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Comonadicity and invertible bimodules ☆

Bachuki Mesablishvili

Razmadze Mathematical Institute, Tbilisi 0193, Georgia

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Abstract

The purpose of this paper is to exhibit the main results of A. Masuoka [A. Masuoka, Corings and invertible bimodules, Tsukuba J. Math. 13 (1989) 353–362] and of L. El Kaoutit and J. Gómez-Torrecillas [L. El Kaoutit, J. Gómez-Torrecillas, Comatrix corings and invertible bimodules, Ann. Univ. Ferrara Ser. VII 51 (2005) 263–280] as special cases of a more general result.
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1. Introduction

Let $i: B \to S$ be an extension of non-commutative rings, let $\mathrm{Inv}_R(S)$ denote the group of invertible B-subbimodules of S, and $\mathrm{Aut}_{S\text{-cor}}(S \otimes_B S)$ the group of S-coring automorphisms of the Sweedler's canonical S-coring $S \otimes_B S$. In [6], Masuoka defined a group homomorphism $\Gamma : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{S\text{-cor}}(S \otimes_B S)$ and showed that if either (a) S is faithfully flat as a right or left B-module, or (b) B is a direct summand of S as a B-bimodule, then Γ is an isomorphism of groups.

This has been further generalized by L. El Kaoutit and J. Gómez-Torrecillas [3], considering extensions of non-commutative rings of the form $B \to S = \mathrm{End}_A(M)$, where M is a B–A-bimodule with M_A finitely generated and projective. They defined a homomorphism

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E-mail address: bachi@rmi.acnet.ge.

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$$\Gamma_0 : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$$

of groups, where $\Sigma = M^* \otimes_B M$ is the so-called comatrix coring corresponding to the B–A-bimodule M (for the notion of comatrix coring, see [4]). This homomorphism extends Γ in the sense that there is a homomorphism of groups

$$\widehat{(-)} : \mathrm{Aut}_{A\text{-cor}}(\Sigma) \to \mathrm{Aut}_{S\text{-cor}}(S \otimes_B S)$$

for which the diagram

$$\begin{array}{ccc} \mathrm{Inv}_B(S) & \xrightarrow{\Gamma_0} & \mathrm{Aut}_{A\text{-cor}}(\Sigma) \\ \downarrow{\scriptstyle \Gamma} & \swarrow{\scriptstyle \widehat{(-)}} & \\ \mathrm{Aut}_{S\text{-cor}}(S \otimes_B S) & & \end{array}$$

commutes. It is then shown in [3] that if (i) ${}_B M$ is faithfully flat, or (ii) M^*_B is faithfully flat, or (iii) ${}_B M_A$ is a separable bimodule, then Γ_0 is an isomorphism of groups.

Examining the proofs of the above results, one observes that they are depend on a descent type argument. For example, it is not hard to show that under any of the conditions (i)–(iii), at least one of the extension-of-scalars functors associated to the ring extension $B \to S = \mathrm{End}_A(M)$ is comonadic. To see this, first note that by a result of [7], the functor $- \otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ (respectively $M^* \otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$) is comonadic iff the functor $- \otimes_B S : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ (respectively $S \otimes_B - : \mathrm{Mod}_B \to \mathrm{Mod}_S$) is so. Next, if ${}_B M$ (respectively M^*_B) is faithfully flat, then the functor $- \otimes_B M$, or equivalently, the functor $- \otimes_B S$ (respectively the functor $M^* \otimes_B -$, or equivalently, the functor $S \otimes_B -$) is comonadic by a simple and well-known application of the (dual of the) Beck theorem. Finally, if ${}_B M_A$ is a separable bimodule, then the ring extension $i : B \to S$ splits (see, for example, [8]), i.e. B is a direct summand of S as a B-bimodule, and thus we can apply Corollary 4.2 of [5] to conclude that both $- \otimes_B S$ and $S \otimes_B -$ are comonadic. It follows that each of the conditions (i)–(iii) guarantees that (at least) one of the extension-of-scalars functors associated to the ring extension $B \to S = \mathrm{End}_A(M)$ is comonadic.

This observation suggests to consider the following question:

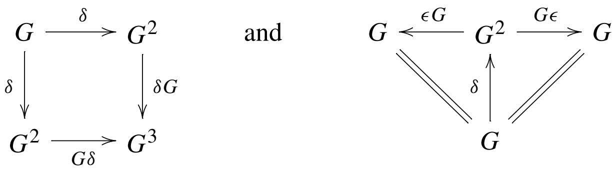
Is it certainly the case that $\Gamma_0 : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$ is an isomorphism of groups when one of the extension-of-scalars functors associated to the ring extension $B \to \mathrm{End}_A(M)$ is comonadic?

The aim of the present paper is to give a positive answer to this question.

We refer to [1] for terminology and general results on (co)monads, and to [2] for a comprehensive introduction to the theory of corings and comodules.

2. Preliminaries

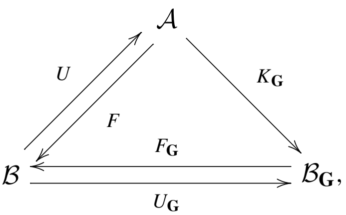
We begin by recalling that a comonad $\mathbf{G}$ on a given category $\mathcal{B}$ is an endofunctor $G : \mathcal{B} \to \mathcal{B}$ equipped with natural transformations $\epsilon : G \to 1$ and $\delta : G \to G^2$ such that the diagrams



are commutative. If $\mathbf{G} = (G, \delta, \epsilon)$ is a comonad on $\mathcal{B}$, then a $\mathbf{G}$-coalgebra is a pair (b, θ_b) with $b \in \mathcal{B}$ and $\theta_b : b \to G(b)$ a morphism in $\mathcal{B}$ for which $\epsilon_b \cdot \theta_b = 1$ and $\delta_b \cdot \theta_b = G(\theta_b) \cdot \theta_b$. If (b, θ_b) and $(b', \theta_{b'})$ are $\mathbf{G}$-coalgebras, then their morphism $f : (b, \theta_b) \to (b', \theta_{b'})$ is a morphism $f : b \to b'$ of $\mathcal{B}$ for which $\theta_{b'} \cdot f = G(f) \cdot \theta_b$.

The $\mathbf{G}$-coalgebras and their morphisms form a category $\mathcal{B}_{\mathbf{G}}$, the category of $\mathbf{G}$-coalgebras (or the *Eilenberg–Moore* category associated to $\mathbf{G}$). There are functors $F_{\mathbf{G}} : \mathcal{B}_{\mathbf{G}} \to \mathcal{B}$ and $U_{\mathbf{G}} : \mathcal{B} \to \mathcal{B}_{\mathbf{G}}$, given on objects by $F_{\mathbf{G}}(b, \theta_b) = b$ and $U_{\mathbf{G}}(b) = (G(b), \delta_b)$. Moreover, $F_{\mathbf{G}}$ is left adjoint to $U_{\mathbf{G}}$.

Recall also that if $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ is an adjunction (so that $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint of $U : \mathcal{B} \to \mathcal{A}$ with unit $\eta : 1 \to UF$ and counit $\epsilon : FU \to 1$), then $\mathbf{G} = (G, \epsilon, \delta)$ is a comonad on $\mathcal{B}$, where $G = FU$, $\epsilon : G = FU \to 1$ and $\delta = F\eta U : G = FU \to FUFU = G^2$, and one has the comparison functor $K_{\mathbf{G}}$ in



where $K_{\mathbf{G}}(a) = (F(a), F(\eta_a))$ and $K_{\mathbf{G}}(f) = F(f)$. Moreover, $F_{\mathbf{G}} \cdot K_{\mathbf{G}} \simeq F$ and $K_{\mathbf{G}} \cdot U \simeq U_{\mathbf{G}}$. One says that the functor F is *precomonadic* if $K_{\mathbf{G}}$ is full and faithful, and it is *comonadic* if $K_{\mathbf{G}}$ is an equivalence of categories.

Theorem 2.1. *(Beck, see [1].) Let* $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ *be an adjunction, and let* $\mathbf{G} = (FU, \epsilon, F\eta U)$ *be the corresponding comonad on* $\mathcal{B}$. *Then*:

1. *The comparison functor* $K_{\mathbf{G}} : \mathcal{A} \to \mathcal{B}_{\mathbf{G}}$ *has a right adjoint* $R_{\mathbf{G}} : \mathcal{B}_{\mathbf{G}} \to \mathcal{A}$ *iff for each* $(b, \theta_b) \in \mathcal{B}_{\mathbf{G}}$*, the pair of morphisms* $(U(\theta_b), \eta_{U(b)})$ *has an equalizer in* $\mathcal{A}$*—one then finds* $R_{\mathbf{G}}(b, \theta_b)$ *as the equalizer*

$$R_{\mathbf{G}}(b, \theta_b) \xrightarrow{e_{(b,\theta_b)}} U(b) \underset{\eta_{U(b)}}{\overset{U(\theta_b)}{\rightrightarrows}} UFU(b). \tag{2.1}$$

2. *Assuming the existence of* $R_{\mathbf{G}}$*,* $K_{\mathbf{G}}$ *is an equivalence of categories* (*in other words,* F *is comonadic*) *iff the functor* F *is conservative* (= *isomorphism-reflecting*) *and preserves* (*or equivalently, preserves and reflects*) *the equalizer* (2.1) *for each* $(b, \theta_b) \in \mathcal{B}_{\mathbf{G}}$.

Let $i : B \to S$ be an arbitrary extension of (non-commutative) rings, $\mathcal{A}$ be the category ${}_B\mathrm{Mod}$ of left B-modules, $\mathcal{B}$ be the category ${}_S\mathrm{Mod}$ of left S-modules,

$$F_S = S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$$

and

$$U_S : {}_S\mathrm{Mod} \to {}_B\mathrm{Mod}$$

be the restriction-of-scalars functor. It is well known that F_S is left adjoint to U_S and that the unit η of this adjunction is given by

$$\eta_X : X \to S \otimes_B X, \quad \eta_X(x) = 1 \otimes_B x.$$

It is also well known that the Eilenberg–Moore category $({}_S\mathrm{Mod})_{\mathbf{G}}$ of $\mathbf{G}$-coalgebras, $\mathbf{G}$ being the comonad on ${}_S\mathrm{Mod}$ associated to the adjunction $F_S \dashv U_S$, is isomorphic to, and thus may be identified with, the category ${}^{S\otimes_B S}({}_S\mathrm{Mod})$ of left comodules over the Sweedler canonical B-coring $S \otimes_B S$ corresponding to the ring extension i. Moreover, module this identification, the comparison functor $K_{\mathbf{G}} : {}_B\mathrm{Mod} \to ({}_S\mathrm{Mod})_{\mathbf{G}}$ corresponds to the functor

$$K_S : {}_B\mathrm{Mod} \longrightarrow {}^{S\otimes_B S}({}_S\mathrm{Mod}), \qquad K_S(X) = (S \otimes_B X, \theta_{S\otimes_B X}),$$

where $\theta_{S\otimes_B X} = S \otimes_B \eta_X$ for all $X \in_B \mathrm{Mod}$. (Note that a left $S \otimes_B S$-comodule is a pair (Y, θ_Y) with $Y \in {}_S\mathrm{Mod}$ and $\theta_Y : Y \to S \otimes_B Y$ a left A-module morphism for which the diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\theta_Y} & S \otimes_B Y \\ & \searrow^{=} & \downarrow{\scriptstyle \alpha_Y} \\ & & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xrightarrow{\theta_Y} & S \otimes_B Y \\ {\scriptstyle \theta_Y}\downarrow & & \downarrow{\scriptstyle S\otimes_B \eta_Y} \\ S \otimes_B Y & \xrightarrow[S\otimes_B \theta_Y]{} & S \otimes_B S \otimes_B Y, \end{array}$$

where α_Y denotes the left S-module structure on Y, are commutative.) So, to say that the functor $F_S = S \otimes_B -$ is comonadic is to say that the functor K_S is an equivalence of categories. Applying Beck's theorem and using that ${}_B\mathrm{Mod}$ has all equalizers, we get:

Theorem 2.2. *The functor* $F_S = S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ *is comonadic if and only if*

(i) *the functor F_S is conservative, or equivalently, the ring extension $i : B \to S$ is a* pure *morphism of right B-modules*;
(ii) *for any* $(Y, \theta_Y) \in {}^{S\otimes_B S}({}_S\mathrm{Mod})$, *$F_S$ preserves the equalizer*

$$R_S(Y, \theta_Y) \xrightarrow{e_{(Y,\theta_Y)}} Y \underset{\theta_Y}{\overset{\eta_Y}{\rightrightarrows}} S \otimes_B Y, \tag{2.2}$$

where $R_S : {}^{S\otimes_B S}({}_S\mathrm{Mod}) \to {}_B\mathrm{Mod}$ *is the right adjoint of the comparison functor* $K_S : {}_B\mathrm{Mod} \to {}^{S\otimes_B S}({}_S\mathrm{Mod})$.

Let A be a ring and Σ be an $\mathcal{A}$-coring. Let us write $\mathrm{End}_{A\text{-cor}}(\Sigma)$ (respectively $\mathrm{Aut}_{A\text{-cor}}(\Sigma)$) for the monoid (respectively group) of A-coring endomorphisms (respectively automorphisms) of Σ. Recall that any $g \in \mathrm{End}_{A\text{-cor}}(\Sigma)$ induces functors:

$$_g(-) : {}^{\Sigma}({}_A\mathrm{Mod}) \to {}^{\Sigma}({}_A\mathrm{Mod}),$$

defined by $_g(Y, \theta_Y) = (Y, (g \otimes_A 1) \circ \theta_Y)$, and

$$(-)_g : (\mathrm{Mod}_A)^{\Sigma} \to (\mathrm{Mod}_A)^{\Sigma}$$

defined by $(Y', \theta_{Y'})_g = (Y', (1 \otimes_A g) \circ \theta_{Y'})$.

It is easy to see that the left S-module S is a left $(S \otimes_B S)$-comodule with left coaction

$$_S\theta : S \to S \otimes_B S, \quad s \to s \otimes_B 1,$$

and that $_g(S, {}_S\theta) = (S, g \circ_S \theta)$. Symmetrically, the right S-module S is a right $S \otimes_B S$-comodule with the right action

$$\theta_S : S \to S \otimes_B S, \quad s \to 1 \otimes_B s,$$

and that $(S, \theta_S)_g = (S, g \circ \theta_S)$.

For a given injective homomorphism $i : B \to S$ of rings, let

- $I_B(S)$ denote the monoid of all B-subbimodules of S, the multiplication being given by

$$IJ = \left\{ \sum_{k \in K} i_k \cdot j_k,\ I, J \in I_B(S),\ i_k \in I,\ j_k \in J,\ \text{and } K \text{ is a finite set} \right\};$$

- $I_B^l(S)$ (respectively $I_B^r(S)$) denote the submonoid of $I_B(S)$ consisting of those $I \in I_B(S)$ for which the map

$$\mathbf{m}_I^l : S \otimes_B I \to S, \quad s \otimes_B i \to si,$$
$$\big(\text{respectively } \mathbf{m}_I^r : I \otimes_B S \to S, \quad i \otimes_B s \to is\big)$$

  is an isomorphism;
- $J(g) = \{s \in S \mid g(s \otimes_B 1) = 1 \otimes_B s\}$ for $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ and let $i_g : J(g) \to S$ be the inclusion map;
- $J'(g) = \{s \in S \mid s \otimes_B 1 = g(1 \otimes_B s)\}$ for $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ and let $i'_g : J'(g) \to S$ be the inclusion map.

It is clear that $J(g), J'(g) \in I_B(S)$ for all $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$.

The following result is verified directly:

Proposition 2.3. *For any* $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$, $R_S({}_g(S, {}_S\theta)) \simeq J(g)$.

3. Main results

In this section we present our main results.

We begin with

Proposition 3.1. *For any $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$, the following conditions are equivalent*:

(i) $J(g) \in I_B^l(S)$;
(ii) *the* ${}_g(S, {}_S\theta)$*-component of the counit* $\varepsilon : K_S R_S \to 1$ *of the adjunction* $K_S \dashv R_S$ *is an isomorphism*;
(iii) *the functor* $S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ *preserves the equalizer*

$$J(g) \xrightarrow{\ i_g\ } S \underset{g \circ {}_S\theta}{\overset{\eta_S}{\rightrightarrows}} S \otimes_B S; \tag{3.1}$$

(iv) *the morphism* $S \otimes_B i_g : S \otimes_B J(g) \to S \otimes_B S$ *is a monomorphism.*

Proof. We know by the general theory of (co)monads (see, for example, [1]) that, for any $(Y, \theta_Y) \in {}^{S \otimes_B S}({}_S\mathrm{Mod})$, the diagram

$$Y \xrightarrow{\ \theta_Y\ } S \otimes_B Y \underset{S \otimes_B \eta_Y}{\overset{S \otimes_B \theta_Y}{\rightrightarrows}} S \otimes_B S \otimes_B Y$$

is an equalizer and that the (Y, θ_Y)-component $\varepsilon_{(Y,\theta_Y)}$ of ε appears as the unique factorization of the morphism $S \otimes_B e_{(Y,\theta_Y)}$ through the morphism θ_Y:

$$\begin{array}{ccccc} S \otimes_B R_S(Y, \theta_Y) & & & & \\ {\scriptstyle \varepsilon_{(Y,\theta_Y)}} \vdots & \searrow^{S \otimes_B e_{(Y,\theta_Y)}} & & & \\ Y & \xrightarrow[\theta_Y]{} & S \otimes_B Y & \underset{S \otimes_B \eta_Y}{\overset{S \otimes_B \theta_Y}{\rightrightarrows}} & S \otimes_B S \otimes_B Y. \end{array} \tag{3.2}$$

Since $\alpha_Y \cdot \theta_Y = 1$, $\varepsilon_{(Y,\theta_Y)} = \alpha_Y \cdot (S \otimes_B e_{(Y,\theta_Y)})$. In particular, when $(Y, \theta_Y) = {}_g(S, {}_S\theta)$ we get that $\varepsilon_{{}_g(S,{}_S\theta)} = \mathbf{m}^l_{J(g)}$. So (i) and (ii) are equivalent.

Since the row of the diagram (3.2) is an equalizer, it follows that the morphism $S \otimes_B e_{(Y,\theta_Y)}$ is an equalizer of the pair of morphisms $(S \otimes_B \theta_Y, S \otimes_B \eta_Y)$ iff $\varepsilon_{(Y,\theta_Y)}$ is an isomorphism. In other words, the functor $S \otimes_B -$ preserves the equalizer (2.2) iff $\varepsilon_{(Y,\theta_Y)}$ is an isomorphism. As a special case we then have that (ii) is equivalent to (iii).

Finally, since the category ${}_B\mathrm{Mod}$ is abelian (and hence coexact in the sense of Barr [1]), and since i_g is the equalizer of the $(S \otimes_B -)$-split pair of morphisms $({}_S\theta, \eta_S)$, it follows from the proof of Duskin's theorem (see, for example, [1]) that the functor $S \otimes_B -$ preserves the equalizer (3.1) iff the morphism $S \otimes_B i_g$ is a monomorphism. So (iii) and (iv) are also equivalent. This completes the proof. □

It is shown in [3] that assigning to each $I \in I_B^l(S)$ (respectively $I \in I_B^r(S)$) the composite $\Gamma(I) = (1 \otimes_B \mathbf{m}_I^r) \circ ((\mathbf{m}_I^l)^{-1} \otimes_B 1)$ (respectively $\Gamma'(I) = (\mathbf{m}_I^l \otimes_B 1) \circ (1 \otimes_B (\mathbf{m}_I^r)^{-1})$) yields an (anti-)homomorphism of monoids $\Gamma : I_B^l(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ (respectively $\Gamma' : I_B^l(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$).

We shall need the following easy consequence of Lemma 2.7 of [6]:

Proposition 3.2. *Assume that $i : B \to S$ is such that any embedding $I \hookrightarrow J$ of B-subbimodules of S is an isomorphism whenever its image under the functor $S \otimes_B -$ is such. Then $\Gamma : I_B^l(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an isomorphism of monoids whose inverse is the map $g \to J(g)$, provided that $J(g) \in I_B^l(S)$ for all $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$.*

Putting Propositions 3.1 and 3.2 together, we get:

Theorem 3.3. *Let $i : B \to S$ be as in Proposition* 3.2. *Then $\Gamma : I_B^l(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an isomorphism of monoids if and only if, for any $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$, the equivalent conditions of Proposition* 3.1 *hold.*

Proposition 3.4. *If the functor $S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ is comonadic, then $J(g) \in \mathrm{I}_B^l(S)$ for all $g \in \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$.*

Proof. Consider the left $(S \otimes_B S)$-comodule $(S, {}_S\theta)$. According to Proposition 2.3 and Theorem 2.1, the pair $(J(g), i_g : J(g) \to S)$ appears as the equalizer

$$J(g) \xrightarrow{i_g} S \underset{g \circ {}_S\theta}{\overset{\eta_S}{\rightrightarrows}} S \otimes_B S,$$

and since the functor $S \otimes_B -$ is assumed to be comonadic, it preserves the equalizer (2.2) for all $(Y, \theta_Y) \in {}^{S \otimes_B S}({}_S\mathrm{Mod})$ and in particular considering $(S, {}_S\theta) \in {}^{S \otimes_B S}({}_S\mathrm{Mod})$, we see that

$$S \otimes_B J(g) \xrightarrow{S \otimes_B i_g} S \otimes_B S \underset{S \otimes_B (g \circ {}_S\theta)}{\overset{S \otimes_B \eta_S}{\rightrightarrows}} S \otimes_B S \otimes_B S$$

is an equalizer diagram. It now follows from Proposition 3.1 that $J(g) \in I_B^l(S)$. $\square$

Recalling that any comonadic functor is conservative, and putting Theorem 3.3 and Proposition 3.4 together, we obtain:

Theorem 3.5. *If the functor $S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ is comonadic, then $\Gamma : I_B^l(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an isomorphism of monoids.*

There is of course a dual result.

Theorem 3.6. *If the functor $- \otimes_B S : \mathrm{Mod}_B \to \mathrm{Mod}_S$ is comonadic, then $\Gamma' : I_B^r(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an anti-isomorphism of monoids.*

It is known (see [6]) that the monoid morphism

$$\Gamma : \mathrm{I}^l_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$$

restricts to a group morphism

$$\mathrm{Inv}_B(S) \to \mathrm{Aut}_{B\text{-cor}}(S \otimes_B S),$$

which is still denoted by Γ. Similarly, the monoid anti-morphism

$$\Gamma' : I^r_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$$

is restricted to the group anti-morphism

$$\mathrm{Inv}_B(S) \to \mathrm{Aut}_{B\text{-cor}}(S \otimes_B S),$$

which is called Γ', too.

Theorem 3.7. *If either*

(i) *the functor* $S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$, *or*
(ii) *the functor* $- \otimes_B S : \mathrm{Mod}_B \to \mathrm{Mod}_S$

is comonadic, then $\Gamma : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{B\text{-cor}}(S \otimes_B S)$ *is an isomorphism of groups.*

Proof. The same argument as in [3] shows that if either $\Gamma : I^l_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an isomorphism of monoids, or $\Gamma' : I^r_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an anti-isomorphism of monoids, then the group homomorphism Γ is an isomorphism. Theorems 3.5 and 3.6 now complete the proof. □

As a special case of this theorem, we obtain the following result of Masuoka (see [6]):

Theorem 3.8. *If either*

(i) ${}_B S$ *is faithfully flat, or*
(ii) B *is a direct summand of* S *as a* B*-bimodule,*

then $\Gamma : I^l_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ *is an isomorphism of monoids.*

Proof. In both cases, the functor $S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$ is comonadic, as we have seen already in the introduction. □

Dually we have:

Theorem 3.9. *If either*

(i) S_B *is faithfully flat, or*

(ii) *B is a direct summand of S as a B-bimodule,*

then $\Gamma' : I^r_B(S) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S)$ is an anti-isomorphism of monoids.

Theorem 3.10.

(i) *${}_BS$ or S_B is faithfully flat, or*
(ii) *B is a direct summand of S as a B-bimodule,*

then $\Gamma : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{B\text{-cor}}(S \otimes_B S)$ is an isomorphism of groups.

Proof. The argument here is the same as in the proof of Theorem 3.7. □

We now consider the following situation: Let A and B be rings, M a (B, A)-bimodule with M_A finitely generated and projective, $S = \mathrm{End}_A(M)$ the ring of right A-endomorphisms of M_A, and $\Sigma = M^* \otimes_B M$ the comatrix A-coring corresponding to ${}_BM_A$. When ${}_BM_A$ is faithful, in the sense that the canonical morphism

$$i : B \to S, \quad s \to [m \to sm]$$

is injective, one has a map

$$\Gamma_0 : I^l_B(S) \to \mathrm{End}_{A\text{-cor}}(\Sigma)$$

of sets defining $\Gamma^l_0(I)$, $I \in I^l_B(S)$, to be the endomorphism

$$m^* \otimes_B m \to \sum_i m^* x_i \otimes_B y_i m,$$

where $(\mathbf{m}^l_I)^{-1}(1) = \sum_i x_i \otimes_B y_i \in I^l_B(S)$.

Theorem 3.11. *Suppose that ${}_BM_A$ is such that the functor*

$$S \otimes_B - : {}_B\mathrm{Mod} \to {}_S\mathrm{Mod}$$

is comonadic. Then the map

$$\Gamma_0 : I^l_B(S) \to \mathrm{End}_{A\text{-cor}}(\Sigma)$$

is in fact an isomorphism of monoids.

Proof. First of all, the morphism $i : B \to S$ is injective (or equivalently, the bimodule ${}_BM_A$ is faithful), since the functor $S \otimes_B -$ is assumed to be comonadic. Next, it is proved in [3] that the assignment

$$g \to \hat{g} = (\xi \otimes_B \xi) \circ (M \otimes_A g \otimes_A M^*) \circ \left(\xi^{-1} \otimes_B \xi^{-1}\right),$$

where $\xi : M \otimes_A M^* \to S = \mathrm{End}_A(M)$ is the canonical isomorphism, yields an injective morphism of monoids

$$\widehat{(-)} : \mathrm{End}_{A\text{-cor}}(\Sigma) \to \mathrm{End}_{B\text{-cor}}(S \otimes_B S).$$

And the same argument as in the proof of Proposition 2.6 of [3] shows that the following diagram of *sets*

$$\begin{array}{ccc} I_B^l(S) & \xrightarrow{\ \Gamma_0\ } & \mathrm{End}_{A\text{-cor}}(\Sigma) \\ \Big\downarrow \Gamma & \swarrow \widehat{(-)} & \\ \mathrm{End}_{B\text{-cor}}(S \otimes_B S) & & \end{array}$$

is commutative. Now, since the functor $S \otimes_B -$ is assumed to be comonadic, it follows from Theorem 3.5 that Γ is an isomorphism of monoids and hence the monoid morphism $\widehat{(-)}$, being injective, is also an isomorphism. Commutativity of the diagram then gives that Γ_0 is an isomorphism of monoids. □

Dually, one can define a map

$$\Gamma_0' : I_B^r(S) \to \mathrm{End}_{A\text{-cor}}(\Sigma)$$

that sends $I \in I_B^r(S)$ to the endomorphism

$$m^* \otimes_B m \to \sum_i m^* y_i \otimes_B x_i m$$

of the A-coring $\mathrm{End}_{A\text{-cor}}(\Sigma)$, where $(\mathbf{m}_I^r)^{-1}(1) = \sum_i y_i \otimes_b x_i \in I \otimes_B S$.

Theorem 3.12. *Suppose that ${}_B M_A$ is such that the functor*

$$- \otimes_B S : \mathrm{Mod}_B \to \mathrm{Mod}_S$$

is comonadic. Then

$$\Gamma_0' : I_B^r(S) \to \mathrm{End}_{A\text{-cor}}(\Sigma)$$

is an anti-isomorphism of monoids.

It is not hard to check that the map

$$\Gamma_0 : I_B^l(S) \to \mathrm{End}_{A\text{-cor}}(\Sigma)$$

of sets restricts to a map

$$\mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$$

which we still call Γ_0. As in [3], it follows from Theorems 3.11 and 3.12 that

Theorem 3.13. *If either*

(i) *the functor* $S \otimes_B -$, *or*
(ii) *the functor* $- \otimes_B S$

is comonadic, then the map

$$\Gamma_0 : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$$

is actually an isomorphism of groups.

It is shown in [7] that the functor $- \otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ (respectively $M^* \otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$) is comonadic iff the functor $- \otimes_B S : {}_B\mathrm{Mod} \to_S \mathrm{Mod}$ (respectively $S \otimes_B - : \mathrm{Mod}_B \to \mathrm{Mod}_S$) is. So we have:

Theorem 3.14. *If either*

(i) *the functor* $- \otimes_B M$, *or*
(ii) *the functor* $M^* \otimes_B -$

is comonadic, then the map

$$\Gamma_0 : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$$

is an isomorphism of groups.

From the last theorem one obtains the following result of L. El Kaoutit and J. Gómez-Torrecillas (see Theorem 2.5 in [3]):

Theorem 3.15. *If*

(i) ${}_BM$ *is faithfully flat, or*
(ii) M^*_B *is faithfully flat, or*
(iii) ${}_BM_A$ *is a separable bimodule,*

then

$$\Gamma_0 : \mathrm{Inv}_B(S) \to \mathrm{Aut}_{A\text{-cor}}(\Sigma)$$

is an isomorphism of groups.

Proof. We have seen in the introduction that under any of the conditions (i)–(iii), at least one of the extension-of-scalars functor associated to the ring extension $B \to S = \mathrm{End}_A(M)$ is comonadic, and applying Theorem 3.13 gives the desired result. □

References

[1] M. Barr, C. Wells, Toposes, Triples, and Theories, Grundlehren Math. Wiss., vol. 278, Springer-Verlag, 1985.
[2] T. Brzezinski, R. Wisbauer, Corings and Comodules, London Math. Soc. Lecture Note Ser., vol. 309, Cambridge University Press, Cambridge, 2003.
[3] L. El Kaoutit, J. Gómez-Torrecillas, Comatrix corings and invertible bimodules, Ann. Univ. Ferrara Ser. VII 51 (2005) 263–280.
[4] L. El Kaoutit, J. Gómez-Torrecillas, Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings, Math. Z. 244 (2003) 887–906.
[5] G. Janelidze, W. Tholen, Facets of descent, III: Monadic descent for rings and algebras, Appl. Categ. Structures 12 (2004) 461–476.
[6] A. Masuoka, Corings and invertible bimodules, Tsukuba J. Math. 13 (1989) 353–362.
[7] B. Mesablishvili, Monads of effective descent type and comonadicity, Theory Appl. Categ. 16 (2006) 1–45.
[8] K. Sugano, Note on separability of endomorphism rings, Hokkaido Math. J. 11 (1982) 111–115.