Proc. A. Razmadze Math. Inst. 151(2009), 139–143

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MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN MORREY SPACES ASSOCIATED WITH GRAND LEBESGUE SPACES

(Reported on 23.09.2009)

Let (X, d, μ) be a space of homogeneous type (SHT). This is a topological space X with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1_\mu(X)$ and there is a non–negative function (quasimetric) $d : X \times X \to \mathbf{R}_+$ which satisfies the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$.

(ii) $d(x, y) > 0$ for all $x \neq y$, $x, y \in X$.

(iii) There exists a positive constant a_0 such that $d(x, y) \leq a_0 d(y, x)$ for every $x, y \in X$.

(iv) There exists a constant a_1 such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for every $x,\ y,\ z \in X$.

(v) For every neighbourhood V of the point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in V.

(vi) Balls $B(x, r)$ are measurable for every $x \in X$ and for arbitrary $r > 0$.

(vii) There exists a constant $b > 0$ such that

$$\mu B(x, 2r) \leq b\mu(B(x, r)) < \infty$$

for every $x \in X$ and $r,\ 0 < r < \infty$.

Throughout the paper we assume that

$$L := \text{ diam } (X) < \infty.$$

This condition obviously implies that $\mu(X) < \infty$.

The grand Lebesgue space $L^{p)}(X)$ $(1 < p < \infty)$ is a rearrangement invariant Banach space defined by the norm

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2000 *Mathematics Subject Classification:* 42B20, 42B25, 46E30, 47B38.

Key words and phrases. Maximal functions, singular integrals, Cauchy singular integral operator, spaces of homogeneous type, Carleson curves, Morrey spaces, grand Lebesgue spaces, boundedness.

$$\|f\|_{L^{p)}(X)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{\mu(X)} \int_X |f(x)|^{p-\varepsilon} d\mu(x) \right)^{1/(p-\varepsilon)}.$$

For some properties and applications of $L^{p)}$ spaces we refer to the papers [8], [9], [4]. It is worth mentioning that the following continuous embeddings hold:

$$L^p(X) \subset L^{p)}(X) \subset L^{p-\varepsilon}(X), \quad 0 < \varepsilon \le p-1.$$

The space $L^{p)}$ on a finite interval was introduced in [10].

Let

$$(Mf)(x) = \sup_{\substack{x\in X \\ 0<r<L}} (\mu B(x,r))^{-1} \int_{B(x,r)} |f(y)| d\mu(y), \quad x \in X,$$

be the Hardy–Littlewood maximal operator defined on X. Further, let $k : X \times X \setminus \{(x,x) : x \in X\} \to \mathbf{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \le \frac{c}{\mu B(x, d(x,y))}, \quad x, y \in X, \quad x \ne y;$$

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big) \frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) > d(x,x_2)$, where ω is a positive, non-decreasing function on $(0,\infty)$ satisfying Δ_2 condition ($\omega(2t) \le c\omega(t)$, $t > 0$) and the Dini condition $\int_0^1 \omega(t)/t dt < \infty$.

We also assume that

$$(Kf)(x) = \text{p.v.} \int_X k(x,y) f(y) d\mu(y)$$

exists almost everywhere on X and that K is bounded in $L^{p_0}(X)$ for some $1 < p_0 < \infty$.

This note is devoted to the boundedness of the operators M and K in Morrey spaces associated with grand Lebesgue spaces $L^{p),\lambda}(X)$, where $1 < p < \infty$ and $0 \le \lambda < 1$. This is a space of measurable functions f defined on X with the norm

$$\|f\|_{L^{p),\lambda}(X)} := \sup_{0<\varepsilon<p-1} \left(\sup_{\substack{x\in X \\ 0<r<L}} \frac{\varepsilon}{\mu(B(x,r))^{\lambda}} \int_{B(x,r)} |f(y)|^{p-\varepsilon} d\mu(y) \right)^{\frac{1}{p-\varepsilon}}.$$

Let us recall that the classical Morrey space defined on X is the class of functions f on X for which

$$\|f\|_{L^{p,\lambda}(X)} := \sup_{\substack{x\in X\\ 0<r<L}} \left(\frac{1}{\mu(B(x,r))^{\lambda}} \int\limits_{B(x,r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty,$$

where $1 < p < \infty$ and $0 \le \lambda < 1$.

For some basic properties of the classical Morrey spaces we refer to the articles [13], [15], [7], [1]. For the boundedness of maximal and singular integrals in the classical Morrey spaces we refer to [2], [6], [11], [16].

It is easy to verify that the following embeddings are valid:

$$L^{p,\lambda}(X) \hookrightarrow L^{p),\lambda}(X) \hookrightarrow L^{p-\varepsilon,\lambda}(X), \quad 0 < \varepsilon \le p-1,$$

Criteria governing the one–weight inequality for the Hardy-Littlewood maximal operator and the Hilbert transform defined on a finite interval in grand Lebesgue spaces was established in [5] and [12] respectively.

In this note our main result is the following statement:

Theorem. *Let $1 < p < \infty$, $0 \le \lambda < 1$. Then the operators T and K are bounded in $L^{p),\lambda}(X)$.*

Let us now discuss the special case of an SHT.

Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be arc-length measure on Γ. By definition, Γ is regular if

$$\nu\big(D(z,r)\cap\Gamma\big) \le r$$

for every $z \in \Gamma$ and all $r > 0$, where $D(z,r)$ is a disc in $\mathbb{C}$ with center z and radius r. The reverse inequality

$$\nu\big(D(z,r)\cap\Gamma\big) \ge cr$$

holds for all $z \in \Gamma$ and $r < L/2$, where L is a diameter of Γ. If we equip Γ with the measure ν and the Euclidean metric, the regular curve becomes an SHT.

The associate kernel in which we are interested is

$$k(z,w) = \frac{1}{z-w}.$$

The Cauchy integral

$$(S_\Gamma f)(t) = \text{p.v.} \int\limits_{\Gamma} \frac{f(\tau)}{t-\tau} d\nu(\tau), \quad t \in \Gamma,$$

is the corresponding singular operator.

The above-mentioned kernel in the case of regular curves is a Calderón-Zygmund kernel. As was proved by G. David [3], a necessary and sufficient condition for continuity of the operator S_Γ in $L^r(\Gamma)$, where r is a constant $(1 < r < \infty)$, is that Γ is regular.

Together with the operator S_Γ we are interested in the Hardy–Littlewood maximal operator defined on Γ

$$(M_\Gamma f)(z) = \sup_{0<r<L} \frac{1}{\nu\big(D(z,r)\cap\Gamma\big)} \int\limits_{D(z,r)\cap\Gamma} |f(t)|d\nu(t).$$

Let $1 < p < \infty$ and let $0 \leq \lambda < 1$. We say that a measurable locally integrable function f on Γ belongs to the class $L^{p),\lambda}(\Gamma)$ if

$$\|f\|_{L^{p),\lambda}(\Gamma)} :=$$
$$= \sup_{0<\varepsilon<p-1} \left(\sup_{\substack{z\in\Gamma \\ 0<r<L}} \frac{\varepsilon}{\nu\big(D(z,r)\cap\Gamma\big)^\lambda} \int\limits_{D(z,r)\cap\Gamma} |f(t)|^{p-\varepsilon} d\nu(t) \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

The theorem formulated above implies the following statement:

Proposition . *Let $L < \infty$, $1 < p < \infty$ and let $0 \leq \lambda < 1$. Suppose that Γ is regular. Then the operators M_Γ and S_Γ are bounded in $\|f\|_{L^{p),\lambda}(\Gamma)}$.*

Remark. In the forthcoming papers we will present the results regarding the boundedness of maximal and singular operators in $L^{p),\lambda}$ spaces defined on quesimetric measure spaces, where the doubling condition is not assumed.

Acknowledgement

The author was partially supported by the Georgian National Science Foundation Grant No. GNSF/ST07/3-169.

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