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Galois functors and generalised Hopf modules

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Dedicated to Professor Hvedri Inassaridze on the occasion of his 80th birthday

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Abstract As shown in a previous paper by the same authors, the theory of Galois functors provides a categorical framework for the characterisation of bimonads on any category as *Hopf monads* and also for the characterisation of opmonoidal monads on monoidal categories as *right Hopf monads* in the sense of Bruguières and Virelizier. Hereby the central part is to describe conditions under which a comparison functor between the base category and the category of Hopf modules becomes an equivalence (*Fundamental Theorem*). For monoidal categories, Aguiar and Chase extended the setting by replacing the base category by a comodule category for some comonoid and considering a comparison functor to *generalised Hopf modules*. For duoidal categories, Böhm, Chen and Zhang investigated a comparison functor to the Hopf modules over a *bimonoid* induced by the two monoidal structures given in such categories. In both approaches fundamental theorems are proved and the purpose of this paper is to show that these can be derived from the theory of Galois functors.

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1 Introduction

Bialgebras A over a commutative ring R induce an endofunctor $A \otimes_R -$ on the category $\mathbb{M}_R$ of R-modules which has a monad and a comonad structure subject to some compatibility conditions. To make the bialgebra A a *Hopf algebra* the comparison functor from $\mathbb{M}_R$ to the category of Hopf modules $\mathbb{M}_A^A$ induced by $A \otimes_R -$ has to be an equivalence (e.g. [5, 7.9]).

Since all these constructions are based on the tensor product in $\mathbb{M}_R$, one may try to extend the notions to monads $\mathcal{T} = (T, m, e)$ on (strictly) monoidal categories $(\mathbb{V}, \otimes, I)$. To ensure that the Eilenberg–Moore category $\mathbb{V}_\mathcal{T}$ is again monoidal, $\mathcal{T}$ has to be an *opmonoidal monad* (e.g. [11]). Such functors consist of two parts: the monad $\mathcal{T}$ and a comonad $- \otimes T(I)$ on $\mathbb{V}$ induced by the coalgebra $T(I)$ which are related by a mixed distributive law (entwining) (e.g. [16, Section 5]). Then a comparison functor between $\mathbb{V}$ and the category of the *entwined modules* determined by this entwining (called *right Hopf $\mathcal{T}$-modules* in [9, Section 4.2]) may be considered. [17, Theorem 4.7] gives a necessary and sufficient condition for this comparison functor to be an equivalence of categories.

In [16], an entwining of a monad $\mathcal{T} = (T, m, e)$ and a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on any category $\mathbb{A}$ is considered, that is, a natural transformation $\lambda : TG \to GT$ subject to certain commutativity conditions (e.g. [21, 5.3]). Then the comonad $\mathcal{G}$ on $\mathbb{A}$ can be lifted to a comonad $\widehat{\mathcal{G}}$ and the λ-entwining modules are just the $\widehat{\mathcal{G}}$-comodules in $\mathbb{A}_\mathcal{T}$ (see Sect. 2.5). For a comparison functor $K : \mathbb{A} \to (\mathbb{A}_\mathcal{T})^{\widehat{G}}$ one requires commutativity of the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K"] \arrow[dr, "\phi_\mathcal{T}"'] & (\mathbb{A}_\mathcal{T})^{\widehat{G}} \arrow[d, "U^{\widehat{G}}"] \\ & \mathbb{A}_\mathcal{T}, \end{tikzcd}$$

where $\phi_\mathcal{T}$ denotes the free $\mathcal{T}$-module functor and $U^{\widehat{G}}$ the forgetful $\widehat{G}$-comodule functor. In [16,17] conditions are given which make K an equivalence.

This setting comprises the opmonoidal monads outlined above and it also applies to the *bimonads* on arbitrary categories introduced in [21, 5.13], [15, Definition 4.1].

To subsume the generalised Hopf modules studied by Aguiar and Chase in [1], one has to add an adjunction $L \dashv R : \mathbb{A} \to \mathbb{B}$ to the picture and observe that the resulting adjunction $\phi_{\mathcal{T}} L \dashv RU_{\mathcal{T}}$ generates a comonad on $\mathbb{A}_{\mathcal{T}}$. Now the results from [16] can be applied to the diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[dr, "\phi_{\mathcal{T}} L"'] & (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}} \arrow[d, "U^{\widehat{\mathcal{G}}}"] \\ & \mathbb{A}_{\mathcal{T}}. \end{tikzcd}$$

This is outlined in Sect. 3 leading to the Fundamental Theorem of generalised Hopf modules from [1].

Having made this extension, also the A-Hopf modules of a bimonoid A in a duoidal category $(\mathbb{D}, \circ, I, *, J)$ and the related comparison functor considered by Böhm, Chen and Zhang in [6] can be handled in our setting: roughly speaking, for a bimonoid A, $- \circ A$ defines a monad while $- * A$ is a comonad on $\mathbb{D}$ and the two functors are related by an entwining. Now it is fairly obvious how our techniques apply and at the end of Sect. 4 we obtain the Fundamental Theorem for A-Hopf modules from [6].

2 Galois functors

2.1 Monads and comonads

Let $\mathcal{T} = (T, m, e)$ be a monad on a category $\mathbb{A}$. We write

- $\mathbb{A}_{\mathcal{T}}$ for the Eilenberg–Moore category of $\mathcal{T}$-modules and

$$\eta_{\mathcal{T}}, \varepsilon_{\mathcal{T}} : \phi_{\mathcal{T}} \dashv U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \to \mathbb{A}$$

  for the corresponding forgetful-free adjunction;
- $\widetilde{\mathbb{A}}_{\mathcal{T}}$ for the Kleisli category for $\mathcal{T}$ (as a full subcategory of $\mathbb{A}_{\mathcal{T}}$, e.g. [5]) and $\overline{\phi}_{\mathcal{T}} \dashv u_{\mathcal{T}} : \widetilde{\mathbb{A}}_{\mathcal{T}} \to \mathbb{A}$ for the corresponding Kleisli adjunction.

Dually, if $\mathcal{G} = (G, \delta, \varepsilon)$ is a comonad on $\mathbb{A}$, we write $\mathbb{A}^{\mathcal{G}}$ for the Eilenberg–Moore category of $\mathcal{G}$-comodules and

$$\eta^{\mathcal{G}}, \varepsilon^{\mathcal{G}} : U^{\mathcal{G}} \dashv \phi^{\mathcal{G}} : \mathbb{A} \to \mathbb{A}^{\mathcal{G}}$$

for the corresponding forgetful-cofree adjunction.

2.2 Comodule functors

Consider an adjunction $\eta, \sigma : F \dashv R : \mathbb{A} \to \mathbb{B}$ and a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$. The functor $F : \mathbb{B} \to \mathbb{A}$ is called a *left* $\mathcal{G}$*-comodule* (e.g. [15, Section 3]) if there exists a natural transformation $\kappa_F : F \to GF$ inducing commutativity of the diagrams

$$\begin{tikzcd} F \arrow[r, "\kappa_F"] \arrow[dr, equal] & GF \arrow[d, "\varepsilon F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\kappa_F"] \arrow[d, "\kappa_F"'] & GF \arrow[d, "\delta F"] \\ GF \arrow[r, "G\kappa_F"'] & GGF. \end{tikzcd}$$

There exist bijective correspondences between

(i) functors $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ with commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[dr, "F"'] & \mathbb{A}^{\mathcal{G}} \arrow[d, "U^{\mathcal{G}}"] \\ & \mathbb{A}\,; \end{tikzcd}$$

(ii) left $\mathcal{G}$-comodule structures $\kappa_F : F \to GF$ on F;
(iii) comonad morphisms $t_K : FR \to G$ from the comonad generated by the adjunction $F \dashv R$ to $\mathcal{G}$.

These bijections are constructed as follows (e.g., [1, Proposition 2.5.1]): Given a functor K making the diagram (Sect. 2.2(i)) commute, $K(X) = (F(X), \kappa_X)$ for some morphism $\kappa_X : F(X) \to GF(X)$ and the collection $\{\kappa_x,\ x \in \mathbb{B}\}$ constitutes a natural transformation $\kappa_F : F \to GF$ making F a $\mathcal{G}$-comodule. Conversely, if $(F, \kappa_F : F \to GF)$ is a $\mathcal{G}$-module, then $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ is defined by $K(X) = (F(X), (\kappa_F)_X)$. Next, for any (left) $\mathcal{G}$-comodule structure $\kappa_F : F \to GF$, the composite

$$t_K : FR \xrightarrow{\kappa_F R} GFR \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by $F \dashv R$ to the comonad $\mathcal{G}$. On the other hand, for any comonad morphism $t : FR \to G$, the composite

$$\kappa_F : F \xrightarrow{F\eta} FRF \xrightarrow{tF} GF$$

defines a $\mathcal{G}$-comodule structure on F.

A left $\mathcal{G}$-comodule functor F is said to be $\mathcal{G}$-*Galois* provided t_K is an isomorphism (e.g. [16, Definition 1.3]).

Proposition 2.1 [13, Theorem 4.4] *The functor K (in Sect. 2.2) is an equivalence of categories if and only if*

(i) *the functor F is comonadic and*
(ii) *t_K is an isomorphism (F is $\mathcal{G}$-Galois).*

2.3 Module functors

For a monad $\mathcal{T} = (T, m, e)$ on $\mathbb{A}$, a (*left*) $\mathcal{T}$-module functor consists of a functor $R : \mathbb{B} \to \mathbb{A}$, equipped with a natural transformation $\alpha_R : TR \to R$ satisfying $\alpha_R \cdot eR = 1$ and $\alpha_R \cdot mR = \alpha_R \cdot T\alpha_R$.

If (R, α_R) is a $\mathcal{T}$-module, then the assignment

$$X \longmapsto (R(X), (\alpha_R)_X)$$

extends uniquely to a functor $K' : \mathbb{B} \to \mathbb{A}_{\mathcal{T}}$ with $U_{\mathcal{T}} K' = R$. This gives a bijection, natural in $\mathcal{T}$, between left $\mathcal{T}$-module structures on $R : \mathbb{B} \to \mathbb{A}$ and the functors $K' : \mathbb{B} \to \mathbb{A}_{\mathcal{T}}$ with $U_{\mathcal{T}} K = R$.

For any $\mathcal{T}$-module $(R : \mathbb{B} \to \mathbb{A}, \alpha_R)$ admitting a left adjoint functor $F : \mathbb{A} \to \mathbb{B}$, the composite

$$t_{K'} : \; T \xrightarrow{T\eta} TRF \xrightarrow{\alpha_R F} RF \;,$$

where $\eta : 1 \to RF$ is the unit of the adjunction $F \dashv R$, is a monad morphism from $\mathcal{T}$ to the monad on $\mathbb{A}$ generated by the adjunction $F \dashv R$.

A left $\mathcal{T}$-module $R : \mathbb{B} \to \mathbb{A}$ with a left adjoint $F : \mathbb{A} \to \mathbb{B}$ is said to be $\mathcal{T}$-*Galois* if the corresponding morphism $t_{K'} : T \to RF$ of monads on $\mathbb{A}$ is an isomorphism.

Expressing the dual of [13, Theorem 4.4] in the present situation gives:

Proposition 2.2 *The functor K' (in (2.3)) is an equivalence of categories if and only if*

(i) *the functor R is monadic and*
(ii) *R is a $\mathcal{T}$-Galois module functor.*

2.4 Mixed distributive laws

Let $\mathcal{T} = (T, m, e)$ be a monad and $\mathcal{G} = (G, \delta, \varepsilon)$ a comonad on a category $\mathbb{A}$.

A *mixed distributive law* or *entwining* from $\mathcal{T}$ to $\mathcal{G}$ is a natural transformation $\lambda : TG \to GT$ with certain commutative diagrams (e.g. [21, 5.3], [22]).

A *lifting* of $\mathcal{G}$ to $\mathbb{A}_{\mathcal{T}}$ is a comonad $\widehat{\mathcal{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ on $\mathbb{A}_{\mathcal{T}}$ for which $GU_{\mathcal{T}} = U_{\mathcal{T}}\widehat{G}$, $U_{\mathcal{T}}\widehat{\delta} = \delta U_{\mathcal{T}}$ and $U_{\mathcal{T}}\widehat{\varepsilon} = \varepsilon U_{\mathcal{T}}$.

The following is a version of [22, Theorem 2.2]:

Theorem 2.3 *Let $\mathcal{T} = (T, m, e)$ be a monad and $\mathcal{G} = (G, \delta, \varepsilon)$ a comonad on a category $\mathbb{A}$. Then there is a one-to-one correspondence between*

- *mixed distributive laws $\lambda : TG \to GT$ from $\mathcal{T}$ to $\mathcal{G}$ and*
- *liftings of $\mathcal{G}$ to a comonad $\widehat{\mathcal{G}}$ on $\mathbb{A}_{\mathcal{T}}$.*

To obtain a lifting $\widehat{\mathcal{G}}$ from a distributive law λ, one defines for $(X, h) \in \mathbb{A}_{\mathcal{T}}$, $\widehat{G}(X, h)$ as the $\mathcal{T}$-module

$$(G(X),\; TG(X) \xrightarrow{\lambda_X} GT(X) \xrightarrow{G(h)} G(X)).$$

Conversely, if one has a lifting comonad $\widehat{\mathcal{G}}$, one defines $\lambda : TG \to GT$ by

$$\begin{tikzcd} TG \arrow[r, "TGe"] \arrow[dr, dashed, "\lambda"'] & TGT \arrow[r, equal] & TGU_{\mathcal{T}}\phi_{\mathcal{T}}TU_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}} \arrow[r, equal] & U_{\mathcal{T}}\phi_{\mathcal{T}}U_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}} \arrow[d, "U_{\mathcal{T}}\varepsilon_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}}"'] \\ & GT \arrow[r, equal] & GU_{\mathcal{T}}\phi_{\mathcal{T}} \arrow[r, equal] & U_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}}. \end{tikzcd}$$

2.5 λ-bimodules

We write $\mathbb{A}_{\mathcal{T}}^{\mathcal{G}}(\lambda)$ (or just $\mathbb{A}_{\mathcal{T}}^{\mathcal{G}}$ when λ is understood) for the category whose objects are triples (X, h, θ), where $(X, h) \in \mathbb{A}_{\mathcal{T}}$ and $(X, \theta) \in \mathbb{A}^{\mathcal{G}}$, with commuting diagram (e.g. [20], [21, 5.7])

$$\begin{tikzcd} T(X) \arrow[r, "h"] \arrow[d, "T(\theta)"'] & X \arrow[r, "\theta"] & G(X) \\ TG(X) \arrow[rr, "\lambda_X"'] & & GT(X). \arrow[u, "G(h)"'] \end{tikzcd}$$

The assignment $(X, h, \theta) \to ((X, h), \theta)$ yields an isomorphism of categories

$$\mathbb{A}_{\mathcal{T}}^{\mathcal{G}}(\lambda) \simeq (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}.$$

2.6 Generalised Galois functors

With the data as given in Theorem 2.3, let $\lambda : TG \to GT$ be a mixed distributive law. Given an adjunction $\nu, \varsigma : L \dashv R : \mathbb{B} \to \mathbb{A}$, assume $K : \mathbb{B} \to (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$ to be a functor with $U^{\widehat{\mathcal{G}}}K = \phi_{\mathcal{T}}L$, i.e. with commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[d, "L"'] & (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}} \arrow[d, "U^{\widehat{\mathcal{G}}}"] \\ \mathbb{A} \arrow[r, "\phi_{\mathcal{T}}"] & \mathbb{A}_{\mathcal{T}}. \end{tikzcd} \tag{2.1}$$

Write $\mathcal{G}'$ for the comonad on the category $\mathbb{A}_{\mathcal{T}}$ generated by the adjunction

$$\phi_{\mathcal{T}}L \dashv RU_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \to \mathbb{B}$$

and write $t_K : \mathcal{G}' \to \widehat{\mathcal{G}}$ for the corresponding comonad morphism (see Sect. 2.2).

Applying Proposition 2.1 to the present situation gives:

Theorem 2.4 *In the setting of Sect. 2.6, the functor $K : \mathbb{B} \to (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$ is an equivalence of categories if and only if*

(i) *the functor* $\phi_{\mathcal{T}} L : \mathbb{B} \to \mathbb{A}_{\mathcal{T}}$ *is comonadic and*
(ii) $\phi_{\mathcal{T}} L$ *is a* $\widehat{\mathcal{G}}$*-Galois comodule functor.*

The following proposition gives a necessary and sufficient condition for the functor $\phi_{\mathcal{T}} L$ to be $\widehat{\mathcal{G}}$-Galois (generalising [17, Proposition 2.10]).

Proposition 2.5 *In the setting of Sect. 2.6,* $\phi_{\mathcal{T}} L$ *is* $\widehat{\mathcal{G}}$*-Galois if and only if the natural transformation* $t_K \phi_{\mathcal{T}}$ *is an isomorphism.*

Proof One direction is clear, so suppose that $t_K \phi_{\mathcal{T}}$ is an isomorphism.

Let $\kappa : \phi_{\mathcal{T}} L \to \widehat{G} \phi_{\mathcal{T}} L$ be the left $\widehat{G}$-comodule structure on $\phi_{\mathcal{T}} L$ corresponding to the diagram (2.1). Then, by Sect. 2.2, $t_K : \mathcal{G}' \to \widehat{\mathcal{G}}$ is the composite

$$\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{\kappa RU_{\mathcal{T}}} \widehat{G}\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{\widehat{G}\phi_{\mathcal{T}}\varsigma U_{\mathcal{T}}} \widehat{G}\phi_{\mathcal{T}} U_{\mathcal{T}} \xrightarrow{\widehat{G}\varepsilon_{\mathcal{T}}} \widehat{G}.$$

Consider the natural transformation $U_{\mathcal{T}} t_K$

$$U_{\mathcal{T}}\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{U_{\mathcal{T}}\kappa RU_{\mathcal{T}}} U_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{U_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}}\varsigma U_{\mathcal{T}}} U_{\mathcal{T}}\widehat{G}\phi_{\mathcal{T}} U_{\mathcal{T}} \xrightarrow{U_{\mathcal{T}}\widehat{G}\varepsilon_{\mathcal{T}}} U_{\mathcal{T}}\widehat{G},$$

and, using $U_{\mathcal{T}}\widehat{G} = GU_{\mathcal{T}}$, rewrite it as

$$U_{\mathcal{T}}\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{U_{\mathcal{T}}\kappa RU_{\mathcal{T}}} GU_{\mathcal{T}}\phi_{\mathcal{T}} LRU_{\mathcal{T}} \xrightarrow{GU_{\mathcal{T}}\phi_{\mathcal{T}}\varsigma U_{\mathcal{T}}} GU_{\mathcal{T}}\phi_{\mathcal{T}} U_{\mathcal{T}} \xrightarrow{GU_{\mathcal{T}}\varepsilon_{\mathcal{T}}} GU_{\mathcal{T}}.$$

By [10, Lemma 2.19], if $U_{\mathcal{T}} t_K \phi_{\mathcal{T}}$ is an isomorphism, then $U_{\mathcal{T}} t_K$ is so. But since $U_{\mathcal{T}}$ is conservative, t_K is an isomorphism, too. This completes the proof. □

In view of Theorem 2.4, it is desirable to find sufficient conditions for the composite $\phi_{\mathcal{T}} L$ to be comonadic. The next proposition gives two such conditions.

Proposition 2.6 *In the setting of Sect. 2.6, suppose that* $\mathbb{A}$ *is Cauchy complete and* L *is comonadic. Then the composite* $\phi_{\mathcal{T}} L$ *is comonadic under any of the conditions*

(i) *the unit* $e : 1 \to T$ *is a split monomorphism, i.e. there is a natural transformation* $e' : T \to I$ *with* $e'e = 1$*;*
(ii) *the monad* $\mathcal{T}$ *is of effective descent type* ($\phi_{\mathcal{T}} : \mathbb{A} \to \mathbb{A}_{\mathcal{T}}$ *is comonadic) and* $\mathbb{A}$ *has and* LR *and* $LRLR$ *preserve equalisers of coreflexive* T*-split pairs.*

Proof (i) Since
- $\mathbb{A}$ is Cauchy complete,
- $e : 1 \to T$ is a split monomorphism, and
- e can be seen as the unit of the adjunction $\phi_{\mathcal{T}} \dashv U_{\mathcal{T}}$,

it follows from [14, Proposition 3.14] that any $\phi_{\mathcal{T}}$-split pair is part of a split equaliser in $\mathbb{A}$, and thus the functor $\phi_{\mathcal{T}}$ creates equalisers of $\phi_{\mathcal{T}}$-split pairs as split equalisers in $\mathbb{A}$, from which it follows that $\phi_{\mathcal{T}} L$ is comonadic whenever L is so.

(ii) Since the functor L is assumed to be comonadic, to say that $\mathbb{A}$ has and LR and $LRLR$ preserve equalisers of coreflexive T-split pairs is to say that the functor L creates equalisers of those pairs whose image under L is part of a T-split equaliser (see, for example, [8, Proposition 4.3.2]). Since $\mathbb{A}$ has and T preserves equalisers of coreflexive T-split pairs if and only if $\mathbb{A}$ has and $\phi_{\mathcal{T}}$ preserves equalisers of coreflexive $\phi_{\mathcal{T}}$-split pairs [14, Proposition 3.11], and since $\mathcal{T}$ is of effective descent type by hypothesis, it follows that $\mathbb{B}$ has and the composite $\phi_{\mathcal{T}} L$ preserves equalisers of coreflexive $\phi_{\mathcal{T}} L$-split pairs. Using now the fact that $\phi_{\mathcal{T}} L$, being a composite of two conservative functors, is conservative, the result follows from the dual of Beck's monadicity theorem (see [12]). □

For later use (in Proposition 4.5) we prove the following technical observation.

Proposition 2.7 *Let $\mathbb{A}$ be Cauchy complete and $L \dashv R : \mathbb{A} \to \mathbb{B}$ an adjunction whose unit is a split monomorphism. Then, in any commutative* (*up to isomorphism*) *diagram*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[dr, "R"'] & \mathbb{E} \arrow[d, "K"] \\ & \mathbb{B}, \end{tikzcd}$$

(i) *the functor $F : \mathbb{A} \to \mathbb{E}$ is conservative;*
(ii) *any coreflexive F-split pair of morphisms has a split equaliser in $\mathbb{A}$;*
(iii) *the functor F is comonadic if and only if it has a right adjoint.*

Proof (i) Since $\mathbb{A}$ is Cauchy complete and since the unit of the adjunction is a split monomorphism, the functor R is comonadic (e.g., [14, Proposition 3.16]) and, in particular, conservative. This implies—since KF is isomorphic to R—that F is conservative, too.

(ii) Suppose that $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is an F-split pair of morphisms in $\mathbb{A}$. Then the morphisms $F(f)$ and $F(g)$ have a split equaliser in $\mathbb{E}$, so that the pair $(F(f), F(g))$ is *contractible* (see [12]). Since contractible pairs, being equationally defined, are preserved by any functors, the pair $(R(f), R(g))$, being isomorphic to the pair $(KF(f), KF(g))$, is also contractible. But since $\mathbb{A}$ is Cauchy complete and since the unit of the adjunction is a split monomorphism (which just means that the functor R is $1_{\mathbb{A}}$-*separable*), it follows from [14, Proposition 3.8] that the pair (f, g) is contractible, too. Then, $\mathbb{A}$ being Cauchy complete, f and g have a split equaliser (e.g. [3]) and this equaliser is clearly preserved by F.

(iii) follows from the fact that split equalisers are preserved by any functor.

3 Generalised Hopf modules

In [1], Aguiar and Chase studied generalised Hopf modules in monoidal categories and proved a *Fundamental Theorem* for them. In this section we show how this result can be obtained as a special case of our approach. We first recall elementary facts about modules and comodules in a monoidal category (e.g. [13,19]).

3.1 Monoids and comonoids in monoidal categories

Let $(\mathbb{V}, \otimes, I, a, l, r)$ be a monoidal category, where a, l, r are the associativity, left identity, and right identity isomorphisms for the monoidal structure on $\mathbb{V}$.

A *monoid* in $\mathbb{V}$ (or $\mathbb{V}$-monoid) consists of an object A of $\mathbb{V}$ endowed with a multiplication $m : A \otimes A \to A$ and a unit morphism $e : I \to A$ such that the usual identity and associative conditions are satisfied. A monoid morphism $f : A \to A'$ is a morphism in $\mathbb{V}$ preserving m and e. The category of monoids in $\mathbb{V}$ is denoted by $\mathbf{Mon}(\mathbb{V})$. The tensor unit I endowed with the obvious structure morphisms is a $\mathbb{V}$-monoid. This $\mathbb{V}$-monoid is called the *trivial* $\mathbb{V}$-*monoid*, denoted by $\mathcal{I}$.

Given a monoid (A, e_A, m_A) in $\mathbb{V}$, a *left* A-*module* is a pair (V, ρ_V), where V is an object of $\mathbb{V}$ and $\rho_V : A \otimes V \to V$ is a morphism in $\mathbb{V}$, called the A-*action* on V, such that $\rho_V(m_A \otimes V)a^{-1}_{A,A,V} = \rho_V(A \otimes \rho_V)$ and $\rho_V(e_A \otimes V) = l_V$.

For any monoid A in $\mathbb{V}$, the left A-modules are the objects of a category ${}_A\mathbb{V}$. A morphism $f : (V, \rho_V) \to (W, \rho_W)$ is a morphism $f : V \to W$ in $\mathbb{V}$ such that $\rho_W(A \otimes f) = f\rho_V$. Analogously, one has the category $\mathbb{V}_A$ of right A-modules.

Let A and B be two monoids in $\mathbb{V}$. An object V in $\mathbb{V}$ is called an (A, B)-*bimodule* if there are morphisms $\rho_V : A \otimes V \to V$ and $\varrho_V : V \otimes B \to V$ in $\mathbb{V}$ such that $(V, \rho_V) \in {}_A\mathbb{V}$, $(V, \varrho_V) \in \mathbb{V}_B$ and $\varrho_V(\rho_V \otimes B) = \rho_V(A \otimes \varrho_V)a_{A,V,B}$. A morphism of (A, B)-bimodules is a morphism in $\mathbb{V}$ which is a morphism of both the left A-modules and right B-modules. Write ${}_A\mathbb{V}_B$ for the corresponding category.

Comonoids and (*left, right, bi-*) *comodules* in $\mathbb{V}$ can be defined as monoids and left (right, bi-) modules in the *opposite* monoidal category $(\mathbb{V}^{\mathrm{op}}, \otimes, I, a^{-1}, l^{-1}, r^{-1})$. The resulting categories are denoted by $\mathbf{Comon}(\mathbb{V})$, ${}^C\mathbb{V}$, $\mathbb{V}^C$ and ${}^C\mathbb{V}^{C'}$, C and C' being comonoids in $\mathbb{V}$.

3.2 Tensor product of modules

If A is a monoid in $\mathbb{V}$, $(V, \varrho_V) \in \mathbb{V}_A$ a right A-module, and $(W, \rho_W) \in {}_A\mathbb{V}$ a left A-module, then their *tensor product* (*over* A) is defined as the object part of the coequaliser (if this exists)

$$\begin{tikzcd}(V \otimes A) \otimes W \arrow[r, "a_{V,A,W}"] \arrow[rr, bend right, "\varrho_V \otimes W"'] & V \otimes (A \otimes W) \arrow[r, "V \otimes \rho_W"] & V \otimes W \arrow[r, "\mathrm{can}"] & V \otimes_A W.\end{tikzcd} \tag{3.1}$$

Given another left A-module $(W', \rho_{W'})$ for which $V \otimes_A W'$ exists, and a morphism $f : W \to W'$ of left A-modules, we form the diagram

$$\begin{tikzcd}(V \otimes A) \otimes W \arrow[r, "a_{V,A,W}"'] \arrow[rr, bend left, "\varrho_V \otimes W"] \arrow[d, "(V\otimes A)\otimes f"] & V \otimes (A \otimes W) \arrow[r, "V \otimes \rho_W"'] \arrow[d, "V\otimes(A\otimes f)"] & V \otimes W \arrow[r, "\mathrm{can}"] \arrow[d, "V\otimes f"] & V \otimes_A W \arrow[d, dotted, "V\otimes_A f"] \\ (V \otimes A) \otimes W' \arrow[r, "a_{V,A,W'}"] \arrow[rr, bend right, "\varrho_V \otimes W'"'] & V \otimes (A \otimes W') \arrow[r, "V \otimes \rho_{W'}"] & V \otimes W' \arrow[r, "\mathrm{can}"'] & V \otimes_A W',\end{tikzcd}$$

in which

- $(V \otimes f)(\varrho_V \otimes W) = (\varrho_V \otimes W')((V \otimes A) \otimes f)$ by functoriality of $\otimes$,
- the left square commutes by naturality of a, and
- the middle square commutes because f is a morphism of left A-modules;

from this one sees that there ia a unique morphism $V \otimes_A f : V \otimes_A W \to V \otimes_A W'$ making the right square commute. It is easy to see that if for $W'' \in {}_A\mathbb{V}$, the tensor product $V \otimes_A W''$ exists, then for any morphism $g : W' \to W''$ in ${}_A\mathbb{V}$,

$$V \otimes_A (gf) = (V \otimes_A g)(V \otimes_A f).$$

If B is another monoid in $\mathbb{V}$ such that the functors $B \otimes -, B \otimes (B \otimes -) : \mathbb{V} \to \mathbb{V}$ both preserve the equaliser (3.1) and if $V \in {}_B\mathbb{V}_A$, then the tensor product $V \otimes_A W$ has the structure of a left B-module such that can : $V \otimes W \to V \otimes_A W$ becomes a morphism of left B-modules. Moreover, if these functors also preserve the equaliser defining $V \otimes_A W'$, then $V \otimes_A f$ also becomes a left B-module morphism.

Recall (for example, from [19]) that the forgetful functor

$${}_AU : {}_A\mathbb{V} \to \mathbb{V}, \quad (V, \rho_V) \mapsto V,$$

is right adjoint, with the left adjoint ${}_A\phi : \mathbb{V} \to {}_A\mathbb{V}$ sending each $V \in \mathbb{V}$ to the "free" left A-module

$$(A \otimes V,\ A \otimes (A \otimes V) \xrightarrow{a_{A,A,V}} (A \otimes A) \otimes V \xrightarrow{m \otimes V} V \otimes V).$$

Write ${}_A\mathcal{T}$ for the monad on $\mathbb{V}$ generated by the adjunction ${}_A\phi \dashv {}_AU : {}_A\mathbb{V} \to \mathbb{V}$. It is well known that the corresponding Eilenberg–Moore category $\mathbb{V}_{{}_A\mathcal{T}}$ of ${}_A\mathcal{T}$-modules is exactly the category ${}_A\mathbb{V}$ of left A-modules.

Lemma 3.1 *Let A be a monoid in $\mathbb{V}$ and $M = A \otimes V$ the free left A-module generated by $V \in \mathbb{V}$. Then*

(1) *for any $N \in \mathbb{V}_A$, the tensor product $N \otimes_A M$ exists and is isomorphic to $N \otimes V$;*
(2) *for $N \in {}_B\mathbb{V}_A$, B any monoid in $\mathbb{V}$, $N \otimes_A M$ is a left B-module;*
(3) *for any morphism $f : A \otimes V \to A \otimes V'$ in $\mathbb{V}$, $N \otimes_A f$ is a morphism of left B-modules;*
(4) *for any morphism : $V \to V'$ in $\mathbb{V}$, the induced morphism $N \otimes_A (A \otimes g)$ of left B-modules is isomorphic to $N \otimes g$.*

Proof Everything follows from Sect. 3.2 and the fact that the equaliser defining the tensor product $N \otimes_A M$ is split and thus is preserved by any functor. □

Remark 3.2 The full subcategory of ${}_A\mathbb{V}$ generated by the left A-modules of the form $A \otimes V$, $V \in \mathbb{V}$, is just the Kleisli category $\widetilde{\mathbb{V}}_{{}_A\mathcal{T}}$ of the monad ${}_A\mathcal{T}$ (e.g. [5, 2.4]). Hence Lemma 3.1 may be alternatively stated as follows:

Let $N \in {}_B\mathbb{V}_A$. Then, for any $X \in \widetilde{\mathbb{V}}_A{}_{\mathcal{T}}$, the tensor product $N \otimes_A X$ exists and has the structure of a left B-module. So the assignment $X \mapsto N \otimes_A X$ yields a functor $N \otimes_A - : \widetilde{\mathbb{V}}_A{}_{\mathcal{T}} \to {}_B\mathbb{V}$ leading to the commutative diagram

$$\begin{tikzcd} \mathbb{V} \arrow[r, "\overline{\phi}_{A\mathcal{T}}"] \arrow[dr, "N\otimes -"'] & \widetilde{\mathbb{V}}_A{}_{\mathcal{T}} \arrow[d, "N\otimes_A -"] \\ & {}_B\mathbb{V} . \end{tikzcd}$$

3.3 Opmonoidal functors

Recall that—following [11]—an *opmonoidal functor* from a monoidal category $(\mathbb{V}, \otimes, I)$ to a monoidal category $(\mathbb{V}', \otimes', I')$ is a triple (S, ω, ξ), where $S : \mathbb{V} \to \mathbb{V}'$ is a functor, $\omega_{V,W} : S(V \otimes W) \to S(V) \otimes S(W)$ is a natural transformation between functors $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$, and $\xi : S(I) \to I'$ is a morphism compatible with the tensor structures. Note that opmonoidal functors S take $\mathbb{V}$-comonoids (i.e. comonoids in $\mathbb{V}$) into $\mathbb{V}'$-comonoids in the sense that if $C = (C, \delta, \varepsilon)$ is a $\mathbb{V}$-comonoid, then it produces a $\mathbb{V}'$-comonoid

$$S(C) = (S(C),\ \omega_{C,C} \cdot S(\delta),\ \xi \cdot S(\varepsilon)).$$

In [11], an *opmonoidal monad* on a monoidal category $(\mathbb{V}, \otimes, I)$ is defined as a monad $\mathcal{T} = (T, m, e)$ on $\mathbb{V}$ such that the functor T and the natural transformations m and e are opmonoidal. Such monads are also called *Hopf monads* in [18, Definition 1.1] or *bimonads* (e.g. in [1, Definition 3.2.1]) but they are different from what are called *bimonads* in [15, Definition 4.1] (compare [16, Section 5]).

The basic property of opmonoidal monads $\mathcal{T}$ is that they lead to a monoidal structure on the Eilenberg–Moore category $\mathbb{V}_{\mathcal{T}}$ of $\mathcal{T}$-modules in such a way that the forgetful functor $U_{\mathcal{T}} : \mathbb{V}_{\mathcal{T}} \to \mathbb{V}$ is strictly monoidal. Explicitly (e.g. [11,18]), for $\mathcal{T}$-modules (V, h) and (W, g), the tensor product $(V, h) \otimes (W, g)$ is given by

$$(V \otimes W,\ T(V \otimes W) \xrightarrow{\omega_{V,W}} T(V) \otimes T(W) \xrightarrow{h \otimes g} V \otimes W)$$

and the unit object of $\mathbb{V}_{\mathcal{T}}$ is the $\mathcal{T}$-module $(I,\ \xi : T(I) \to I)$. The unitary and associativity isomorphisms for $\mathbb{V}_{\mathcal{T}}$ are inherited from $\mathbb{V}$.

3.4 $\mathcal{T}$-module-comonoids

Given an opmonoidal monad $\mathcal{T}$ on $\mathbb{V}$, a comonoid in the monoidal category $\mathbb{V}_{\mathcal{T}}$ is called a *$\mathcal{T}$-module-comonoid*. Explicitly, a $\mathcal{T}$-module-comonoid $\mathcal{Z} = ((Z, \sigma), \delta, \varepsilon)$ consists of an object $(Z, \sigma) \in \mathbb{V}_{\mathcal{T}}$ and $\mathbb{V}$-morphisms $\delta : Z \to Z \otimes Z$ and $\varepsilon : Z \to I$ such that $U_{\mathcal{T}}(\mathcal{Z}) = (Z, \delta, \varepsilon)$ is a $\mathbb{V}$-comonoid and that δ and ε are morphism of $\mathcal{T}$-modules.

For any $\mathbb{V}$-comonoid (C, δ, ε), $T(C)$ allows for a module-comonoid structure with the morphisms (e.g. [1])

- $TT(C) \xrightarrow{m_C} T(C)$,
- $T(C) \xrightarrow{T(\delta)} T(C \otimes C) \xrightarrow{\omega_{C,C}} T(C) \otimes T(C)$,
- $T(C) \xrightarrow{T(\varepsilon)} T(I) \xrightarrow{\xi} I$.

We write $\mathcal{T}(C)$ for this module-comonoid.

3.5 $\mathbb{V}$-categories

A *left* $\mathbb{V}$-*category* is a category $\mathbb{A}$ equipped with a bifunctor

$$- \diamond - : \mathbb{V} \times \mathbb{A} \to \mathbb{A},$$

called the *action of* $\mathbb{V}$ *on* $\mathbb{A}$, and invertible natural transformations

$$\alpha_{V,W,X} : (V \otimes W) \diamond X \to V \diamond (W \diamond X) \quad \text{and} \quad \lambda_X : I \diamond X \to X,$$

called the *associativity* and *unit* isomorphisms, respectively, satisfying two coherence axioms (see Bénabou [4]). Note that $\mathbb{V}$ has a canonical (left) action on itself, given by taking $V \diamond W = V \otimes W$, $\alpha = a$, and $\lambda = l$.

Given a left $\mathbb{V}$-category $\mathbb{A}$ and a monoid (A, e_A, m_A) in $\mathbb{V}$, one has a monad $\mathcal{T}_A^l$ on $\mathbb{A}$ defined on any $X \in \mathbb{A}$ by

- $\mathcal{T}_A^l(X) = A \diamond X$,
- $(e_{\mathcal{T}_A^l})_X : X \xrightarrow{\lambda_X^{-1}} I \diamond X \xrightarrow{e_A \diamond X} A \diamond X = \mathcal{T}_A^l(X)$,
- $(m_{\mathcal{T}_A^l})_X : \mathcal{T}_A^l(\mathcal{T}_A^l(X)) = A \diamond (A \diamond X) \xrightarrow{\alpha_{A,A,X}^{-1}} (A \otimes A) \diamond X \xrightarrow{m_A \diamond X} A \diamond X = \mathcal{T}_A^l(X)$,

and we write ${}_A\mathbb{A}$ for the Eilenberg–Moore category $\mathbb{A}_{\mathcal{T}_A^l}$ of $\mathcal{T}_A^l$-modules. For the canonical left action of $\mathbb{V}$ on itself, ${}_A\mathbb{A}$ is just the category ${}_A\mathbb{V}$ of (left) A-modules.

Dually, for any $\mathbb{V}$-coalgebra $(C, \varepsilon_C, \delta_C)$, the endofunctor $C \diamond - : \mathbb{A} \to \mathbb{A}$ is the functor-part of a comonad $\mathcal{G}_C^l$ on $\mathbb{A}$ and one has the corresponding Eilenberg–Moore category ${}^C\mathbb{A} = \mathbb{A}^{\mathcal{G}_C^l}$; for $\mathbb{A} = \mathbb{V}$ this is just the category ${}^C\mathbb{V}$ of (left) C-comodules. We sometimes write ${}_A\phi$ and ${}^C\phi$ for the functors $\phi_{\mathcal{T}_A^l}$ and $\phi_{\mathcal{G}_C^l}$, respectively.

Symmetrically, one has the monad $\mathcal{T}_A^r = - \diamond A$ (resp. comonad $\mathcal{G}_C^r = - \diamond C$) on $\mathbb{A}$, the corresponding Eilenberg–Moore category $\mathbb{A}_A$ (resp. $\mathbb{A}^C$) of $\mathcal{T}_A^r$-modules (resp. $\mathcal{G}_C^r$-comodules), and the functor $\phi_A : \mathbb{A} \to \mathbb{A}_A$ (resp. $\phi^C : \mathbb{A} \to \mathbb{A}^C$).

3.6 Comodules over opmonoidal functors

Let $- \diamond - : \mathbb{V} \times \mathbb{A} \to \mathbb{A}$ be a left action of a monoidal category $\mathbb{V}$ on a category $\mathbb{A}$ and let $\mathcal{F} : \mathbb{V} \to \mathbb{V}$ be an opmonoidal functor on $\mathbb{V}$. A *comodule* over $\mathcal{F}$ is a pair (H, χ), where $H : \mathbb{A} \to \mathbb{A}$ is a functor and $\chi_{V,X} : H(V \diamond X) \to \mathcal{F}(V) \diamond H(X)$ is a natural transformation satisfying two axioms (e.g. [1, Definition 3.3.1]).

Suppose that $\mathcal{T} = (T, m_{\mathcal{T}}, e_{\mathcal{T}})$ is an opmonoidal monad on $\mathbb{V}$ (with structure $\omega_{V,W} : T(V \otimes W) \to T(V) \otimes T(W)$ and $\xi : T(I) \to I$) and that $\mathcal{S} = (S, m_{\mathcal{S}}, e_{\mathcal{S}})$

is a monad on $\mathbb{A}$ such that the functor S is a comodule over the opmonoidal functor (T, ω, ξ) via $\chi_{V,X} : S(V \diamond X) \to T(V) \diamond S(X)$. One says that $(\mathcal{S}, \chi)$ is a *comodule-monad* over the bimonad $\mathcal{T}$ if χ is compatible with the monad structures [1, Definition 3.5.1]. Considering $\mathcal{T}$ as a monad on the left $\mathbb{V}$-category $\mathbb{V}$, it follows from the definition of an opmonoidal monad that the pair $(\mathcal{T}, \omega)$ is a comodule-monad over the opmonoidal monad $\mathcal{T}$.

There is a left action of the monoidal category $\mathbb{V}_{\mathcal{T}}$ (with the monoidal structure from Sect. 3.3) on the category $\mathbb{A}_{\mathcal{S}}$: given a $\mathcal{T}$-module (V, f) and an $\mathcal{S}$-module (X, h), $(V, f) \diamond (X, h)$ is the pair (e.g. [1, Proposition 3.5.3])

$$(V \diamond X,\ S(V \diamond X) \xrightarrow{\chi_{V,X}} T(V) \diamond S(X) \xrightarrow{f \diamond h} V \diamond X).$$

Assumption 3.3 We henceforth suppose that

- $\mathcal{T} = ((T, m_{\mathcal{T}}, e_{\mathcal{T}}), \omega, \xi)$ is an opmonoidal monad on a monoidal category $(\mathbb{V}, \otimes, I, a, l, r)$;
- $(\mathbb{A}, \diamond, \alpha, \lambda)$ is a left $\mathbb{V}$-category;
- $\mathcal{S} = (S, m_{\mathcal{S}}, e_{\mathcal{S}})$ is a $\mathcal{T}$-comodule-monad on $\mathbb{A}$ via

$$\chi_{V,-} : S(V \diamond -) \to T(V) \diamond S(-);$$

- $\mathcal{Z} = ((Z, \sigma), \delta, \varepsilon)$ is a $\mathcal{T}$-module-comonoid.

Since $\mathcal{Z}$ is a comonoid in the monoidal category $\mathbb{V}_{\mathcal{T}}$ and since $\mathbb{V}_{\mathcal{T}}$ acts from the left on $\mathbb{A}_{\mathcal{S}}$, one has the $\mathbb{A}_{\mathcal{S}}$-comonad $\mathcal{G}^l_{\mathcal{Z}}$. Moreover, since $\mathcal{Z}_0 = U_{\mathcal{T}}(\mathcal{Z})$ is a comonoid in the monoidal category $\mathbb{V}$, one has the $\mathbb{A}$-comonad $\mathcal{G}^l_{\mathcal{Z}_0}$, and it is not hard to check that $\mathcal{G}^l_{\mathcal{Z}}$ is a lifting of $\mathcal{G}^l_{\mathcal{Z}_0}$ to $\mathbb{A}_{\mathcal{S}}$. It follows from Theorem 2.3 that there is a mixed distributive law $\underline{\lambda}$ from the $\mathbb{A}$-monad $\mathcal{S}$ to the $\mathbb{A}$-comonad $\mathcal{G}^l_{\mathcal{Z}_0}$.

The following result without proof appears in [1, Remark 3.6.5]:

Proposition 3.4 *With the data considered in Sect. 3.6, $\underline{\lambda}$ is the composite*

$$S(Z \diamond -) \xrightarrow{\chi_{Z,-}} T(Z) \diamond S(-) \xrightarrow{\sigma \diamond S(-)} Z \diamond S(-).$$

Proof By Sect. 3.6, for any $(X, h) \in \mathbb{A}_{\mathcal{T}}$,

$$\mathcal{G}^l_{\mathcal{Z}}(X, h) = (Z \diamond X,\ S(Z \diamond X) \xrightarrow{\chi_{Z,X}} T(Z) \diamond S(X) \xrightarrow{\sigma \diamond h} Z \diamond X),$$

and it follows that $(\varepsilon_{\mathcal{S}})_{\mathcal{G}^l_{\mathcal{Z}}(X,h)} = (\sigma \diamond h) \cdot \chi_{Z,X}$, thus

$$(\varepsilon_{\mathcal{S}})_{\mathcal{G}^l_{\mathcal{Z}} \phi_{\mathcal{S}}(X)} = (\sigma \diamond (m_{\mathcal{S}})_X) \cdot \chi_{Z,S(X)},$$

since $\phi_{\mathcal{S}}(X) = (S(X), (m_{\mathcal{S}})_X)$. By Theorem 2.3, for $X \in \mathbb{A}$, $\underline{\lambda}_X$ is the composite

$$S(Z \diamond X) \xrightarrow{S(Z \diamond (e_{\mathcal{S}})_X)} S(Z \diamond S(X)) \xrightarrow{\chi_{Z,S(X)}} T(Z) \diamond SS(X) \xrightarrow{\sigma \diamond (m_{\mathcal{S}})_X} Z \diamond S(X).$$

In the diagram

$$\begin{tikzcd}
S(Z \diamond X) \arrow[r, "S(Z\diamond(e_{\mathcal{S}})_X)"] \arrow[d, "\chi_{Z,X}"'] & S(Z \diamond S(X)) \arrow[d, "\chi_{Z,S(X)}"] \\
T(Z) \diamond S(X) \arrow[r, "T(Z)\diamond(e_{\mathcal{S}})_X"] \arrow[dr, "\sigma\diamond S(X)"'] & T(Z) \diamond SS(X) \arrow[d, "\sigma\diamond(m_{\mathcal{S}})_X"] \\
& Z \diamond S(X)\,,
\end{tikzcd}$$

the rectangle commutes by naturality of χ, while $m_{\mathcal{S}} \cdot e_{\mathcal{S}} = 1$ implies commutativity of the triangle; it follows that $\underline{\lambda}_X = (\sigma \diamond S(X)) \cdot \chi_{Z,X}$. □

3.7 Generalised Hopf modules

${}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}}) = \mathbb{A}_{\mathcal{S}}^{\mathcal{G}^l_{\mathcal{Z}_0}}(\lambda)$ is the category of $\underline{\lambda}$-bimodules (see Sect. 2.5); the objects are triples (X, h, ϑ), where $X \in \mathbb{A}$, $(X,\, h : S(X) \to X) \in \mathbb{A}_{\mathcal{S}}$, $(X,\, \vartheta : X \to Z \diamond X) \in {}^{\mathcal{Z}_0}\mathbb{A}$ with commuting diagram

$$\begin{tikzcd}
S(X) \arrow[r, "h"] \arrow[d, "S(\vartheta)"'] & X \arrow[r, "\vartheta"] & Z \diamond X \\
S(Z \diamond X) \arrow[rr, "\underline{\lambda}_X"'] & & Z \diamond S(X). \arrow[u, "Z\diamond h"']
\end{tikzcd}$$

In [1, Definition 3.6.1], these are called *generalised Hopf modules* and the category ${}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$ is denoted by Hopf$(\mathcal{T}, \mathcal{S}, Z)$.

Assumption 3.5 We henceforth suppose that (C, δ, ε) is a comonoid in $\mathbb{V}$ and that $\mathcal{Z} = \mathcal{T}(C)$ is the corresponding $\mathcal{T}$-module-comonoid (see Sect. 3.4).

Lemma 3.6 *In the situation considered above, the assignment*

$$(X, \theta) \rightsquigarrow (S(X), (m_{\mathcal{S}})_X, \vartheta),$$

where $\vartheta : S(X) \to Z \diamond S(X)$ *is the composite*

$$S(X) \xrightarrow{S(\theta)} S(C \diamond X) \xrightarrow{\chi_{C,X}} \mathcal{T}(C) \diamond S(X) = Z \diamond S(X),$$

yields a functor

$$K : {}^{C}\mathbb{A} \to {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$$

yielding commutativity of the diagram

$$\begin{tikzcd} {}^{C}\mathbb{A} \arrow[r, "K"] \arrow[d, "{}^{C}U"'] & {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}}) \arrow[d, "{}^{\mathcal{Z}}U"] \\ \mathbb{A} \arrow[r, "\phi_{\mathcal{S}}"'] & \mathbb{A}_{\mathcal{S}}. \end{tikzcd} \tag{3.2}$$

Proof To show that $(X, \vartheta) \in {}^{\mathcal{Z}_0}\mathbb{A}$ is to show commutativity of the diagrams

$$\begin{tikzcd} S(X) \arrow[r, "\vartheta"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{(I)}"] & Z \diamond S(X) \arrow[d, "\overline{\varepsilon} \diamond S(X)"] \\ S(X) & I \diamond S(X) \arrow[l, "\lambda_{S(X)}"], \end{tikzcd} \qquad \begin{tikzcd} S(X) \arrow[r, "\vartheta"] \arrow[d, "\vartheta"'] \arrow[dr, phantom, "\mathrm{(II)}"] & Z \diamond S(X) \arrow[d, "\overline{\delta} \diamond S(X)"] \\ Z \diamond S(X) \arrow[r, "Z \diamond \vartheta"'] & Z \diamond Z \diamond S(X), \end{tikzcd}$$

where $\overline{\varepsilon} = \xi \cdot T(\varepsilon)$ and $\overline{\delta} = \omega_{C,C} \cdot T(\delta)$ are the counit and the comultiplication for the $\mathbb{V}_{\mathcal{T}}$-module-comonoid $\mathcal{Z} = \mathcal{T}(C)$ (see Sect. 3.4). In the diagram

$$\begin{tikzcd} S(X) \arrow[r, "S(\theta)"] \arrow[dr, "S(\lambda_X^{-1})"'] & S(C \diamond X) \arrow[r, "\chi_{C,X}"] \arrow[d, "S(\varepsilon \diamond X)"] & T(C) \diamond S(X) \arrow[d, "T(\varepsilon) \diamond S(X)"] \\ & S(I \diamond X) \arrow[r, "\chi_{I,X}"'] \arrow[d, "S(\lambda_X)"'] & T(I) \diamond S(X) \arrow[d, "\xi \diamond S(X)"] \\ & S(X) \arrow[r, "\lambda_{S(X)}^{-1}"'] & I \diamond S(X), \end{tikzcd}$$

- the triangle commutes since $(X, \theta) \in {}^{C}\mathbb{A}$,
- the top rectangle commutes by naturality of χ,
- the bottom rectangle commutes since $\mathcal{S}$ is a $\mathcal{T}$-comodule-monad (see diagram (3.6) in [1]);

it follows that diagram (I) is commutative. To show that (II) is also commutative, consider the diagram

$$\begin{tikzcd} S(X) \arrow[r, "S(\theta)"] \arrow[d, "S(\theta)"'] \arrow[dr, phantom, "(1)"] & S(C \diamond X) \arrow[r, "\chi_{C,X}"] \arrow[d, "S(\delta \diamond X)"] \arrow[dr, phantom, "(2)"] & T(C) \diamond S(X) \arrow[d, "T(\delta) \diamond S(X)"] \\ S(C \diamond X) \arrow[r, "S(C \diamond \theta)"'] \arrow[d, "\chi_{C,X}"'] \arrow[dr, phantom, "(3)"] & S(C \diamond C \diamond X) \arrow[r, "\chi_{C \diamond C,X}"'] \arrow[d, "\chi_{C,C \diamond X}"] \arrow[dr, phantom, "(4)"] & T(C \diamond C) \diamond S(X) \arrow[d, "\omega_{C,C} \diamond S(X)"] \\ T(C) \diamond S(X) \arrow[r, "T(C) \diamond S(\theta)"'] & T(C) \diamond S(C \diamond X) \arrow[r, "T(C) \diamond \chi_{C,X}"'] & T(C) \diamond T(C) \diamond S(X), \end{tikzcd}$$

in which

- rectangle (1) commutes since $(X, \theta) \in {}^{C}\mathbb{A}$,
- rectangle (2) and (3) commute by naturality of χ,
- rectangle (4) commutes since $\mathcal{S}$ is a $\mathcal{T}$-comodule-monad (see diagram (3.5) in [1]);

therefore the outer square (and hence (II)) is commutative. Thus, $(X, \vartheta) \in {}^{\mathcal{Z}_0}\mathbb{A}$, and since $(S(X), (m_{\mathcal{S}})_X) \in \mathbb{A}_{\mathcal{S}}$, in order to show that $(S(X), (m_{\mathcal{S}})_X, \vartheta) \in {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$, we need commutativity of the diagram

$$\begin{tikzcd}
SS(X) \arrow[r, "(m_{\mathcal{S}})_X"] \arrow[d, "SS(\theta)"'] & S(X) \arrow[d, "S(\theta)"] & \\
SS(C \diamond X) \arrow[r, "(m_{\mathcal{S}})_{C\diamond X}"] \arrow[d, "S(\chi_{C,X})"'] & S(C \diamond X) \arrow[dr, "\chi_{C,X}"] & \\
S(T(C) \diamond S(X)) \arrow[r, "\chi_{T(C),T(X)}"'] & T(T(C)) \diamond SS(X) \arrow[r, "(m_{\mathcal{T}})_C \diamond (m_{\mathcal{S}})_C"'] & T(C) \diamond S(X);
\end{tikzcd}$$

since the rectangle commutes by naturality of $m_{\mathcal{S}}$, while the trapezoid commutes since $\mathcal{S}$ is a $\mathcal{T}$-comodule-monad (see diagram (3.10) in [1]) the outer paths commute, too. □

As an immediate consequence we obtain from Sect. 2.2:

Corollary 3.7 *For $(X, \theta) \in {}^{C}\mathbb{A}$, the (X, θ)-component $\kappa_{(X,\theta)} : S(X) \to Z \diamond S(X)$ of the $\mathcal{G}^l_{\mathcal{Z}}$-comodule structure on the composite $\phi_{\mathcal{S}}\, {}^{C}U : {}^{C}\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ induced by the commutative diagram (3.2) is the composite*

$$S(X) \xrightarrow{S(\theta)} S(C \diamond X) \xrightarrow{\chi_{C,X}} \mathcal{T}(C) \diamond S(X) = Z \diamond S(X).$$

Write $\mathcal{G}$ for the comonad on the category $\mathbb{A}_{\mathcal{S}}$ generated by the adjunction

$$\phi_{\mathcal{S}}\, {}^{C}U \dashv {}^{C}\phi\, U_{\mathcal{S}} : \mathbb{A}_{\mathcal{S}} \to {}^{C}\mathbb{A}.$$

Proposition 3.8 *For any $(X, h) \in \mathbb{A}_{\mathcal{S}}$, the (X, h)-component of the comonad morphism $t_K : \mathcal{G} \to \mathcal{G}^l_{\mathcal{Z}}$ induced by the commutative diagram (3.2) is the composite*

$$\mathcal{G}(X, h) = S(C \diamond X) \xrightarrow{\chi_{C,X}} Z \diamond S(X) \xrightarrow{Z\diamond h} Z \diamond X = \mathcal{G}^l_{\mathcal{Z}}(X, h).$$

Proof Let $(X, h) \in \mathbb{A}_{\mathcal{S}}$. The (X, h)-component of the counit of the adjunction $\phi_{\mathcal{S}}\, {}^{C}U \dashv {}^{C}\phi\, U_{\mathcal{S}} : \mathbb{A}_{\mathcal{S}} \to {}^{C}\mathbb{A}$ is the composite

$$S(C \diamond X) \xrightarrow{S(\varepsilon\diamond X)} S(I \diamond X) \xrightarrow{S(\lambda_X)} S(X) \xrightarrow{h} X;$$

it follows from Sect. 2.2 that the morphism

$$(t_K)_{(X,h)} : S(C \diamond X) = \mathcal{G}(X,h) \to \mathcal{G}^l_{\mathcal{Z}}(X,h) = Z \diamond X$$

is the composite

$$S(C \diamond X) \xrightarrow{\kappa_{\mathcal{C}_{\phi(X)}}} Z \diamond S(C \diamond X) \xrightarrow{Z \diamond S(\varepsilon \diamond X)} Z \diamond S(I \diamond X) \xrightarrow{Z \diamond S(\lambda_X)} Z \diamond S(X) \xrightarrow{Z \diamond h} Z \diamond X.$$

From

$$\mathcal{C}_\phi(X) = (C \diamond X,\ C \diamond X \xrightarrow{\delta \diamond X} (C \otimes C) \diamond X \xrightarrow{\alpha_{C,C,X}} C \diamond (C \diamond X)),$$

we obtain by Corollary 3.7 that $\kappa_{\mathcal{C}_{\phi(X)}}$ is the composite

$$S(C \diamond X) \xrightarrow{S(\delta \diamond X)} S((C \otimes C) \diamond X) \xrightarrow{S(\alpha_{C,C,X})} S(C \diamond (C \diamond X)) \xrightarrow{\chi_{C,C \diamond X}} Z \diamond S(C \diamond X).$$

In the diagram

$$\begin{tikzcd}
S(C \diamond X) \arrow[r, "S(\delta \diamond X)"] \arrow[dr, "S(r_C^{-1} \diamond X)"'] & S((C \otimes C) \diamond X) \arrow[r, "S(\alpha_{C,C,X})"] \arrow[d, "S((C \otimes \varepsilon) \diamond X)"] & S(C \diamond (C \diamond X)) \arrow[r, "\chi_{C,C \diamond X}"] \arrow[d, "S(C \diamond (\varepsilon \diamond X))"] & Z \diamond S(C \diamond X) \arrow[d, "Z \diamond S(\varepsilon \diamond X)"] \\
 & S((C \otimes I) \diamond X) \arrow[r, "S(\alpha_{C,I,X})"'] \arrow[dr, "S(r_C \diamond X)"'] & S(C \diamond (I \diamond X)) \arrow[r, "\chi_{C,I \diamond X}"'] \arrow[d, "S(C \diamond \lambda_X)"] & Z \diamond S(I \diamond X) \arrow[d, "Z \diamond S(\lambda_X)"] \\
 & & S(C \diamond X) \arrow[r, "\chi_{C,X}"'] & Z \diamond S(X),
\end{tikzcd}$$

- the three rectangles are commutative by naturality of α and χ,
- the top triangle commutes since ε is the counit for the coalgebra C,
- the bottom triangle commutes because $\diamond$ is a left action of $\mathbb{V}$ on $\mathbb{A}$.

Hence the outer paths commute and we have

$$\begin{aligned}
(t_K)_{(X,h)} &= (Z \diamond h)(Z \diamond S(\lambda_X))(Z \diamond S(\varepsilon \diamond X))\kappa_{\mathcal{C}_{\phi(X)}} \\
&= (Z \diamond h)(Z \diamond S(\lambda_X))(Z \diamond S(\varepsilon \diamond X))\chi_{C,C \diamond X} S(\alpha_{C,C,X})(S(\delta \diamond X)) \\
&= (Z \diamond h)\chi_{C,X} S(r_C \diamond X) S(r_C^{-1} \diamond X) = (Z \diamond h)\chi_{C,X}.
\end{aligned}$$

□

Note that the above result is contained in the proof of [1, Lemma 5.2.1].

Since for any $X \in \mathbb{A}$, $\phi_{\mathcal{S}}(X) = (S(X), (m_{\mathcal{S}})_X)$, the following is immediate:

Corollary 3.9 *For any $X \in \mathbb{A}$, the $\phi_{\mathcal{S}}(X)$-component $(t_K)_{\phi_{\mathcal{S}}(X)}$ of the comonad morphism $t_K : \mathcal{G} \to \mathcal{G}^l_{\mathcal{Z}}$ is the composite*

$$S(C \diamond S(X)) \xrightarrow{\chi_{C,S(X)}} Z \diamond SS(X) \xrightarrow{Z \diamond (m_{\mathcal{S}})_X} Z \diamond S(X).$$

Combining this with Theorem 2.4 and with Proposition 2.5 and using $(t_K)_{\phi_{\mathcal{S}}(X)} = (t_K \phi_{\mathcal{S}})_X$ yields:

Theorem 3.10 *Under the Assumptions 3.3, 3.5, the functor $K : {}^C\mathbb{A} \to {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$ in a commutative diagram (3.2) is an equivalence of categories if and only if*

(i) *the functor $\phi_{\mathcal{S}}{}^C U : {}^C\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is comonadic and*
(ii) *the composite*

$$S(C \diamond S(X)) \xrightarrow{\chi_{C,S(X)}} Z \diamond SS(X) \xrightarrow{Z\diamond (m_{\mathcal{S}})_X} Z \diamond S(X)$$

is an isomorphism for all $X \in \mathbb{A}$, or, equivalently, $\phi_{\mathcal{S}}{}^C U : {}^C\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is a $\mathcal{G}^l_{\mathcal{Z}}$-Galois comodule functor.

In view of Proposition 2.6(i), the preceding theorem implies:

Theorem 3.11 *Assume that $\mathbb{A}$ is Cauchy complete and that $e_{\mathcal{S}} : I \to S$ is a split monomorphism. Under the Assumptions 3.3, 3.5, the functor $K : {}^C\mathbb{A} \to {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$ with commutative diagram (3.2) is an equivalence of categories if and only if the functor $\phi_{\mathcal{S}}{}^C U : {}^C\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is $\mathcal{G}^l_{\mathcal{Z}}$-Galois.*

We now obtain the Fundamental Theorem of generalised Hopf modules (see [1, Theorem 5.3.1]) as a particular case of Theorem 3.10.

Theorem 3.12 *Under the Assumptions 3.3, 3.5, suppose that*

(i) *$\mathbb{A}$ is Cauchy complete and admits equalisers of coreflexive $\phi_{\mathcal{S}}$-split pairs,*
(ii) *the functors S, $C \diamond -$, $C \diamond (C \diamond -) : \mathbb{A} \to \mathbb{A}$ preserve these equalisers, and*
(iii) *the functor S is conservative.*

Then the functor $K : {}^C\mathbb{A} \to {}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$ in a commutative diagram (3.2) is an equivalence of categories if and only if the functor $\phi_{\mathcal{S}}{}^C U : {}^C\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is $\mathcal{G}^l_{\mathcal{Z}}$-Galois.

Proof Since the functor S is conservative and the category $\mathbb{A}$ admits—and the functor S preserves—equalisers of coreflexive $\phi_{\mathcal{S}}$-split pairs, it follows from the dual of Beck's monadicity theorem (see [12]) that the functor $\phi_{\mathcal{S}} : \mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is comonadic, or equivalently, the monad $\mathcal{S}$ is of effective descent type. Since any $\phi_{\mathcal{S}}$-split pair is automatically S-split, we may apply Proposition 2.6(ii) to deduce that the functor $\phi_{\mathcal{S}}{}^C U : {}^C\mathbb{A} \to \mathbb{A}_{\mathcal{S}}$ is comonadic. The result now follows from Theorem 3.10. □

Remark 3.13 It is pointed out in [1, Remark 3.6.2.] that, when $\mathbb{A} = \mathbb{V}$, $\mathcal{S} = \mathcal{T}$ and $\mathcal{Z} = \mathcal{T}(\mathcal{I})$ is the $\mathcal{T}$-module-comonoid corresponding to the trivial $\mathbb{V}$-monoid $\mathcal{I}$ (see Sect. 3.4), then the category ${}^{\mathcal{Z}}(\mathbb{A}_{\mathcal{S}})$ is nothing but the category of *Hopf modules* in the sense of [10]. Moreover, in this case, to say that Condition (ii) of Theorem 3.10 is satisfied is to say that $\mathcal{T}$ is a *left pre-Hopf monad* in the sense of [10]. It is not hard to see now that [10, Theorem 6.11] is a particular instance of Theorem 3.10.

4 Bimonoids in duoidal categories

In [2], Aguiar and Mahajan generalised bialgebras over fields to *bimonoids* in *duoidal categories*, that is, categories with two monoidal structures $*$ and $\circ$. Any object A in such a category induces endofunctors $- * A$ and $- \circ A$ and for A being a bimonoid these functors have to be a comonad and a monad, respectively, related by a mixed distributive law ([2, Definition 6.25], compare [15, Proposition 6.3]). In [6], Böhm, Chen and Zhang studied which structures are required to define *Hopf monoids* in such categories. In this section we outline how their Fundamental Theorem for Hopf modules (see [6, Section 3.4]) can be seen as special case of the results in the Sects. 2 and 3.

Recall from [2] that duoidal categories $\mathbb{D}$ are equipped with two monoidal structures $(\mathbb{D}, \circ, I)$ and $(\mathbb{D}, *, J)$, along with a natural transformation

$$\zeta_{W,X,Y,Z} : (W * X) \circ (Y * Z) \to (W \circ Y) * (X \circ Z),$$

called the *interchange law*, and three morphisms

$$\Delta : I \to I * I, \quad \mu : J \circ J \to J, \quad \tau : I \to J,$$

such that the conditions for *associativity, unitality* and *compatibility of the units* are satisfied. For example, the compatibility of the units means that the monoidal units I and J satisfy

- (J, μ, τ) is a monoid in the monoidal category $(\mathbb{D}, \circ, I)$ and
- (I, Δ, τ) is a comonoid in the monoidal category $(\mathbb{D}, *, J)$.

It is pointed out in [2] that if $(\mathbb{D}, \circ, I, *, J)$ is a duoidal category with interchange law ζ, then $(\mathbb{D}^{\mathrm{op}}, *, J, \circ, I)$ is also a duoidal category, called the *opposite* duoidal category of $\mathbb{D}$. The interchange law $\overline{\zeta}_{W,X,Y,Z} : (W \circ X) * (Y \circ Z) \to (W * Y) \circ (X * Z)$ for this is given by the $\mathbb{D}$-morphism $\zeta_{W,Y,X,Z} : (W * Y) \circ (X * Z) \to (W \circ X) * (Y \circ Z)$.

4.1 Bimonoids

A *bimonoid* in a duoidal category $\mathbb{D}$ is an object A with a monoid structure (A, m, e) in $(\mathbb{D}, \circ, I)$ and a comonoid structure (A, δ, ε) in $(\mathbb{D}, *, J)$ inducing commutativity of the diagrams

$$\begin{tikzcd}
A \circ A \arrow[r, "m"] \arrow[d, "\delta\circ\delta"'] & A \arrow[r, "\delta"] \arrow[d, phantom, "\mathrm{(I)}"] & A \circ A \\
(A * A) \circ (A * A) \arrow[rr, "\zeta"'] & & (A \circ A) * (A \circ A), \arrow[u, "m*m"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \circ A \arrow[r, "\varepsilon\circ\varepsilon"] \arrow[d, "m"'] \arrow[dr, phantom, "\mathrm{(II)}"] & J \circ J \arrow[d, "\mu"] \\
A \arrow[r, "\varepsilon"'] & J,
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\mathrm{(III)}"] & A \arrow[d, "\delta"] \\
I * I \arrow[r, "e*e"'] & A * A,
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "\tau"'] & A \arrow[d, "\varepsilon"] \arrow[dl, phantom, "\mathrm{(IV)}"] \\
 & J.
\end{tikzcd}$$

A morphism of bimonoids is a morphism of the underlying monoids and comonoids.

Recall [2, Proposition 6.27] that the tensor units I and J carry a unique bimonoid structure and that the morphism $\tau : I \to J$ is a morphism of bimonoids.

Fix a duoidal category $(\mathbb{D}, \circ, I, *, J)$ and a bimonoid $(A, m, e, \delta, \varepsilon)$ in $\mathbb{D}$. Since $(A, A \circ A \xrightarrow{m} A, e : I \xrightarrow{e} A)$ is a monoid in the monoidal category $(\mathbb{D}, \circ, I)$, while $(A, A \xrightarrow{\delta} A * A, A \xrightarrow{\varepsilon} J)$ is a comonoid in the monoidal category $(\mathbb{D}, *, J)$, one has by Sect. 3.5 the monad $\mathcal{T}^r_A$ and the comonad $\mathcal{G}^r_A$ on $\mathbb{D}$. Recall that the functor part of the monad $\mathcal{T}^r_A$ (resp. comonad $\mathcal{G}^r_A$) is the functor $- \circ A : \mathbb{D} \to \mathbb{D}$ (resp. $- * A : \mathbb{D} \to \mathbb{D}$).

It is shown in [7] that $\mathcal{T}^r_A$ is an opmonoidal monad on the monoidal category $(\mathbb{D}, *, J)$, with the structure morphisms

$$\overline{\omega}_{X,Y} : (X * Y) \circ A \xrightarrow{(X*Y)\circ\delta} (X * Y) \circ (A * A) \xrightarrow{\zeta} (X \circ A) * (Y \circ A),$$

$$\overline{\xi} : J \circ A \xrightarrow{J\circ\varepsilon} J \circ I \xrightarrow{\simeq} J.$$

It follows that the category $\mathbb{D}_A = \mathbb{D}_{\mathcal{T}^r_A}$ is monoidal. Note that $((A, m), \delta, \varepsilon)$ is an object of $\mathbf{Comon}(\mathbb{D}_A)$: Clearly $(A, m) \in \mathbb{D}_A$ and the comultiplication δ and the counit ε of A are morphisms of right A-comodules by the diagrams (I) and (II) in Sect. 4.1, respectively. Hence $((A, m), \delta, \varepsilon)$ is a $\mathcal{T}^r_A$-module-comonoid.

Thus, we have

- the opmonoidal monad $\mathcal{T}^r_A$ on the monoidal category $(\mathbb{D}, *, J)$,
- the left $(\mathbb{D}, *, J)$-category structure on $\mathbb{D}$ given by $X \diamond Y = X * Y$,
- the $\mathcal{T}^r_A$-comodule-monad $(\mathcal{T}^r_A, \overline{\omega})$ on $\mathbb{A}$, and
- the $\mathcal{T}^r_A$-module-comonoid $((A, m), \delta, \varepsilon)$.

Hence, we may apply the results of Sect. 3 to the present situation. In particular, Proposition 3.4 gives (see also [6, Section 2]):

Proposition 4.1 *The natural transformation*

$$\lambda : (- * A) \circ A \xrightarrow{(-*A)\circ\delta} (- * A) \circ (A * A) \xrightarrow{\zeta} (- \circ A) * (A \circ A) \xrightarrow{(-\circ A)*m} (- \circ A) * A$$

is a mixed distributive law from the monad $\mathcal{T}^r_A$ to the comonad $\mathcal{G}^r_A$.

We write $\mathbb{D}^A_A$ for the category $(\mathbb{D}_{\mathcal{T}^r_A})^{\widehat{\mathcal{G}^r_A}} = (\mathbb{D}_A)^{\widehat{\mathcal{G}^r_A}}$, where $\widehat{\mathcal{G}^r_A}$ is the lifting of $\mathcal{G}^r_A$ to $\mathbb{D}_{\mathcal{T}^r_A} = \mathbb{D}_A$ corresponding to the mixed distributive law λ. This is called the category of A-*Hopf modules* in [6, Section 3]. Thus, an A-Hopf module is a triple (X, h, ϑ) such that $(X,\ h : X \circ A \to A)$ is a right A-module in the monoidal category $(\mathbb{D}, \circ, I)$, $(X,\ \vartheta : X \to X * A)$ is a right A-comodule in the monoidal category $(\mathbb{D}, *, J)$ with commutative diagram

$$\begin{tikzcd}
X \circ A \arrow[r, "h"] \arrow[d, "\vartheta\circ A"'] & X \arrow[r, "\vartheta"] & X * A \\
(X * A) \circ A \arrow[rr, "\lambda_X"'] & & (X \circ A) * A. \arrow[u, "h*A"']
\end{tikzcd}$$

Since (I, Δ, τ) is a comonoid in the monoidal category $(\mathbb{D}, *, J)$, one has the category $\mathbb{D}^I$ of I-comodules on this monoidal category. Recall that $\mathbb{D}^I$ is the Eilenberg–Moore category of $\mathcal{G}_I^r$-comodules. Now Lemma 3.6 implies:

4.2 Comparison functor $- \circ A : \mathbb{D}^I \to \mathbb{D}_A^A$

The assignment

$$(X, \theta) \rightsquigarrow (X \circ A, \overline{m}, \overline{\vartheta}),$$

where $\overline{m} : (X \circ A) \circ A \to X \circ A$ is the composite

$$(X \circ A) \circ A \xrightarrow{a_{X,A,A}} X \circ (A \circ A) \xrightarrow{X \circ m} X \circ A,$$

while $\overline{\vartheta} : X \circ A \to (X \circ A) * A$ is the composite

$$X \circ A \xrightarrow{\vartheta \circ A} (X * I) \circ A \xrightarrow{(X*I)\circ\delta} (X * I) \circ (A * A) \xrightarrow{\zeta} (X \circ A) * (I \circ A) \simeq (X \circ A) * A,$$

yields a comparison functor $K = - \circ A : \mathbb{D}^I \to \mathbb{D}_A^A$ with commutative diagram

$$\begin{tikzcd} \mathbb{D}^I \arrow[r, "K"] \arrow[d, "U^I"'] & \mathbb{D}_A^A \arrow[d, "U^{\widehat{\mathcal{G}_A^r}}"] \\ \mathbb{D} \arrow[r, "\phi_A"] & \mathbb{D}_A . \end{tikzcd}$$

Write $\mathcal{G}$ for the comonad on the category $\mathbb{D}_A$ generated by the adjunction

$$\phi_A U^I \dashv \phi^I U_A : \mathbb{D}_A \to \mathbb{D}^I.$$

Proposition 4.2 *For any $(X, h) \in \mathbb{D}_A$, the (X, h)-component of the comonad morphism $t_K : \mathcal{G} \to \widehat{\mathcal{G}_A^r}$ induced by the diagram in Sect. 4.2, is the composite*

$$(X * I) \circ A \xrightarrow{(X*I)\circ\delta} (X * I) \circ (A * A) \xrightarrow{\zeta} (X \circ A) * (I \circ A) \simeq (X \circ A) * A \xrightarrow{h*A} X * A.$$

For any $X \in \mathbb{D}$, $\phi_A(X) = (X \circ A,\ (X \circ m) \cdot a_{X,A,A} : (X \circ A) \circ A \to X \circ A)$, and the $\phi_A(X)$-component $(t_K)_{\phi_A(X)}$ of t_K is the composite

$$\begin{tikzcd} ((X \circ A) * I) \circ A \arrow[r, "((X\circ A)*I)\circ\delta"] \arrow[d, dotted, "(t_K)_{\phi_A(X)}"'] & ((X \circ A) * I) \circ (A * A) \arrow[r, "\zeta"] & ((X \circ A) \circ A) * (I \circ A) \arrow[d, "\simeq"] \\ (X \circ A) * A & (X \circ (A \circ A)) * A \arrow[l, "(X\circ m)*A"] & ((X \circ A) \circ A) * A . \arrow[l, "\simeq"] \end{tikzcd}$$

Applying now Theorem 3.10 yields:

Theorem 4.3 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category and $(A, m, e, \delta, \varepsilon)$ a bimonoid in $\mathbb{D}$. Then the comparison functor $K : \mathbb{D}^I \to \mathbb{D}_A^A$ is an equivalence of categories if and only if*

(i) *the functor $\phi_A U^I : \mathbb{D}^I \to \mathbb{D}_A$ is comonadic and*
(ii) *the morphism $(t_K)_{\phi_A(X)}$ (in Proposition 4.2) is an isomorphism for any $X \in \mathbb{D}$, or, equivalently, $\phi_A U^I : \mathbb{D}^I \to \mathbb{D}_A$ is a $\mathcal{G}_A^r$-Galois comodule functor.*

Theorem 4.4 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category with Cauchy complete $\mathbb{D}$. If the morphism $\tau : I \to J$ is a split monomorphism, then for any bimonoid $(A, m, e, \delta, \varepsilon)$ in $\mathbb{D}$, the functor $K : \mathbb{D}^I \to \mathbb{D}_A^A$ (in Sect. 4.2) is an equivalence of categories if and only if $\phi_A U^I : \mathbb{D}^I \to \mathbb{D}_A$ is $\mathcal{G}_A^r$-Galois.*

Proof Since $\varepsilon \cdot e = \tau$ (see Diagram (IV) in Sect. 4.1) and since τ is a split monomorphism by hypothesis, so also is the morphism $e : I \to A$. It then follows that the unit of the monad $\mathcal{T}_A^r = - \circ A$ is a split monomorphism and Theorem 3.11 completes the proof. □

The following elementary observation is of use for our investigations.

Proposition 4.5 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category with Cauchy complete $\mathbb{D}$. If the unit of the adjunction*

$$\begin{tikzcd} \mathbb{D}^I \arrow[r, bend left, "U^I"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "\phi^I"] \arrow[r, bend left, "\phi_J"] \arrow[r, phantom, "\perp"] & \mathbb{D}_J \arrow[l, bend left, "U_J"] \end{tikzcd}$$

is a split monomorphism, then for any bimonoid $(A, m, e, \delta, \varepsilon)$ in $\mathbb{D}$, the functor $\mathbb{D}^I \xrightarrow{U^I} \mathbb{D} \xrightarrow{\phi_A} \mathbb{D}_A$ is comonadic.

Proof Note first that the commutativity of the diagrams (II) and (IV) in Sect. 4.1 allow to consider $\varepsilon : A \to J$ as a morphism from the monoid (A, m, e) to the monoid (J, μ, τ) in the monoidal category $(\mathbb{D}, \circ, I)$. Then the composites $A \circ J \xrightarrow{\varepsilon \circ J} J \circ J \xrightarrow{\mu} J$ and $J \circ A \xrightarrow{J \circ \varepsilon} J \circ J \xrightarrow{\mu} J$ give the structure of an (A, A)-bimodule on J, and so, by Remark 3.2, the triangle in the diagram

$$\begin{tikzcd} \mathbb{D}^I \arrow[r, "U^I"] \arrow[d, equal] & \mathbb{D} \arrow[r, "\overline{\phi}_A"] \arrow[dr, "\phi_J = J \otimes -"'] & \widetilde{\mathbb{D}}_{\mathcal{T}_A^l} \arrow[d, "J \otimes_A -"] \\ \mathbb{D}^I \arrow[r, "U^I"'] & \mathbb{D} \arrow[r, "\phi_J"'] & \mathbb{D}_J \end{tikzcd}$$

is commutative, implying that the outer diagram is also commutative. Since $\mathbb{D}$ is assumed to be Cauchy complete, so also is $\mathbb{D}^I$. Now apply Proposition 2.7 to conclude that the composite $\overline{\phi}_A U^I$ is conservative, and that any coreflexive $(\overline{\phi}_A U^I)$-split pair of morphisms has a split equaliser in $\mathbb{D}^I$.

Next, since $\iota\overline{\phi}_A = \phi_A$, where $\iota : \widetilde{\mathbb{D}}_{\mathcal{T}_A^l} \to \mathbb{D}_A$ is the canonical embedding, the composite $\overline{\phi}_A U^I$ is conservative if and only if $\iota\overline{\phi}_A U^I = \phi_A U^I$ is conservative, and a pair of morphisms in D^I is $\overline{\phi}_A U^I$-split if and only if it is $\phi_A U^I$-split. Thus, $\overline{\phi}_A U^I$ is conservative and any $\overline{\phi}_A U^I$-split pair of morphisms has a split equaliser in $\mathbb{D}^I$. It follows – since the composite $\phi_A U^I$ admits as a right adjoint the composite $\phi^I U_A$ – that $\overline{\phi}_A U^I$ is comonadic. □

Combining Propositions 4.3 and 4.5 we obtain:

Theorem 4.6 *In the situation of Proposition 4.5, the functor $K : \mathbb{D}^I \to \mathbb{D}_A^A$ (in Sect. 4.2) is an equivalence of categories if and only if $\phi_A U^I : \mathbb{D}^I \to \mathbb{D}_A$ is $\mathcal{G}_A^r$-Galois.*

Since a left adjoint functor is full and faithful if and only if the unit of the adjunction is an isomorphism (hence a split monomorphism), it follows immediately:

Corollary 4.7 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category with Cauchy complete $\mathbb{D}$. If the composite $\phi_J U^I : \mathbb{D}^I \to \mathbb{D}_J$ is full and faithful, then, for any bimonoid $(A, m, e, \delta, \varepsilon)$ in $\mathbb{D}$, the functor $K : \mathbb{D}^I \to \mathbb{D}_A^A$ (in Sect. 4.2) is an equivalence of categories if and only if $\phi_A U^I : \mathbb{D}^I \to \mathbb{D}_A$ is $\mathcal{G}_A^r$-Galois.*

Considering any bimonoid $(A, m, e, \delta, \varepsilon)$ in $\mathbb{D}$ as a bimonoid in $\mathbb{D}^{\mathrm{op}}$ (see [2, Remark 6.26]) and applying the duality explained in [2], the Theorems 4.3, 4.6 and Corollary 4.7 yield the following:

Theorem 4.8 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category and $(A, m, e, \delta, \varepsilon)$ a bimonoid in $\mathbb{D}$. Then the comparison functor*

$$K' = - * A : \mathbb{D}_J \to \mathbb{D}_A^A$$

is an equivalence of categories if and only if

(i) *the functor $\phi^A U_J : \mathbb{D}_J \to \mathbb{D}^A$ is monadic and*
(ii) *$\phi^A U_J : \mathbb{D}_J \to \mathbb{D}^A$ is a $\mathcal{T}_A^r$-Galois module functor, or, equivalently, the following composite is an isomorphism for all $X \in \mathbb{D}$:*

$$\begin{tikzcd}
(X * A) \circ A \arrow[r, "(X*\delta)\circ A"] \arrow[d, dotted, "(t_{K'})_{\phi^A(X)}"'] & (X * (A * A)) \circ A \arrow[r, "\simeq"] & ((X * A) * A) \circ A \arrow[d, "\simeq"] \\
((X * A) \circ J) * A & ((X * A) \circ J) * (A \circ A) \arrow[l, "((X*A)\circ J)*m"'] & ((X * A) * A) \circ (J * A). \arrow[l, "\zeta"']
\end{tikzcd}$$

Cauchy completeness of $\mathbb{D}$ allows for the following characterisations.

Theorem 4.9 *Let $(\mathbb{D}, \circ, I, *, J)$ be a duoidal category with Cauchy complete $\mathbb{D}$. Assume either of the conditions*

(1) *the morphism* $\tau : I \to J$ *is a split epimorphism, or*

(2) *the unit of the adjunction* $\begin{tikzcd} \mathbb{D}_J \arrow[r, bend right, "U_J"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "\phi_J"'] \arrow[r, bend right, "\phi^I"'] \arrow[r, phantom, "\perp"] & \mathbb{D}^I \arrow[l, bend right, "U^I"'] \end{tikzcd}$ *is a split epimorphism, or*

(3) *the composite* $\phi^I U_J : \mathbb{D}_J \to \mathbb{D}^I$ *is full and faithful.*

Then, for any bimonoid $(A, m, e, \delta, \varepsilon)$ *in* $\mathbb{D}$*, the functor* $K' = - * A : \mathbb{D}_J \to \mathbb{D}_A^A$ *is an equivalence of categories if and only if* $\phi^A U_J : \mathbb{D}_J \to \mathbb{D}^A$ *is* $\mathcal{T}_A^r$*-Galois.*

Note that Corollary 4.7 subsumes [6, Theorem 3.11], while [6, Theorem 3.13] is a consequence of Theorem 4.9 (iii).

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