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AZUMAYA ALGEBRAS AS GALOIS COMODULES

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Abstract. It is shown that Azumaya algebras (over commutative rings) are special examples of Galois comodules. This leads to a new characterization of Azumaya algebras.

CONTENTS

1. Preliminaries

Theorem 1.1 (distributive laws and adjoint functors)**.** *Let $\lambda : TS \to ST$ be a distributive law from a monad $\boldsymbol{T}$ to a comonad $\boldsymbol{S}$ on any category $\mathbb{A}$. It is proved in* [3] (*compare also* [7]) *that if the monad $\boldsymbol{T}$ admits a right adjoint comonad $\boldsymbol{T}^\diamond$, then the composite*

$$\lambda^\diamond : ST^\diamond \xrightarrow{\overline{\eta}ST^\diamond} T^\diamond TST^\diamond \xrightarrow{T^\diamond\lambda T^\diamond} T^\diamond STT^\diamond \xrightarrow{T^\diamond S\overline{\varepsilon}} T^\diamond S,$$

where $\overline{\eta} : 1 \to T^\diamond T$ and $\overline{\varepsilon} : TT^\diamond \to 1$ are the unit and counit of the adjunction $T \dashv T^\diamond$, is a mixed distributive law from the monad $\boldsymbol{S}$ to the comonad $\boldsymbol{T}^\diamond$. Moreover, there is an isomorphism of categories

$$\mathbb{A}_{\underline{\boldsymbol{ST}}} \simeq (\mathbb{A}_{\boldsymbol{S}})^{\widetilde{\boldsymbol{T}^\diamond}},$$

where $\widetilde{\boldsymbol{T}^\diamond}$ is the lifting of the comonad $\boldsymbol{T}^\diamond$ to $\mathbb{A}_{\boldsymbol{S}}$.

Note that for a monad $\boldsymbol{T}$ on $\mathbb{A}$, the category of $\boldsymbol{T}$-modules is denoted $\mathbb{A}_{\boldsymbol{T}}$, while for a comonad $\boldsymbol{G}$ on $\mathbb{A}$, the category of $\boldsymbol{G}$-comodules is denoted by $\mathbb{A}^{\boldsymbol{G}}$.

Theorem 1.2 (Galois functors)**.** *Given a comonad $\boldsymbol{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$, a functor $F : \mathbb{B} \to \mathbb{A}$ is a* left **G**-comodule *if there exists a natural transformation $\alpha : F \to GF$ with the commutative diagrams*

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & GF \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon F} \\ & & F, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\alpha} & GF \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \delta F} \\ GF & \xrightarrow[G\alpha]{} & GGF. \end{array}$$

According to the dual of [4, Proposition II.1.1], a $\boldsymbol{G}$-comodule structure on $F : \mathbb{B} \to \mathbb{A}$ is equivalent to the existence of a functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ leading to the commutative diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\overline{F}} & \mathbb{A}^G \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle U^G} \\ & & \mathbb{A}. \end{array}$$

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If a $\boldsymbol{G}$-comodule (F, β) admits a right adjoint $R : \mathbb{A} \to \mathbb{B}$, with counit $\sigma : FR \to 1$, then (see [4, Propositions II.1.1 and II.1.4]) *the composite*

$$t_{\overline{F}} : FR \xrightarrow{\alpha R} GFR \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by the adjunction $F \dashv R$ to the comonad $\boldsymbol{G}$.

*A left $\boldsymbol{G}$-comodule $F : \mathbb{B} \to \mathbb{A}$ with a right adjoint $R : \mathbb{A} \to \mathbb{B}$ is said to be $\boldsymbol{G}$-*Galois *comodule if the corresponding morphism $t_{\overline{F}} : FR \to G$ of comonads on $\mathbb{A}$ is an isomorphism.*

If the category $\mathbb{B}$ has equalizers of coreflexive pairs, the functor $\overline{F}$ has a right adjoint.

Proposition 1.3 (see [8, Theorem 1.6])**.** *The functor $\overline{F}$ is an equivalence of categories if and only if the functor F is $\boldsymbol{G}$-Galois and comonadic.*

Throughout R will denote a commutative ring with unit. The tensor product $\otimes$ will always mean $\otimes_R$. For any R-algebra A, $\mathbb{M}_A$, ${}_A\mathbb{M}$, and ${}_A\mathbb{M}_A$ denote the categories of left A-modules, right A-modules, and (A, A)-bimodules.

Theorem 1.4 (Galois comodules)**.** *Consider an A-coring $\mathfrak{C}$, A an associative R-algebra, and write $\mathbb{M}^{\mathfrak{C}}$ for the category of right $\mathfrak{C}$-comodules. Take an arbitrary right $\mathfrak{C}$-comodule (P, ρ) and set*

$$S = End^{\mathfrak{C}}(P, \rho) = \mathbb{M}^{\mathfrak{C}}((P, \rho), (P, \rho)).$$

Then S is an R-algebra with respect to the composition of endomorphisms and P is a left S-module, and since ρ is S-linear, there is a functor $K : \mathbb{M}_S \to \mathbb{M}^{\mathfrak{C}}$ that takes $N \in \mathbb{M}_S$ to $(N \otimes_S P, N \otimes_S \rho)$. Observe that for the forgetful functor $U^{\mathfrak{C}} : \mathbb{M}^{\mathfrak{C}} \to \mathbb{M}_A$, one has $U^{\mathfrak{C}}K(N) = N \otimes_S P$. Thus, the diagram

$$\begin{array}{ccc} \mathbb{M}_S & \xrightarrow{K} & \mathbb{M}^{\mathfrak{C}} \\ & {\scriptstyle -\otimes_S P}\searrow & \downarrow {\scriptstyle U^{\mathfrak{C}}} \\ & & \mathbb{M}_A \end{array}$$

commutes. Since $\mathbb{M}^{\mathfrak{C}}$ is isomorphic to the Eilenberg–Moore category of $\boldsymbol{G}$-comodules, $\boldsymbol{G}$ being the comonad on $\mathbb{M}_A$ induced by the A-coalgebra $\mathfrak{C}$, and since the functor $\mathrm{Hom}_A(P, -) : \mathbb{M}_A \to \mathbb{M}_S$ *is a right adjoint to the functor $- \otimes_S P : \mathbb{M}_S \to \mathbb{M}_A$, there is a comonad morphism*

$$t_K : \mathrm{Hom}_A(P, -) \otimes_S P \to - \otimes_A \mathfrak{C}.$$

Write $\rho(p) = \sum p_{(0)} \otimes_A p_{(1)}$, $p \in P$; a direct calculation shows that

$$(t_K)_M(f \otimes_S p) = \sum f(p_{(0)}) \otimes_A p_{(1)}$$

for all $f : P \to M$, $M \in \mathbb{M}_A$. P is called a Galois comodule *provided $(t_K)_M$ is an isomorphism for any right A-module M, that is, the functor $- \otimes_S P : \mathbb{M}_S \to \mathbb{M}^{\mathfrak{C}}$ is a $- \otimes_A \mathfrak{C}$-Galois comodule* (see [9, Definiton 4.1])*.*

Theorem 1.5 (monads related to an algebra)**.** *For any R-algebra A, one has the monads*

$$T := A \otimes - : \quad \mathbb{M}_R \to \mathbb{M}_R,$$
$$S := A^{op} \otimes - : \quad \mathbb{M}_R \to \mathbb{M}_R,$$

and it is clear that the ordinary twist map

$$\tau : A \otimes A^{op} \to A^{op} \otimes A, \quad a \otimes \bar{b} \mapsto \bar{b} \otimes a,$$

induces a distributive law from the monad $T = A \otimes -$ to the monad $S = A^{op} \otimes -$. Note that the category $(\mathbb{M}_R)_{\boldsymbol{ST}}$ is just the category of left $(A^{op} \otimes A)$-modules, which is in turn isomorphic to the category ${}_A\mathbb{M}_A$ of (A, A)-bimodules.

Theorem 1.6 (adjunction $(A \otimes -, \mathrm{Hom}_R(A,-))$)**.** *For $X, Y \in \mathbb{M}_R$, we write $[X,Y]$ for $\mathrm{Hom}_R(X,Y)$. Since A is an R-algebra, the functor $T^\diamond = [A,-] : \mathbb{M}_R \to \mathbb{M}_R$ is a comonad that is right adjoint to the monad $T = A \otimes -$. Recall that the unit η and counit ε of this adjunction are defined as follows: For any $X \in \mathbb{M}_R$,*

$$\eta_X : X \to T^\diamond T(X) = [A, A \otimes X], \quad x \mapsto (a \mapsto a \otimes x),$$
$$\varepsilon_X : TT^\diamond(X) = A \otimes [A,X] \to X, \quad a \otimes f \to f(a).$$

Then for any $X \in \mathbb{M}_R$, the X-component $\tau^\diamond_X$ of the mixed distributive law $\tau^\diamond : ST^\diamond \to T^\diamond S$ corresponding to the distributive law τ is the composite

$$A^{op} \otimes [A,X] \xrightarrow{\eta_{A^{op}\otimes[A,X]}} [A, A \otimes A^{op} \otimes [A,X]]$$
$$\xrightarrow{[A,\tau\otimes[A,X]]} [A, A^{op} \otimes A \otimes [A,X]] \xrightarrow{[A,A^{op}\otimes\varepsilon_X]} [A, A^{op} \otimes X].$$

It is easy to see that for all $\overline{a} \in A^{op}$, $f \in [A,X]$, $\tau^\diamond_X(\overline{a} \otimes f)$ is the homomorphism $A \to A^{op} \otimes X$ that takes an arbitrary $a' \in A$ to $\overline{a} \otimes f(a')$.

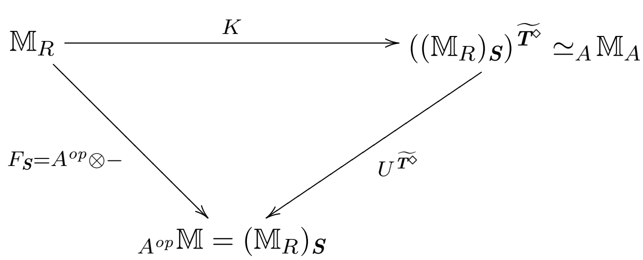
Theorem 1.7 ((A,A)-bimodules)**.** *For any $X \in \mathbb{M}_R$, $A^{op} \otimes X$ has the structure of an (A,A)-bimodule. The left action corresponds under the isomorphisms ${}_A\mathbb{M}_A \simeq (\mathbb{M}_R)_{\boldsymbol{ST}} \simeq ((\mathbb{M}_R)_{\boldsymbol{S}})^{\widetilde{\boldsymbol{T}^\diamond}}$ to a left $\widetilde{\boldsymbol{T}^\diamond}$-comodule structure*

$$\alpha_X : F_{\boldsymbol{S}}(X) = A^{op} \otimes X \to \widetilde{T^\diamond} F_{\boldsymbol{S}}(X) = [A, A^{op} \otimes X], \quad \overline{a} \otimes x \mapsto [a' \mapsto \overline{aa'} \otimes x].$$

This leads to the functor

$$K : \mathbb{M}_R \to ((\mathbb{M}_R)_{\boldsymbol{S}})^{\widetilde{\boldsymbol{T}^\diamond}} \simeq {}_A\mathbb{M}_A, \quad X \mapsto A^{op} \otimes X$$

that makes the diagram



(1.1)

commutative. Now it is easy to see that for any $\boldsymbol{X} = (X,h) \in_{A^{op}} \mathbb{M}$, the $\boldsymbol{X}$-component $t_{\boldsymbol{X}}$ of the comonad morphism $t : F_{\boldsymbol{S}} U_{\boldsymbol{S}} \to \widetilde{T^\diamond}$ is the composite

$$t_{\boldsymbol{X}} : F_{\boldsymbol{S}} U_{\boldsymbol{S}}(\boldsymbol{X}) = A^{op} \otimes X \xrightarrow{\alpha_X} [A, A^{op} \otimes X] \xrightarrow{[A,h]} [A,X] = \widetilde{T^\diamond}(X)\ .$$

Thus, for all $\overline{a} \in A^{op}$, $a' \in A$ and $x \in X$, $t_{\boldsymbol{X}}(\overline{a} \otimes x) = (a' \mapsto \overline{a'a} \cdot x)$.

2. Azumaya Algebras

Applying [5, Theorem 4.4] and using that A is an Azumaya R-algebra if and only if the functor K is an equivalence of categories, we obtain the following assertion.

Theorem 2.1. *An R-algebra A is an Azumaya R-algebra if and only if*

(1) *the functor $A^{\mathrm{op}} \otimes - : \mathbb{M}_R \to_{A^{\mathrm{op}}} \mathbb{M}$ is comonadic, and*
(2) *for any $M \in_{A^{\mathrm{op}}} \mathbb{M}$, the map*

$$A^{op} \otimes M \to [A, M], \quad \overline{a} \otimes m \mapsto [a' \mapsto aa'm],$$

is an isomorphism.

Since the canonical morphism $i : R \to A$ factorizes through the centre of A, it follows from [6, Theorem 8.11] that the functor

$$A^{\mathrm{op}} \otimes - : \mathbb{M}_R \to {}_{A^{\mathrm{op}}}\mathbb{M}$$

is comonadic if and only if i is a pure morphism of R-modules. Thus we have the following result.

Theorem 2.2. *An R-algebra A is an Azumaya R-algebra if and only if*

(1) *the canonical morphism $i : R \to A$ is pure in $\mathbb{M}_R$, and*
(2) *for any $M \in {}_{A^{op}}\mathbb{M}$, the map*

$$A^{op} \otimes M \to [A, M], \quad \overline{a} \otimes m \mapsto [a' \mapsto aa'm],$$

is an isomorphism.

Since for any R-module M, $A^{\mathrm{op}} \otimes M$ with the left A^{op}-module structure $\overline{a} \cdot (\bar{b} \otimes m) = \overline{ba} \cdot m$ is a free $\mathbf{S}$-module $\phi_{\mathbf{T}}(M)$, $A \otimes M$ can be viewed as the free $\mathbf{S}$-module $\phi_{\mathbf{T}}(M)$, where the left action on $A \otimes M$ is given by $\overline{a} \cdot b \otimes m) = \overline{ba} \cdot m$. Applying now [2, Lemma 2.19] we have the following assertion.

Theorem 2.3. *An R-algebra A is an Azumaya R-algebra if and only if*

(1) *the canonical morphism $i : R \to A$ is pure in $\mathbb{M}_R$, and*
(2) *for any $M \in \mathbb{M}_R$, the map*

$$A^{op} \otimes A \otimes M \to [A, A \otimes M], \quad \overline{a} \otimes b \otimes m \mapsto [a' \mapsto ba'a \otimes m],$$

is an isomorphism.

Let M be an (A, A)-bimodule. Define

$$M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

Theorem 2.4. *An R-algebra A is an Azumaya R-algebra if and only if*

(1) *the canonical morphism $i : R \to A$ is pure in $\mathbb{M}_R$, and*
(2) *for any (A, A)-bimodule M, the evaluation map*

$$A^{op} \otimes M^A \to M, \quad a \otimes m \mapsto am,$$

is an isomorphism.

Noting that for a (von Neumann) regular ring (e.g., a field) R, $i : R \to A$ is always pure, we have the following theorem.

Theorem 2.5. *Let R be a regular ring. For an R-algebra A, the following are equivalent*:

(a) *A is an Azumaya R-algebra*;
(b) *for any $M \in {}_{A^{\mathrm{op}}}\mathbb{M}$, the map*

$$A^{op} \otimes M \to [A, M], \quad \overline{a} \otimes m \mapsto [a' \to \overline{a'a} \cdot m]$$

is an isomorphism;
(c) *for any $M \in \mathbb{M}_R$, the map*

$$A^{op} \otimes A \otimes M \to [A, A \otimes M], \quad \overline{a} \otimes b \otimes m \mapsto [a' \mapsto ba'a \otimes m],$$

is an isomorphism;
(d) *for any (A, A)-bimodule M, the evaluation map* $\mathrm{A}^{op} \otimes \mathrm{M}^A \to \mathrm{M}$ *is an isomorphism.*

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