2 = (B_2 \setminus B_1) \cap \overline{B}, \ldots, E_k = (B_k \setminus \bigcup_{i=1}^{k-1} B_i) \cap \overline{B}, \ldots$). We define

$$Pf(x) = \sum_{i=1}^n f_{E_i} \chi_{E_i}(x), \quad f_{E_i} = |E_i|^{-1} \int_{E_i} f(x)dx. \tag{2.16}$$

Then,

$$\begin{aligned}
\|g_i - Pg_i\|^p_{L^p(\overline{B})} &= \sum_{j=1}^n \int_{E_j} \left|\frac{1}{|E_j|}\int_{E_j} [g_i(x) - g_i(y)]dy\right|^p dx \\
&\leq \sum_{j=1}^m \int_{E_j} \frac{1}{|E_j|} \int_{E_j} |g_i(x) - g_i(y)|^p dy\, dx \\
&\leq \sup_{r(z)\leq 2c_o h} \int_{\overline{B}} |g_i(x) - g_i(zx)|^p dx \longrightarrow 0
\end{aligned} \tag{2.17}$$

as $h \to 0$. The latter fact follows from the continuity of the norm $L_p(G)$ (see, e.g., [33, page 19]).

From this and (2.14), we find that

$$\|g_i - Pg_i\|_{L^p(G)} < \delta, \quad i = 1, 2, 3, \ldots, N, \tag{2.18}$$

when h is sufficiently small. Further,

$$\begin{aligned}
\|Pf\|^p_{L^p(G)} &= \sum_{j=1}^n \int_{E_j} \left||E_j|^{-1} \int_{\bar{E}_j} f(y)dy\right|^p dx \\
&\leq \sum_{j=1}^n \int_{\bar{E}_j} |E_j|^{-1} \int_{\bar{E}_j} |f(y)|^p dy\, dx \\
&\leq \|f\|^p_{L^p(\overline{B})} \\
&\leq \|f\|^p_{L^p(G)}.
\end{aligned} \tag{2.19}$$

It is also clear that the functionals $f \to f_{E_i}$ belong to $(L^p(G))^* \cap M$. Hence, $P \in F_L(L^p(G))$. Finally, (2.14)–(2.15) and (2.18) yield

$$\begin{aligned}
\|f - Pf\|_{L^p(G)} &\leq \|f - g_i\|_{L^p(G)} + \|g_i - Pg_i\|_{L^p(G)} + \|P(g_i - f)\|_{L^p(G)} \\
&< \varepsilon + \delta + \|g_i - f\|_{L^p(G)} \leq 2\varepsilon + \delta.
\end{aligned} \tag{2.20}$$

Since δ is arbitrarily small, we have the desired result. □

Lemma A. *Let $1 \leq p < \infty$ and assume that a set $K \subset L^p(G)$ is compact. Then for any given $\varepsilon > 0$, there exist an operator $P_\varepsilon \in F_L(L^p(G))$ such that for all $f \in K$,*

$$\left\| f - P_\varepsilon f \right\|_{L^p(G)} \leq \varepsilon . \tag{2.21}$$

Proof. Let K be a compact set in $L^p(G)$. Using Theorem A, we see that $\Psi(K) = 0$. Hence for $\varepsilon > 0$, there exists $P_\varepsilon \in F_L(L^p(G))$ such that

$$\sup \left\{ \left\| f - P_\varepsilon f \right\|_{L^p(G)} : f \in K \right\} \leq \varepsilon. \tag{2.22}$$

□

Lemma B. *Let $T : X \to L^p(G)$ be compact, order-preserving, and sublinear operator, where $1 \leq p < \infty$. Then, $\alpha(T) = 0$.*

Proof. Let $U_X = \{f : \|f\|_X \leq 1\}$. From the compactness of T, it follows that $T(U_X)$ is relatively compact in $L^p(G)$. Using Lemma A, we have that for any given $\varepsilon > 0$ there exists an operator $P_\varepsilon \in F_L(L^p(G))$ such that for all $f \in U_X$,

$$\left\| Tf - P_\varepsilon T f \right\|_{L^p(G)} \leq \varepsilon. \tag{2.23}$$

Let $\widetilde{P}_\varepsilon = P_\varepsilon \circ T$. Then, $\widetilde{P}_\varepsilon \in F_S(X, L^p(G))$. Indeed, there exist functionals $\alpha_j \in X^* \cap M$, $j \in \{1, 2, \ldots, m\}$, and linearly independent functions $u_j \in L^p(G)$, $j \in \{1, 2, \ldots, m\}$, such that

$$\widetilde{P}_\varepsilon f(x) = P_\varepsilon(Tf)(x) = \sum_{j=1}^{m} \alpha_j(Tf) u_j(x) = \sum_{j=1}^{m} \beta_j(f) u_j(x), \tag{2.24}$$

where $\beta_j = \alpha_j \circ T$ belongs to $S(X) \cap M$. Since by (2.23),

$$\left\| Tf - \widetilde{P}_\varepsilon f \right\|_{L^p(G)} \leq \varepsilon \tag{2.25}$$

for all $f \in U_X$, it follows immediately that $\alpha(T) = 0$. □

We will also need the following lemma.

Lemma C. *Let T be a bounded, order-preserving, and sublinear operator from X to $L^q(G)$, where $1 \leq q < \infty$. Then,*

$$\|T\|_\kappa = \alpha(T). \tag{2.26}$$

Proof. Let $\delta > 0$. Then, there exists an operator $K \in \mathcal{K}(X, L^q(G))$, such that $\|T - K\| \leq \|T\|_\kappa + \delta$. By Lemma B there is $P \in F_S(X, L^q(G))$ for which the inequality $\|K - P\| < \delta$ holds. This gives

$$\|T - P\| \leq \|T - K\| + \|K - P\| \leq \|T\|_\kappa + 2\delta. \tag{2.27}$$

Hence, $\alpha(T) \leq \|T\|_\kappa$. Moreover, it is obvious that

$$\|T\|_\kappa \leq \alpha(T). \tag{2.28}$$

□

Lemma D. *Let $1 \le q < \infty$ and let $P \in F_S(X, L^q(G))$. Then for every $a \in G$ and $\varepsilon > 0$, there exist an operator $R \in F_S(X, L^q(G))$ and positive numbers α, $\overline{\alpha}$ such that for all $f \in X$, the inequality*

$$\left\| (P-R)f \right\|_{L^q(G)} \le \varepsilon \|f\|_X \tag{2.29}$$

holds and $\operatorname{supp} Rf \subset B(a, \overline{\alpha}) \setminus B(a, \alpha)$.

Proof. There exist linearly independent nonnegative functions $u_j \in L^q(G)$, $j \in \{1, 2, \ldots, N\}$, such that

$$Pf(x) = \sum_{j=1}^{N} \beta_j(f) u_j(x), \quad f \in X, \tag{2.30}$$

where β_j are bounded, order-preserving, sublinear functionals $\beta_j : X \to \mathbb{R}$. On the other hand, there is a positive constant c for which

$$\sum_{j=1}^{N} \left| \beta_j(f) \right| \le c \|f\|_X. \tag{2.31}$$

Let us choose linearly independent $\Phi_j \in L^q(G)$ and positive real numbers α_j, $\overline{\alpha}_j$ such that

$$\left\| u_j - \Phi_j \right\|_{L^q(G)} < \varepsilon, \quad j \in \{1, 2, \ldots, N\} \tag{2.32}$$

and $\operatorname{supp} \Phi_j \subset B(a, \overline{\alpha}_j) \setminus B(a, \alpha_j)$. If

$$Rf(x) = \sum_{j=1}^{N} \beta_j(f)\, \Phi_j(x), \tag{2.33}$$

then it is obvious that $R \in F_S(X, L^q(G))$ and moreover,

$$\|Pf - Rf\|_{L^q(G)} \le \sum_{j=1}^{N} \left| \beta_j(f) \right| \left\| u_j - \Phi_j \right\|_{L^q(G)} \le c\varepsilon \|f\|_X \tag{2.34}$$

for all $f \in X$. Besides this, $\operatorname{supp} Rf \subset B(a, \overline{\alpha}) \setminus B(a, \alpha)$, where $\alpha = \min\{\alpha_j\}$ and $\overline{\alpha} = \max\{\overline{\alpha}_j\}$. □

Lemmas C and D for Lebesgue spaces defined on Euclidean spaces have been proved in [35] for the linear case and in [2] for sublinear operators.

Lemma E. *Let $1 < p, q < \infty$, and let T be a bounded, order-preserving, and sublinear operator from $L^p_w(G)$ to $L^q_v(G)$. Suppose that $\lambda > \|T\|_{\kappa(L^p_w(G), L^q_v(G))}$, and a is a point of G. Then, there exist constants β_1, β_2, $0 < \beta_1 < \beta_2 < \infty$, such that for all τ and r with $r > \beta_2$, $\tau < \beta_1$, the following inequalities hold:*

$$\begin{gathered} \|Tf\|_{L^q_v(B(a,\tau))} \le \lambda \|f\|_{L^p_w(G)}, \\ \|Tf\|_{L^q_v(B(a,r)^c)} \le \lambda \|f\|_{L^p_w(G)}, \end{gathered} \tag{2.35}$$

where $f \in L^p_w(G)$.

Proof. Let T be bounded from $L^p_w(G)$ to $L^q_v(G)$. Let $T^{(v)}$ be the operator given by

$$T^{(v)}f = v^{1/q}Tf. \tag{2.36}$$

Then, it is easy to see that

$$\|T^{(v)}\|_{\kappa(L^p_w(G)\to L^q(G))} = \|T\|_{\kappa(L^p_w(G)\to L^q_v(G))}. \tag{2.37}$$

By Lemma C, we have that

$$\lambda > \alpha(T^{(v)}). \tag{2.38}$$

Consequently, there exists $P \in F_S(L^p_w(G), L^q(G))$ such that

$$\|T^{(v)} - P\| < \lambda. \tag{2.39}$$

Fix $a \in G$. According to Lemma D, there are positive constants β_1 and β_2, $\beta_1 < \beta_2$, and $R \in F_S(L^p_w(G), L^q_v(G))$ for which

$$\|P - R\| \le \frac{\lambda - \|T^{(v)} - P\|}{2} \tag{2.40}$$

and supp $Rf \subset B(a,\beta_2) \setminus B(a,\beta_1)$ for all $f \in L^p_w(G)$. Hence,

$$\|T^{(v)} - R\| < \lambda. \tag{2.41}$$

From the last inequality, it follows that if $0 < \tau < \beta_1$ and $r > \beta_2$, then (2.35) holds for f, $f \in L^p_w(G)$. □

The following lemmas are taken from [2] (for the linear case see [35]).

Lemma F. *Let Ω be a domain in $\mathbb{R}^n$, and let T be a bounded, order-preserving, and sublinear operator from $L^r_w(\Omega)$ to $L^p(\Omega)$, where $1 < r$, $p < \infty$, and w is a weight function on Ω. Then,*

$$\|T\|_{\kappa(L^r_w(\Omega),L^p(\Omega))} = \alpha(T). \tag{2.42}$$

Lemma G. *Let Ω be a domain in $\mathbb{R}^n$ and let $P \in F_S(X, L^p(\Omega))$, where $X = L^r_w(\Omega)$ and $1 < r$, $p < \infty$. Then for every $a \in \Omega$ and $\varepsilon > 0$, there exist an operator $R \in F_S(X, L^p(\Omega))$ and positive numbers β_1 and β_2, $\beta_1 < \beta_2$ such that for all $f \in X$, the inequality*

$$\|(P - R)f\|_{L^p(\Omega)} \le \varepsilon\|f\|_X \tag{2.43}$$

holds and supp $Rf \subset D(a,\beta_2) \setminus D(a,\beta_1)$, *where $D(a,s) := \Omega \bigcap B(a,s)$.*

Lemmas F and G yield the next statement which follows in the same manner as Lemma E was proved; therefore we give it without proof.

Lemma H. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that $1 < p$, $q < \infty$, and that T is bounded, order-preserving, and sublinear operator from $L^p_w(\Omega)$ to $L^q_v(\Omega)$. Assume that $\lambda > \|T\|_{\kappa(L^p_w(\Omega),L^q_v(\Omega))}$ and $a \in \Omega$. Then, there exist constants β_1, β_2, $0 < \beta_1 < \beta_2 < \infty$ such that for all τ and r with $r > \beta_2$, $\tau < \beta_1$, the following inequalities hold:*

$$\|Tf\|_{L^q_v(B(a,\tau))} \le \lambda\|f\|_{L^p_w(\Omega)}; \qquad \|Tf\|_{L^q_v(\Omega\setminus B(a,r))} \le \lambda\|f\|_{L^p_w(\Omega)}, \tag{2.44}$$

where $f \in L^p_w(\Omega)$.

Lemma I (see [36, Chapter IX]). *Let $1 < p,\ q < \infty$, and let (X,μ) and (Y,ν) be σ-finite measure spaces. If*

$$\Big\| \|k(x,y)\|_{L^{p'}_\nu(Y)} \Big\|_{L^q_\mu(X)} < \infty, \quad p' = \frac{p}{p-1}, \tag{2.45}$$

then the operator

$$Kf(x) = \int_Y k(x,y) f(y) d\nu(y), \quad x \in X, \tag{2.46}$$

is compact from $L^p_\nu(Y)$ into $L^q_\mu(X)$.

3. Main results

3.1. Maximal functions

Let G be a homogeneous group and let

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)| dy, \quad x \in G,\ 0 \le \alpha < Q, \tag{3.1}$$

where the supremum is taken over all balls B containing x. If $\alpha = 0$, then M_α becomes the Hardy-Littlewood maximal function which will be denoted by M.

It is known (see, e.g., [17, 18] for $\alpha = 0$, and [19], [33, Chapter 6], for $\alpha > 0$) that if $1 < p < \infty$ and $0 \le \alpha < Q/p$, then the operator M_α is bounded from $L^p_{\rho^p}(G)$ to $L^q_{\rho^q}(G)$, where $q = Qp/(Q - \alpha p)$, if and only if $\rho \in A_{p,q}(G)$, that is,

$$\sup_B \left(\frac{1}{|B|} \int_B \rho^q \right)^{1/q} \left(\frac{1}{|B|} \int_B \rho^{-p'} \right)^{1/p'} < \infty. \tag{3.2}$$

Now, we formulate the main results of this subsection.

Theorem 3.1. *Let $1 < p < \infty$. Suppose that the maximal operator M is bounded from $L^p_w(G)$ to $L^p_v(G)$. Then, there is no weight pair (v,w) such that M is compact from $L^p_w(G)$ to $L^p_v(G)$. Moreover, the inequality*

$$\|M\|_{\kappa(L^p_w(G), L^p_v(G))} \ge \sup_{a \in G} \overline{\lim_{\tau \to 0}} \frac{1}{|B(a,\tau)|} \left(\int_{B(a,\tau)} v(x) dx \right)^{1/p} \left(\int_{B(a,\tau)} w^{1-p'}(x) dx \right)^{1/p'} \tag{3.3}$$

holds.

Proof. Suppose that $\lambda > \|M\|_{\kappa(L^p_w \to L^p_v)}$ and $a \in G$. By Lemma E, we have that

$$\int_{B(a,\tau)} v(x) \left(\sup_{B \ni x} \frac{1}{|B(a,\tau)|} \int_{B(a,\tau)} |f(y)| dy \right)^p dx \le \lambda^p \int_{B(a,\tau)} |f(x)|^p w(x) dx \tag{3.4}$$

for all τ ($\tau \le \beta$) and all f supported in $\overline{B}(a,\tau)$. Substituting $f(y) = \chi_{B(a,r)}(y)\, w^{1-p'}(y)$ in the latter inequality and taking into account that $\int_{B(a,\tau)} w^{1-p'}(x) dx < \infty$ (see, e.g., [17, 18], [25, Chapter 4]) for all $\tau > 0$ we find that

$$\frac{1}{|B(a,\tau)|^p} \left(\int_{B(a,\tau)} v(x) dx \right) \left(\int_{B(a,\tau)} w^{1-p'}(x) dx \right)^{p-1} \le \lambda^p. \tag{3.5}$$

This inequality and Lebesgue differentiation theorem (see [33, page 67]) yield the desired result. □

For the fractional maximal functions, we have the following theorem.

Theorem 3.2. *Let $1 < p < \infty$, $0 < \alpha < Q/p$ and let $q = Qp/(Q-\alpha p)$. Suppose that M_α is bounded from $L^p_w(G)$ to $L^q_v(G)$. Then, there is no weight pair (v,w) such that M_α is compact from $L^p_w(G)$ to $L^q_v(G)$. Moreover, the inequality*

$$\|M_\alpha\|_\kappa \geq \sup_{a\in G} \overline{\lim_{\tau\to 0}} \frac{1}{|B(a,\tau)|^{\alpha/Q-1}} \left(\int_{B(a,\tau)} v(x)dx\right)^{1/q} \left(\int_{B(a,\tau)} w^{1-p'}(x)dx\right)^{1/p'} \tag{3.6}$$

holds.

The proof of this statement is similar to that of Theorem 3.1; therefore the proof is omitted.

Example 3.3. Let $1 < p < \infty$, $v(x) = w(x) = r(x)^\gamma$, where $-Q < \gamma < (p-1)Q$. Then,

$$\|M\|_{\kappa(L^p_w(G))} \geq Q\left[(\gamma+Q)^{1/p}(\gamma(1-p')+Q)^{1/p'}\right]^{-1}. \tag{3.7}$$

Indeed, first observe that the fact $|B(e,1)| = 1$ and Proposition A implies $\sigma(S) = Q$, where S is the unit sphere in G and $\sigma(S)$ is its measure. By Theorem 3.1 and Proposition A, we have

$$\begin{aligned} \|M\|_{\kappa(L^p_w(G))} &\geq \lim_{\tau\to 0} \frac{1}{|B(e,\tau)|} \left(\int_{B(e,\tau)} w(x)dx\right)^{1/p} \left(\int_{B(e,\tau)} w^{1-p'}(x)dx\right)^{1/p'} \\ &= \sigma(S) \lim_{\tau\to 0} \tau^{-Q} \left(\int_0^\tau t^{\gamma+Q-1}dt\right)^{1/p} \left(\int_0^\tau t^{\gamma(1-p')+Q-1}dt\right)^{1/p'} \\ &= Q\left[(\gamma+Q)^{1/p}(\gamma(1-p')+Q)^{1/p'}\right]^{-1}. \end{aligned} \tag{3.8}$$

3.2. Riesz potentials

Let G be a homogeneous group and let

$$I_\alpha f(x) = \int_G \frac{f(y)}{r(xy^{-1})^{Q-\alpha}} dy, \quad 0 < \alpha < Q, \tag{3.9}$$

be the Riesz potential operator. It is well known (see [33, Chapter 6]) that I_α is bounded from $L^p(G)$ to $L^q(G)$, $1 < p$, $q < \infty$, if and only if $q = Qp/(Q-\alpha p)$.

Theorem 3.4. *Let $1 < p \leq q < \infty$, $0 < \alpha < Q$. Let I_α be bounded from $L^p_w(G)$ to $L^q_v(G)$. Then, the following inequality holds*

$$\|I_\alpha\|_\kappa \geq C_{\alpha,Q} \max\{A_1, A_2, A_3\}, \tag{3.10}$$

where

$$C_{\alpha,Q} = \frac{1}{(2c_o)^{Q-\alpha}},$$

$$A_1 = \sup_{\alpha \in G} \overline{\lim_{r\to 0}}\, r^{\alpha-Q} \left(\int_{B(a,r)} v(x)dx\right)^{1/q} \left(\int_{B(a,r)} w^{1-p'}(x)dx\right)^{1/p'},$$

$$A_2 = \sup_{a \in G} \overline{\lim_{r\to 0}} \left(\int_{B(a,r)} v(x)dx\right)^{1/q} \left(\int_{(B(a,r))^c} r\left(ay^{-1}\right)^{(\alpha-Q)p'} w^{1-p'}(y)dy\right)^{1/p'},$$

$$A_3 = \sup_{a \in G} \overline{\lim_{r\to 0}} \left(\int_{B(a,r)} w^{1-p'}(x)dx\right)^{1/p'} \left(\int_{(B(a,r))^c} r\left(ay^{-1}\right)^{(\alpha-Q)q} v(y)dy\right)^{1/q}. \tag{3.11}$$

(c_o is the constant from the triangle inequality for the homogeneous norms.)

The next statement is formulated for the Riesz potentials defined on domains in $\mathbb{R}^n$:

$$J_{\Omega,\alpha} f(x) = \int_{\Omega} f(y)|x-y|^{\alpha-n} dy, \quad x \in \Omega. \tag{3.12}$$

Theorem 3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. Let $1 < p \leq q < \infty$. If $J_{\Omega,\alpha}$ is bounded from $L^p_w(\Omega)$ to $L^q_v(\Omega)$, then one has*

$$\|J_{\Omega,\alpha}\|_\kappa \geq 2^{\alpha-n} B_1, \tag{3.13}$$

where

$$B_1 = \sup_{a \in \Omega} \overline{\lim_{r\to 0}}\, r^{\alpha-n} \left(\int_{B(a,r)} v\right)^{1/q} \left(\int_{B(a,r)} w^{1-p'}\right)^{1/p'}. \tag{3.14}$$

In particular, if $\Omega \equiv \mathbb{R}^n$, then

$$\|J_{\Omega,\alpha}\|_\kappa \geq 2^{\alpha-n} \max\{B_2, B_3\}, \tag{3.15}$$

where

$$B_2 = \sup_{a \in \mathbb{R}^n} \overline{\lim_{r\to 0}} \left(\int_{B(a,r)} v(x)dx\right)^{1/q} \left(\int_{\mathbb{R}^n \setminus B(a,r)} |a-y|^{(\alpha-n)p'} w^{1-p'}(y)dy\right)^{1/p'},$$

$$B_3 = \sup_{a \in \mathbb{R}^n} \overline{\lim_{r\to 0}} \left(\int_{B(a,r)} w^{1-p'}(x)dx\right)^{1/p'} \left(\int_{\mathbb{R}^n \setminus B(a,r)} |a-y|^{(\alpha-n)q} v(y)dy\right)^{1/q}. \tag{3.16}$$

Corollary 3.6. *Let $1 < p < \infty$, $1 < p < Q/\alpha$, $q = pQ/(Q-\alpha p)$, then there is no weight pair (v, w) for which I_α is compact from $L^p_w(G)$ to $L^q_v(G)$. Moreover, if I_α is bounded from $L^p_w(G)$ to $L^q_v(G)$, then*

$$\|I_\alpha\|_\kappa \geq C_{\alpha,Q} A_1, \tag{3.17}$$

where $C_{\alpha,Q}$ and A_1 are defined in Theorem 3.4.

Proof of Theorem 3.4. By Lemma E, we have that for $\lambda > \|I_\alpha\|_{\kappa(L^p_w(G),L^q_v(G))}$ and $a \in G$, there are positive constants β_1 and β_2 ($\beta_1 < \beta_2$) such that for all τ, s ($\tau < \beta_1$, $s > \beta_2$),

$$\int_{B(a,\tau)} v(x)\left|I_\alpha f(x)\right|^q dx \le \lambda^q \left(\int_G \left|f(x)\right|^p w(x)dx\right)^{q/p} \tag{3.18}$$

for $f \in L^p_w(G)$, and

$$\int_{B(a,s)^c} v(x)\left|I_\alpha f(x)\right|^q dx \le \lambda^q \left(\int_{B(a,s)} |f(x)|^p w(x)dx\right)^{q/p} \tag{3.19}$$

for supp $f \subset B(a,s)$.

Now taking $f(x) = \chi_{B(a,r)}(x)w^{1-p'}(x)$ in (3.18) and observing that $\int_{B(a,r)} w^{1-p'}(x)dx < \infty$ for all $r > 0$ (see also [25, Chapter 3]), we find that

$$\int_{B(a,r)} v(x)\left(\int_{B(a,r)} \frac{w^{1-p'}(y)}{r(xy^{-1})^{Q-\alpha}}dy\right)^q dx \le \lambda^q \left(\int_{B(a,r)} w^{1-p'}(x)dx\right)^{q/p} < \infty. \tag{3.20}$$

Further if $x, y \in B(a,\tau)$, then

$$r(xy^{-1}) \le c_o(r(xa^{-1}) + r(ay^{-1})) \le 2c_o\tau. \tag{3.21}$$

Hence,

$$\|I_\alpha\|_\kappa \ge C_{\alpha,Q}A_1. \tag{3.22}$$

If $f(x) = \chi_{B(a,\tau)^c}(x)(w^{1-p'}(x)/r(ay^{-1})^{(Q-\alpha)(p'-1)})$, then

$$\int_{B(a,\tau)} v(x)\left(\int_{B(a,\tau)^c} \frac{w^{1-p'}(y)dy}{r(xy^{-1})^{Q-\alpha} r(ay^{-1})^{(Q-\alpha)(p'-1)}}\right)^q dx \le \lambda^q \left(\int_{B(a,\tau)^c} \frac{w^{1-p'}(x)dx}{r(ay^{-1})^{(Q-\alpha)p'}}\right)^{q/p} < \infty. \tag{3.23}$$

Let $r(xa^{-1}) < \tau$ and $r(ya^{-1}) > \tau$. Then,

$$r(xy^{-1}) \le c_o(r(xa^{-1}) + r(ay^{-1})) \le c_o(\tau + r(ay^{-1})) \le 2c_o r(ay^{-1}). \tag{3.24}$$

Hence, by (3.18) we have

$$\frac{1}{(2c_o)^{q(Q-\alpha)}}\left(\int_{B(a,\tau)} v(x)dx\right)\left(\int_{B(a,\tau)^c} \frac{w^{1-p'}(y)dy}{r(ay^{-1})^{(Q-\alpha)p'}}\right)^q \le \lambda^q \left(\int_{B(a,\tau)^c} \frac{w^{1-p'}(x)dx}{r(ay^{-1})^{(Q-\alpha)p'}}\right)^{q/p}. \tag{3.25}$$

The latter inequality implies

$$\|I_\alpha\|_\kappa \ge \frac{1}{(2c_o)^{Q-\alpha}}A_2. \tag{3.26}$$

Further, observe that (3.19) means that the norm of the operator

$$\overline{I}_\alpha f(x) = \int_{B(a,s)} \frac{f(y)dy}{r(y^{-1}a)^{Q-\alpha}} \tag{3.27}$$

can be estimated as follows:

$$\|\overline{I}_\alpha\|_{L^p_w(B(a,s)) \to L^q_v(B(a,s)^c)} \le \lambda. \tag{3.28}$$

Now by duality, we find that

$$\|\overline{I}_\alpha\|_{L^p_w(B(a,s)) \to L^q_v(B(a,s)^c)} = \|\widetilde{I}_\alpha\|_{L^{q'}_{v^{1-q'}}(B(a,s)^c) \to L^{p'}_{w^{1-p'}}(B(a,s))}, \tag{3.29}$$

where

$$\widetilde{I}_\alpha g(y) = \int_{B(a,s)^c} \frac{g(x)dx}{r(xy^{-1})^{Q-\alpha}}. \tag{3.30}$$

Indeed, by Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}
\|\overline{I}_\alpha f\|_{L^q_v(B(a,s)^c)} &\le \sup_{\|g\|_{L^{q'}_v(B(a,s)^c)} \le 1} \int_{B(a,s)^c} \left|g(x)\left(\overline{I}_\alpha f(x)\right)\right| dx \\
&\le \sup_{\|g\|_{L^{q'}_{v^{1-q'}}(B(a,s)^c)} \le 1} \int_{B(a,s)} |f(y)|\, \widetilde{I}_\alpha(|g|)(y)\, dy \\
&\le \sup_{\|g\|_{L^{q'}_{v^{1-q'}}(B(a,s)^c)} \le 1} \left(\int_{B(a,s)} |f|^p w\right)^{1/p} \left(\int_{B(a,s)} \left(\widetilde{I}_\alpha(|g|)\right)^{p'} w^{1-p'}\right)^{1/p'} \\
&\le \|\widetilde{I}_\alpha\| \left(\int_{B(a,s)} |f|^p w\right)^{1/p}.
\end{aligned} \tag{3.31}$$

Hence, $\|\overline{I}_\alpha\| \le \|\widetilde{I}_\alpha\|$. Analogously, $\|\widetilde{I}_\alpha\| \le \|\overline{I}_\alpha\|$.

Further, (3.19) implies

$$\int_{B(a,s)} w^{1-p'}(x) \left| \int_{(B(a,s))^c} \frac{g(y)dy}{r(xy^{-1})^{Q-\alpha}} dx \right|^{p'} \le \lambda^{p'} \left(\int_{(B(a,s))^c} |g(x)|^{q'} v^{1-q'}(x) dx \right)^{p'/q'}. \tag{3.32}$$

Now, taking $g(x) = \chi_{B(a,s)^c}(x) r(xa^{-1})^{(Q-\alpha)(1-q)} v(x)$ in the last inequality we conclude that $\|I_\alpha\|_\kappa \ge (1/(2c_o)^{Q-\alpha}) A_3$. □

Theorem 3.5 follows in the same manner as Theorem 3.4 was obtained. We only need to use Lemma H.

3.3. Truncated potentials

This subsection is devoted to the two-sided estimates of the essential norm for the operator:

$$T_\alpha f(x) = \int_{B(e,2r(x))} \frac{f(y)}{r(xy^{-1})^{Q-\alpha}}, \quad x \in G. \tag{3.33}$$

A necessary and sufficient condition guaranteeing the trace inequality for T_α in Euclidean spaces was established in [37]. This result was generalized in [38], [10, Chapter 6], for the spaces of homogeneous type. From the latter result as a corollary, we have the following proposition.

Proposition B. *Let $1 < p \le q < \infty$ and let $\alpha > Q/p$. Then,*

(i) *T_α is bounded from $L^p(G)$ to $L^q_v(G)$ if and only if*

$$B := \sup_{t>0} B(t) := \sup_{t>0} \left(\int_{r(x)>t} v(x) r(x)^{(\alpha-Q)q} dx \right)^{1/q} t^{Q/p'} < \infty; \tag{3.34}$$

(ii) *T_α is compact from $L^p(G)$ to $L^q_v(G)$ if and only if*

$$\lim_{t\to 0} B(t) = \lim_{t\to\infty} B(t) = 0. \tag{3.35}$$

Theorem 3.7. *Let $1 < p \le q < \infty$ and let $0 < \alpha < Q$. Suppose that T_α is bounded from $L^p_w(G)$ to $L^q_v(G)$. Then, the inequality*

$$\|T_\alpha\|_{\kappa(L^p_w(G)\to L^q_v(G))} \ge C_{Q,\alpha} \left(\lim_{a\to 0} A^{(a)} + \lim_{b\to\infty} A_{(b)} \right) \tag{3.36}$$

holds, where

$$\begin{aligned} C_{Q,\alpha} &= (2c_o)^{\alpha-Q}, \\ A^{(a)} &= \sup_{0<t<a} \left(\int_{B(e,a)\setminus B(e,t)} v(x) r(x)^{(\alpha-Q)q} dx \right)^{1/q} \left(\int_{B(e,t)} w^{1-p'}(x) dx \right)^{1/p'}, \\ A_{(b)} &= \sup_{t>b} \left(\int_{B(e,t)^c} v(x) r(x)^{(\alpha-Q)q} dx \right)^{1/q} \left(\int_{B(e,t)\setminus B(e,b)} w^{1-p'}(x) dx \right)^{1/p'}. \end{aligned} \tag{3.37}$$

To prove Theorem 3.7 we need the following lemma.

Lemma 3.8. *Let p, q, and α satisfy the conditions of Theorem 3.7 Then from the boundedness of T_α from $L^p_w(G)$ to $L^q_v(G)$, it follows that $w^{1-p'}$ is locally integrable on G.*

Proof. Let

$$I(t) = \int_{B(e,t)} w^{1-p'}(x) dx = \infty \tag{3.38}$$

for some $t > 0$. Then, there exists $g \in L^p(B(e,t))$ such that $\int_{B(e,t)} g w^{-1/p} = \infty$. Let us assume that $f_t(y) = g(y) w^{-1/p}(y) \chi_{B(e,t)}(y)$. Then, we have

$$\begin{aligned} \left\| T_\alpha f_t \right\|_{L^q_v(G)} &\geq \left\| \chi_{B(e,t)^c} T_\alpha f_t \right\|_{L^q_v(G)} \\ &\geq c \left(\int_{B(e,t)^c} v(x) r(x)^{(\alpha-Q)q} dx \right)^{1/q} \int_{B(e,t)} g(y) w^{-1/p'}(y) dy = \infty. \end{aligned} \tag{3.39}$$

On the other hand,

$$\left\| f_t \right\|_{L^p_w(G)} = \int_{B(e,t)} g^p(x) dx < \infty. \tag{3.40}$$

Finally, we conclude that $I(t) < \infty$ for all t, $t > 0$. □

Proof of Theorem 3.7. Let $\lambda > \|T_\alpha\|_{\kappa(L^p_w(G), L^q_v(G))}$. Then by Lemma E, there exists a positive constant β such that for all τ_1, τ_2, $0 < \tau_1 < \tau_2 < \beta$ and f, $\operatorname{supp} f \subset B(e, \tau_1)$, the inequality

$$\left\| T_\alpha f \right\|_{L^q_v(B(e,\tau_2) \setminus B(e,\tau_1))} \leq \lambda \| f \|_{L^p_w(B(e,\tau_1))} \tag{3.41}$$

holds. Observe that if $r(x) > \tau_1$ and $r(y) < \tau_1$, then $r(xy^{-1}) \leq 2c_o r(x)$. Consequently, taking $f = w^{1-p'} \chi_{B(e,\tau_1)}$ and using Lemma 3.8, we find that

$$\frac{1}{(2c_o)^{Q-\alpha}} \left(\int_{B(e,\tau_2) \setminus B(e.\tau_1)} v(x) (r(x))^{(\alpha-Q)q} dx \right)^{1/q} \left(\int_{B(e,\tau_1)} w^{1-p'}(x) dx \right)^{1/p'} \leq \lambda, \tag{3.42}$$

from which it follows that

$$\frac{1}{(2c_o)^{(Q-\alpha)q}} \lim_{a \to 0} A^{(a)} \leq \lambda. \tag{3.43}$$

Further, by virtue of Lemma E there exists β_2 such that for all s_1, s_2 with $\beta_2 < s_1 < s_2$ the inequality

$$\left\| T_\alpha f \right\|_{L^q_v(B(e,s_2)^c)} \leq \lambda \| f \|_{L^p_w(B(e,s_2) \setminus B(e,s_1))} \tag{3.44}$$

holds, where $\operatorname{supp} f \subset B(e,s_2) \setminus B(e,s_1)$. Hence by Lemma 3.8, we find that

$$\frac{1}{(2c_o)^{Q-\alpha}} \left(\int_{B(e,s_2)^c} v(x) (r(x))^{(\alpha-Q)q} dx \right)^{1/q} \left(\int_{B(e,s_2) \setminus B(e,s_1)} w^{1-p'}(x) dx \right)^{1/p'} \leq \lambda, \tag{3.45}$$

which leads us to

$$\frac{1}{(2c_o)^{Q-\alpha}} \lim_{b \to 0} A_{(b)} \leq \lambda. \tag{3.46}$$

Thus, we have the desired result. □

Theorem 3.9. *Let $1 < p \le q < \infty$ and let $Q/p < \alpha < Q$. Suppose that* (3.34) *holds. Then, there is a positive constant C such that*

$$\|T_\alpha\|_{\kappa(L^p(G)\to L^q_v(G))} \le C\Big(\lim_{a\to 0} B^{(a)} + \lim_{b\to 0} B_{(b)}\Big), \tag{3.47}$$

where

$$\begin{aligned} B^{(a)} &= \sup_{t\le a}\left(\int_{\overline{B}(e,a)\setminus B(e,r)} v(x) r(x)^{(\alpha-Q)q} dx\right)^{1/q} r^{Q/p'}, \\ B_{(b)} &= \sup_{t\ge b}\left(\int_{B(e,t)^c} v(x) r(x)^{(\alpha-Q)q} dx\right)^{1/q} \left(r^Q - b^Q\right)^{1/p'}. \end{aligned} \tag{3.48}$$

Proof. Let $0 < a < b < \infty$ and represent $T_\alpha f$ as follows:

$$\begin{aligned} T_\alpha f &= \chi_{\overline{B}(e,a)} T_\alpha\big(f\chi_{\overline{B}(e,a)}\big) + \chi_{\overline{B}(e,b)\setminus\overline{B}(e,a)} T_\alpha\big(f\chi_{\overline{B}(e,b)}\big) \\ &\quad + \chi_{G\setminus\overline{B}(e,b)} T_\alpha\big(f\chi_{\overline{B}(e,b/2c_0)}\big) + \chi_{G\setminus\overline{B}(e,b)} T_\alpha\big(f\chi_{G\setminus\overline{B}(e,b/2c_0)}\big) \\ &\equiv P_1 f + P_2 f + P_3 f + P_4 f. \end{aligned} \tag{3.49}$$

For P_2, we have

$$P_2 f(x) = \int_G k(x,y)\,dy, \tag{3.50}$$

where $k(x,y) = \chi_{\overline{B}(e,b)\setminus\overline{B}(e,a)}(x)\chi_{\overline{B}(e,2r(x))}(y) r(xy^{-1})^{\alpha-Q}$.

Further observe that

$$\begin{aligned} \int_G\left(\int_G (k(x,y))^{p'} dy\right)^{q/p'} v(x)\,dx &= \int_{\overline{B}(e,b)\setminus\overline{B}(e,a)}\left(\int_{\overline{B}(e,2r(x))} \left(r(xy^{-1})\right)^{(\alpha-Q)p'} dy\right)^{q/p'} v(x)\,dx \\ &\le c\int_{\overline{B}(e,b)\setminus\overline{B}(e,a)}\left(\int_{\overline{B}(e,r(x)/2c_0)} \left(r(xy^{-1})\right)^{(\alpha-Q)p'} dy\right)^{q/p'} v(x)\,dx \\ &\le c\int_{\overline{B}(e,b)\setminus\overline{B}(e,a)} r(x)^{(\alpha-Q)q+q/p'} v(x)\,dx < \infty. \end{aligned} \tag{3.51}$$

Hence by Lemma I, we conclude that P_2 is compact for every a and b. Now we observe that if $r(x) > b$ and $r(y) < b/2c_o$, then $r(x) \le 2c_o r(xy^{-1})$. Due to Proposition A we have that P_3 is compact.

Further, we know that (see [38], [10, Chapter 6])

$$\|P_1\| \le C_1 B^{(a)}, \qquad \|P_4\| \le C_2 B_{(b/2c_o)}, \tag{3.52}$$

where the constants C_1 and C_2 depend only on p, q, Q, and α.

Therefore,

$$\|T_\alpha - P_2 - P_3\| \le \|P_1\| + \|P_4\| \le c\big(B^{(a)} + B_{(b)}\big). \tag{3.53}$$

The last inequality completes the proof. □

Theorem 3.10. *Let p and q satisfy the conditions of Theorem 3.9. Suppose that* (3.18) *holds. Then, one has the following two-sided estimate:*

$$c_2\Big(\lim_{a\to 0} B^{(a)} + \lim_{b\to\infty} B_{(b)}\Big) \le \|T_\alpha\|_{\kappa(L^p(G),L^q_v(G))} \le c_1\Big(\lim_{a\to 0} B^{(a)} + \lim_{b\to\infty} B_{(b)}\Big) \tag{3.54}$$

for some positive constants c_1 and c_2 depending only on Q, α, p, and q.

Theorem 3.10 follows immediately from Theorems 3.7 and 3.9.

3.4. Partial sums of Fourier series

Here, we investigate the lower estimate of the essential norm for the partial sums of the Fourier series:

$$S_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n(t)\,dt, \tag{3.55}$$

where $D_n = 1/2 + \sum_{k=1}^{n} \cos kt$.

One-weighted inequalities for S_n were obtained in [32] (see also [25, Chapter 6]). For basic properties of S_n in unweighted case; see, for example, [39].

Theorem 3.11. *Let $1 < p < \infty$. Then, there is no $n \in \mathbb{N}$ and weight pair (w, v) on $T := (-\pi, \pi)$ such that S_n is compact from $L^p_w(T)$ to $L^p_v(T)$. Moreover, if S_n is bounded from $L^p_w(T)$ to $L^p_v(T)$, then*

$$\|S_n\| \ge \frac{\left(2+2^{1/2}\right)^{1/2}}{2\pi} \sup_{a\in T} \overline{\lim_{r\to 0}} \left(\frac{1}{2r}\int_{a-r}^{a+r} v\right)^{1/p} \left(\frac{1}{2r}\int_{a-r}^{a+r} w^{1-p'}\right)^{1/p'}, \tag{3.56}$$

where $I = (a - r, a + r)$.

Proof. Taking $\lambda > \|S_n\|_{\kappa(L^p_w(T),L^p_v(T))}$, by Lemma H we find that

$$\int_I v(x)\left|S_n f(x)\right|^p dx \le \lambda^p \int_I \left|f(x)\right|^p w(x)\,dx \tag{3.57}$$

for all intervals $I = (a - r, a + r)$, where r is a small positive number.

Let

$$J_1 = \int_I v(x)\left|S_n f(x)\right|^p dx, \qquad J_2 = \int_I \left|f(x)\right|^p w(x)\,d(x). \tag{3.58}$$

Suppose that $|I| \le \pi/4$, and let n be the greatest integer less than or equal to $\pi/4|I|$. Then for $x \in I$ (see [32]),

$$\left|S_n f(x)\right| \ge \frac{1}{\pi}\int_I \frac{\left|f(\theta)\right| \sin(3\pi/8)}{\pi/4n}\,d\theta. \tag{3.59}$$

Using this estimate and taking $f := w^{1-p'}(x)\chi_I(x)$, we find that

$$J_1 \ge \left(\frac{1}{\pi}\sin\frac{3\pi}{8}\right)^p |I|^{-p} \left(\int_I v\right)\left(\int_I w^{1-p'}\right)^p. \tag{3.60}$$

On the other hand, it is easy to see that $J_2 = \int_I w^{1-p'} < \infty$.

Hence, by (3.57) we conclude that

$$\lambda \geq \frac{1}{\pi} \sin \frac{3\pi}{8} \left(\frac{1}{|I|} \int_I v \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^{1/p'}. \tag{3.61}$$

Now passing r to 0, taking supremum over $a \in T$, and using the fact $\sin(3\pi/8) = (2 + 2^{1/2})^{1/2}/2$, we find that (3.56) holds. □

Corollary 3.12. *Let $1 < p < \infty$ and let $n \in \mathbb{N}$. Then*

$$\|S_n\|_{\kappa(L^p(T))} \geq \frac{(2 + 2^{1/2})^{1/2}}{2\pi}. \tag{3.62}$$

Corollary 3.13. *Let $1 < p < \infty$ and let $n \in \mathbb{N}$. Suppose that $w(x) = v(x) = |x|^{\alpha}$. Then, one has*

$$\|S_n\|_{\kappa(L^p_w(T))} \geq \frac{(2 + 2^{1/2})^{1/2}}{2\pi} \left(\frac{1}{\alpha + 1} \right)^{1/p} \left(\frac{1}{\alpha(1 - p') + 1} \right)^{1/p'}. \tag{3.63}$$

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