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ON DESCENT COHOMOLOGY

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Abstract. The zeroth and first descent cohomology sets for a (co)monad on arbitrary base category with coefficients in a (co)algebra are introduced and their basic properties are studied. These sets generalize those for a coring with coefficients in a comodule. It is shown that under this generalization, essential properties and relationships are preserved.

1. Introduction

The aim of this paper is to introduce and investigate low-dimensional descent cohomology sets of comonads with coefficients in coalgebras over the comonad. These include not only the non-abelian descent cohomology for Hopf modules with coefficients in comodule algebras in the sense of P. Nuss and M. Wambst ([23], [24]), and hence the classical non-abelian group cohomology of Serre [25], but also their generalization by T. Brzeziński to comodules over corings ([4]).

Recall that P. Nuss and M. Wambst in [23] and [24] introduced the zeroth and first descent cohomology sets for Hopf modules with coefficients in comodule algebras, and showed that

- their cohomology generalizes the non-abelian group cohomology of Serre [25], and
- the first descent cohomology pointed set classifies twisted forms of Hopf modules and Hopf torsors.

Based on the descent theory for corings (see, [6], [9]) and the fact that an arbitrary Hopf module can be considered as a special case of an entwining module and hence a comodule over an appropriate coring, T. Brzeziński [4] gave a coring approach to the descent cohomology theory. In particular, he introduced the zeroth and first descent cohomology pointed sets for a coring with values in a comodule over the coring and showed that the first descent cohomology pointed sets still classify twisted forms and suitably defined torsors. Brzeziński's definition of these sets involves a k-algebra A (k being a commutative ring with unit), an A-coring $\mathcal{C}$ and a (right) $\mathcal{C}$-comodule (M, ϱ). Given these data, the *zeroth descent cohomology set of* $\mathcal{C}$ *with coefficients in* (M, ϱ) (which is in fact a group) is defined as the group of $\mathcal{C}$-comodule automorphisms of (M, ϱ), while the *first descent cohomology set of* $\mathcal{C}$ *with coefficients in* M as the set of equivalence classes of $\mathcal{C}$-comodule structures on M, where two $\mathcal{C}$-comodule structures are equivalent if they are isomorphic as $\mathcal{C}$-comodules. Since an A-coring can be defined as an A-bimodule $\mathcal{C}$ such that the endofunctor $\mathbb{G}_{\mathcal{C}} = - \otimes_{\mathsf{A}} \mathcal{C}$ on the category of right A-modules $\mathbf{M}_{\mathsf{A}}$ is a comonad, and since right $\mathcal{C}$-comodules are the same as $\mathbb{G}_{\mathcal{C}}$-coalgebras, the concepts of the zeroth and first descent cohomology sets of an A-coring $\mathcal{C}$ with coefficients in a $\mathcal{C}$-comodule can obviously be formulated in pure categorical terms. One may then ask whether the results of [23], [24] and [4] are valid in other categories than the category of (co)modules over a (co)ring. The motivation and main purpose of this paper is to show that this is indeed the case. We demonstrate in particular that several aspects of descent cohomology sets for corings can be generalized in the context of (co)monads on general categories in such a way that their essential properties and relationships with (appropriately generalized to this context) twisted forms and torsors are maintained. We should point out that our method of obtaining this generalization do not use sophisticated machinery of (co)ring and (co)module theory (which is not applicable to our situation due to the great generality of the context we are working in). Our proofs are, in fact, based only on two elementary results concerning pseudo-pullbacks. The first result states that pseudo-pullbacks preserve equivalences of categories, while the second one states that the comparison functor from the pullback to the pseudo-pullback of

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any functor along a functor that lifts isomorphism uniquely, is an equivalence of categories. For the convenience of the reader, we have recalled these results in Section 2.

The outline of this paper is as follows. After recalling in Section 2 some notions and aspects of the theory of (co)monads and (pseudo-)pullbacks, we obtain some categorical results that will be needed for proving our results in the next sections.

In Section 3, we introduce the zeroth and first descent cohomology (pointed) sets of a comonad with coefficients in a coalgebra and study their elementary properties. We close the section by giving two examples of calculating these pointed sets for some comonads.

In Section 4, we introduce twisted forms of an object w.r.t. functors and show how to describe the first descent cohomology sets using them.

Section 5 is concerned with the description of the first cohomology pointed set in terms of subobjects of a certain object.

In Section 6, the first cohomology sets are related with isomorphism classes of suitable defined torsors.

Finally, in the last section, we formally dualize the notions of descent cohomology sets of comonads and define descent cohomology sets of monads. As an application, we calculate descent cohomology sets for some monads.

We refer to S. MacLane [15] and F. Borceux [2,3] for terminology and general results on (co)monads and on (pseudo-)pullbacks, and to T. Brzezinski and R. Wisbauer [5] for coring and comodule theory.

2. Preliminaries

This section introduces the categorical preliminaries necessary for the other sections.

2.1. Monads and comonads. We write $\eta, \varepsilon \colon F \dashv U \colon \mathscr{B} \to \mathscr{A}$ to denote that $F : \mathscr{A} \to \mathscr{B}$ and $U : \mathscr{B} \to \mathscr{A}$ are the functors, where F is left adjoint to U with unit $\eta : 1 \to UF$ and counit $\varepsilon : FU \to 1$.

For a monad $\mathbb{T} = (T, m, e)$ on a category $\mathscr{A}$, we write

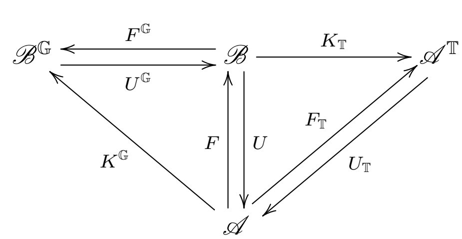
- $\mathscr{A}_{\mathbb{T}}$ for the Eilenberg-Moore category of $\mathbb{T}$-algebras;
- $U_{\mathbb{T}} : \mathscr{A}_{\mathbb{T}} \to \mathscr{A},\ (a, h) \to a$, for the forgetful functor;
- $F_{\mathbb{T}} : \mathscr{A} \to \mathscr{A}_{\mathbb{T}},\ a \to (T(a), m_a)$, for the free $\mathbb{T}$-algebra functor, and
- $\eta_{\mathbb{T}}, \varepsilon_{\mathbb{T}} : F_{\mathbb{T}} \dashv U_{\mathbb{T}} : \mathscr{A}_{\mathbb{T}} \to \mathscr{A}$ for the corresponding forgetful-free adjunction, in which $\eta_{\mathbb{T}} = \eta$ and $(\varepsilon_{\mathbb{T}})_{(a,h)} = h$ for each $\mathbb{T}$-algebra (a, h).

Dually, for a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on $\mathscr{A}$, we write

- $\mathscr{A}^{\mathbb{G}}$ for the category of the Eilenberg-Moore category of $\mathbb{G}$-coalgebras;
- $U^{\mathbb{G}} : \mathscr{A}^{\mathbb{G}} \to \mathscr{A},\ (a, \theta) \to a$, for the forgetful functor;
- $F^{\mathbb{G}} : \mathscr{A} \to \mathscr{A}^{\mathbb{G}},\ a \to (G(a), \delta_a)$, for the cofree $\mathbb{G}$-coalgebra functor, and
- $\eta^{\mathbb{G}}, \varepsilon^{\mathbb{G}} : U^{\mathbb{G}} \dashv F^{\mathbb{G}} : \mathscr{A} \to \mathscr{A}^{\mathbb{G}}$ for the forgetful-cofree adjunction, in which $\varepsilon^{\mathbb{G}} = \varepsilon$ and $(\eta^{\mathbb{G}})_{(a,\theta)} = \theta$ for each $\mathbb{G}$-coalgebra (a, θ).

It is well known (e.g., [15]) that any adjunction $\eta, \varepsilon \colon F \dashv U \colon \mathscr{B} \to \mathscr{A}$ generates a monad $\mathbb{T} = (T, m, e)$ on $\mathscr{A}$, where $T = UF$, $m = U\varepsilon F$, $e = \eta$, and a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on $\mathscr{B}$, where $\mathbb{G} = FU,\ \delta = F\eta U$.

Let $\eta, \varepsilon : F \dashv U \colon \mathscr{B} \to \mathscr{A}$ be an adjunction and $\mathbb{T}$ and $\mathbb{G}$ be the associated monad and comonad on $\mathscr{A}$ and $\mathscr{B}$, respectively. Then one has the comparison functors $K_{\mathbb{T}} : \mathscr{B} \to \mathscr{A}_{\mathbb{T}}$ and $K^{\mathbb{G}} : \mathscr{A} \to \mathscr{B}^{\mathbb{G}}$ and a diagram of categories and functors



where the functors $K_{\mathbb{T}} : \mathscr{B} \to \mathscr{A}_{\mathbb{T}}$ and $K^{\mathbb{G}} : \mathscr{A} \to \mathscr{B}^{\mathbb{G}}$ are defined by

$$K_{\mathbb{T}}(b) = (U(b), U(\varepsilon_b)) \text{ and } K_{\mathbb{T}}(f) = U(f) : (U(b), U(\varepsilon_b)) \to (U(b'), U(\varepsilon_{b'}))$$

and

$$K^{\mathbb{G}}(a) = (F(a), F(\eta_a)) \text{ and } K^{\mathbb{G}}(g) = F(g) : (F(a), F(\eta_a)) \to (F(a'), F(\eta_{a'})).$$

Thus

$$K_{\mathbb{T}}F = F_{\mathbb{T}}, U_{\mathbb{T}}K_{\mathbb{T}} = U, U^{\mathbb{G}}K^{\mathbb{G}} = F \text{ and } K^{\mathbb{G}}U = F^{\mathbb{G}}.$$

One says that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is a *monadic* (resp. *premonadic*) adjunction if $K_{\mathbb{T}}$ is an equivalence of categories (resp. full and faithful). Dually, one says that the adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is *comonadic* (resp. *precomonadic*) if $K^{\mathbb{G}}$ is an equivalence of categories (resp. full and faithful).

When $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is (co)monadic, we write $L_{\mathbb{T}}$ (resp. $R^{\mathbb{G}}$) for the adjoint inverse of $K_{\mathbb{T}}$ (resp. $K^{\mathbb{G}}$) and write $\underline{\eta} : 1 \to K_{\mathbb{T}}L_{\mathbb{T}}$ and $\underline{\varepsilon} : L_{\mathbb{T}}K_{\mathbb{T}} \to 1$ (resp. $\overline{\eta} : 1 \to R^{\mathbb{G}}K^{\mathbb{G}}$ and $\overline{\varepsilon} : K^{\mathbb{G}}R^{\mathbb{G}} \to 1$) for the unit and counit of the adjunction $L_{\mathbb{T}} \dashv K_{\mathbb{T}}$ (resp. $K^{\mathbb{G}} \dashv R^{\mathbb{G}}$).

2.2. Pullbacks and pseudo-pullbacks. We begin with recalling (for example, from [15]) that the *comma category* $(F_1 \downarrow F_2)$ of the functors

$$F_1 : \mathscr{A}_1 \to \mathscr{A} \text{ and } F_2 : \mathscr{A}_2 \to \mathscr{A},$$

is the category whose objects are the triples (a_1, f, a_2), where a_1 is an object of $\mathscr{A}_1$, a_2 one of $\mathscr{A}_2$, and $f : F_1(a_1) \to F_2(a_2)$ is a morphism in $\mathscr{A}$, and whose morphisms $(a_1, f, a_2) \to (a_1', f', a_2')$ are pairs (α_1, α_2), where $\alpha_1 : a_1 \to a_1'$ is a morphism in $\mathscr{A}_1$ and $\alpha_2 : a_2 \to a_2'$ is a morphism in $\mathscr{A}_2$ such that the diagram

$$\begin{array}{ccc} F_1(a_1) & \xrightarrow{f} & F_2(a_2) \\ {\scriptstyle F_1(\alpha_1)}\downarrow & & \downarrow{\scriptstyle F_2(\alpha_2)} \\ F_1(a_1') & \xrightarrow[f']{} & F_2(a_2') \end{array}$$

commutes. Composition and identities in $(F_1 \downarrow F_2)$ are inherited from $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$. It follows that a morphism (α_1, α_2) is an isomorphism in $(F_1 \downarrow F_2)$ if and only if the morphisms α_1 and α_2 are both isomorphisms.

The comma category $(F_1 \downarrow F_2)$ is equipped with the obvious projections

$$P_1 : (F_1 \downarrow F_2) \to \mathscr{A}_1, \;\; P_2 : (F_1 \downarrow F_2) \to \mathscr{A}_2$$

and a natural transformation $\omega : F_1P_1 \to F_2P_2$ defined by $\omega_{(a_1,\, f,\, a_2)} = f$. Then the square

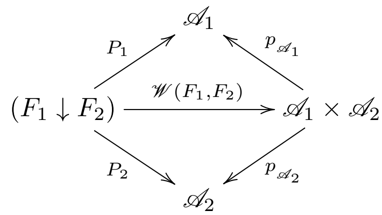
$$\begin{array}{ccc} (F_1 \downarrow F_2) & \xrightarrow{P_2} & \mathscr{A}_2 \\ {\scriptstyle P_1}\downarrow & \overset{\omega}{\Nearrow} & \downarrow{\scriptstyle F_2} \\ \mathscr{A}_1 & \xrightarrow[F_1]{} & \mathscr{A} \end{array}$$

is universal among such squares in the sense that given any other such square

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{Q} & \mathscr{A}_2 \\ {\scriptstyle P}\downarrow & \overset{\varpi}{\Nearrow} & \downarrow{\scriptstyle F_2} \\ \mathscr{A}_1 & \xrightarrow[F_1]{} & \mathscr{A} \end{array}$$

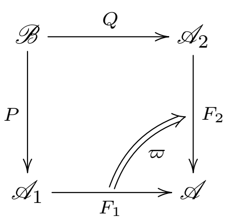
where $\varpi : F_1P \to F_2Q$ is a natural transformation, then there is a unique functor $F : \mathscr{B} \to (F_1 \downarrow F_2)$ such that $P_1F = P$, $P_2F = Q$ and $\omega F = \varpi$.

We write $\mathscr{W}(F_1, F_2)$ for the unique functor $(F_1 \downarrow F_2) \to \mathscr{A}_1 \times \mathscr{A}_2$ making the diagram



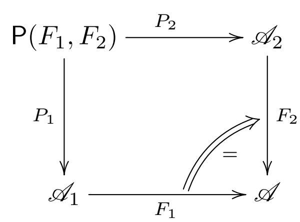
where $p_{\mathscr{A}_1}$ and $p_{\mathscr{A}_2}$ are the projections, commute. Comma categories are also sometimes called *lax pullbacks.*

Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, their *pullback* $\mathsf{P}(F_1, F_2)$ (resp., *pseudo-pullback* $\mathsf{Ps}(F_1, F_2)$) is the full subcategory of $(F_1 \downarrow F_2)$ consisting of those (a_1, f, a_2) for which f is an identity morphism (resp., an isomorphism). The restrictions of P_1 and P_2 on $\mathsf{P}(F_1, F_2)$ (resp., $\mathsf{Ps}(F_1, F_2)$) are denoted again by P_1 and P_2, respectively. Then $\omega : F_1P_1 \to F_2P_2$ is an identity (resp., invertible) natural transformation and $\mathsf{P}(F_1, F_2)$ (resp., $\mathsf{Ps}(F_1, F_2)$) is universal among those diagrams



in which ω is an identity (resp., invertible) natural transformation.

2.3. Comparing pullbacks and pseudo-pullbacks. Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, we write K_{P} for the functor $\mathsf{P}(F_1, F_2) \to \mathsf{Ps}(F_1, F_2)$ induced by the universal property of the pseudo-pullback $\mathsf{Ps}(F_1, F_2)$ and the defining diagram for $\mathsf{P}(F_1, F_2)$:



K_{P} is called the *canonical comparison functor.* It takes (a_1, a_2) to $(a_1, 1, a_2)$, where 1 is the identity morphism of $F_1(a_1) = F_2(a_2)$ in $\mathscr{A}$, and takes (α_1, α_2) to (α_1, α_2).

While clearly fully faithful, K_{P} need not be an equivalence of categories, in general. The following proposition provides a sufficient condition for K_{P} to be an equivalence. Recall (for example, from [1]) that a functor $F : \mathscr{B} \to \mathscr{A}$ *lifts isomorphisms uniquely* if for any isomorphism $f : F(b) \to a$ in $\mathscr{A}$, there exists a unique isomorphism $g : b \to b'$ in $\mathscr{B}$ such that $F(b') = a$ and $F(g) = f$.

2.4. Proposition. ([12]) *Given functors $F_1 : \mathscr{A}_1 \to \mathscr{A}$ and $F_2 : \mathscr{A}_2 \to \mathscr{A}$, the comparison fun ctor*

$$\mathrm{K}_{\mathsf{P}} : \mathsf{P}(F_1, F_2) \to \mathsf{Ps}(F_1, F_2)$$

is an equivalence of categories provided F_1 lifts isomorphism uniquely. When this is the case, an adjoint inverse for K_{P} is the functor sending (a_1, f, a_2) to (a'_1, a_2), where a'_1 is the unique object of $\mathscr{A}_1$ for which there is an isomorphism $g : a_1 \to a'_1$ in $\mathscr{A}_1$ with $F_1(g) = f$.

Note that $(g, 1)$ is an isomorphism from (a_1, f, a_2) to $\mathrm{K}_{\mathsf{P}}(a'_1, a_2) = (a'_1, 1, a_2)$.

Recall (for example, from [1]) that if $\mathbb{G}$ is a comonad on a category $\mathscr{A}$, then the forgetful functor $U^{\mathbb{G}} : \mathscr{A}^{\mathbb{G}} \to \mathscr{A}$ lifts isomorphisms as follows. If $(a, \theta) \in \mathscr{A}^{\mathbb{G}}$ is such that there exists an isomorphism $f : U^{\mathbb{G}}(a, \theta) = a \to a'$ in $\mathscr{A}$, then the pair (a', θ'), where θ' is the composite

$$a' \xrightarrow{f^{-1}} a \xrightarrow{\theta} G(a) \xrightarrow{G(f)} G(a'),$$

is a $\mathbb{G}$-coalgebra and f is an isomorphism from (a, θ) to (a', θ'). Quite obviously, $U^{\mathbb{G}}(f : (a, \theta) \to (a', \theta')) = f$. Therefore, as a special case of Proposition 2.4, we have

2.5. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{A}$. Then for any functor $F : \mathscr{B} \to \mathscr{A}$, the comparison functor*

$$\mathrm{K}_{\mathsf{P}} : \mathsf{P}(U^{\mathbb{G}}, F) \to \mathsf{Ps}(U^{\mathbb{G}}, F)$$
$$\mathrm{K}_{\mathsf{P}}((a, \theta), b) = ((a, \theta), 1, b), \ \ \mathrm{K}_{\mathsf{P}}(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2)$$

is an equivalence of categories whose adjoint inverse is the functor

$$\overline{\mathrm{K}}_{\mathsf{P}} : \mathsf{Ps}(U^{\mathbb{G}}, F) \to \mathsf{P}(U^{\mathbb{G}}, F)$$
$$\overline{\mathrm{K}}_{\mathsf{P}}((a, \theta), f : U^{\mathbb{G}}(a, \theta) = a \approx F(b), b) = ((F(b), G(f) \cdot \theta \cdot f^{-1}), b).$$

2.6. Functoriality of pseudo-pullbacks. The category $\mathsf{Ps}(F_1, F_2)$ depends functorially both on F_1 and on F_2. The dependence on F_1 is as follows. Given any pair of functors $H : \mathscr{A}_1 \to \mathscr{C}$ and $F : \mathscr{C} \to \mathscr{A}$ and any invertible natural transformation $\omega : FH \approx F_1$, it follows from the universal property of pseudo-pullback that the assignments

$$(a_1, f, a_2) \longmapsto (H(a_1), f \cdot \omega_{a_1}, a_2)$$

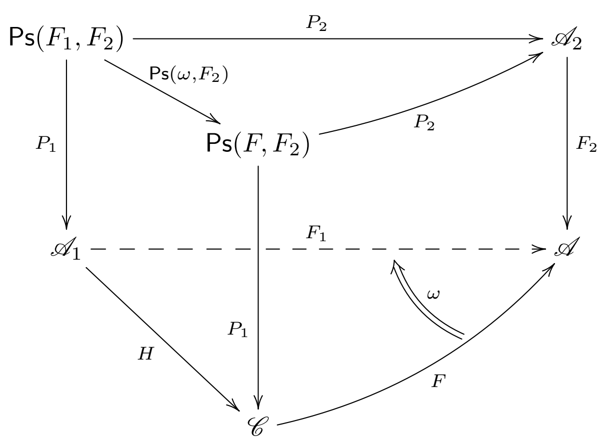
and

$$(\alpha_1, \alpha_2) \longmapsto (H(\alpha_1), \alpha_2)$$

define a functor

$$\mathsf{Ps}(\omega, F_2) : \mathsf{Ps}(F_1, F_2) \to \mathsf{Ps}(F, F_2).$$

The situation may be pictured by the following diagram:



(2.1)

in which all the rectangles and the top triangle commute.

When H is an equivalence of categories, i.e. when there exists a functor $H' : \mathscr{C} \to \mathscr{A}_1$ with natural isomorphisms $\sigma : HH' \approx 1$ and $\varsigma : 1 \approx H'H$, then the composite

$$\omega^* : F_1 H' \xrightarrow{\omega^{-1} H'} FHH' \xrightarrow{F\sigma} F$$

is an isomorphism and the induced functor

$$\mathsf{Ps}(\omega^*, F_2) : \mathsf{Ps}(F, F_2) \to \mathsf{Ps}(F_1, F_2),$$

that takes an object $(b, g, a) \in \mathsf{Ps}(F, F_2)$ to $(H'(b), g \cdot F(\sigma_b) \cdot (\omega_{H'(b)})^{-1}, a)$, is an adjoint inverse of the functor $\mathsf{Ps}(\omega, F_2)$.

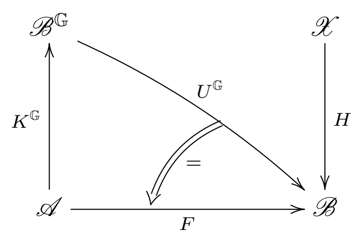
In particular, in the case where $\omega = 1_{F_1}$,

$$\mathsf{Ps}(1_{F_1}, F_2)(a_1, f, a_2) = (H(a_1), f, a_2)$$

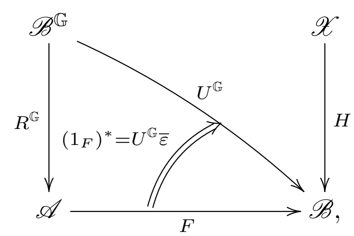
and

$$\mathsf{Ps}((1_{F_1})^*, F_2)(b, g, a) = (H'(b), g \cdot F(\sigma_b), a).$$

Suppose now that the adjunction $\eta, \varepsilon : F \dashv U$ is comonadic. Then $\overline{\eta}, \overline{\varepsilon} : K^{\mathbb{G}} \dashv R^{\mathbb{G}}$ is an adjoint equivalence and considering the diagrams



and



where $H : \mathscr{X} \to \mathscr{B}$ is an arbitrary functor, from the facts above follows

2.7. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and $H : \mathscr{X} \to \mathscr{B}$ be an arbitrary functor. Then the functor*

$$\mathsf{Ps}(1_F, H) : \mathsf{Ps}(F, H) \to \mathsf{Ps}(U^{\mathbb{G}}, H)$$
$$(a, f : F(a) \simeq H(x), x) \longrightarrow (K^{\mathbb{G}}(a), f, x) = ((F(a), F(\eta_a)), f, x)$$

is an equivalence of categories. Its adjoint inverse is the functor

$$\mathsf{Ps}(U^{\mathbb{G}}\overline{\varepsilon}, H) : \mathsf{Ps}(U^{\mathbb{G}}, H) \to \mathsf{Ps}(F, H)$$
$$((b', \theta_{b'}), g : b' \simeq H(x), x) \longrightarrow (R^{\mathbb{G}}(b', \theta_{b'}), g \cdot U^{\mathbb{G}}(\overline{\varepsilon}_{(b', \theta_{b'})}), x).$$

Combining Propositions 2.5 and 2.7, we have

2.8. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction. Then for any functor $H : \mathscr{X} \to \mathscr{B}$, the composite*

$$\mathsf{Ps}(F, H) \xrightarrow{\mathsf{Ps}(1_F, H)} \mathsf{Ps}(U^{\mathbb{G}}, H) \xrightarrow{\overline{K}_{\mathsf{P}}} \mathsf{P}(U^{\mathbb{G}}, H)$$
$$(a, f : F(a) \simeq H(x), x) \to ((H(x), G(f) \cdot F(\eta_a) \cdot f^{-1}), x)$$

is an equivalence of categories. Its adjoint inverse takes $((b, \theta), x) \in \mathsf{P}(U^{\mathbb{G}}, H)$ to $(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), x) \in \mathsf{Ps}(F, H)$.

The following special case will be the most important for us.

Given a category $\mathscr{X}$ and object x of $\mathscr{X}$, we write $\langle x \rangle$ the full subcategory of $\mathscr{X}$ generated by the object x; this means that $\langle x \rangle$ has only one object x and $\langle x \rangle(x, x) = \mathscr{X}(x, x)$. The canonical inclusion $\langle x \rangle \to \mathscr{X}$ will be denoted by $\iota_{\langle x \rangle}$.

Fixing now an object $b \in \mathscr{B}$ and applying Proposition 2.8 to the case where $H = \iota_{\langle b \rangle}$, we obtain

2.9. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and $b \in \mathscr{B}$. Then the functor*

$$P_{\langle b \rangle} : \mathsf{Ps}(F, \iota_{\langle b \rangle}) \to \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$$
$$(a, f, b) \to (b, G(f) \cdot F(\eta_a) \cdot f^{-1})$$

is an equivalence of categories. Its adjoint inverse takes $(b, \theta) \in \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$ to $(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), b) \in \mathsf{Ps}(F, \iota_{\langle b \rangle})$.

For a category $\mathscr{X}$, we write $\pi_0(\mathscr{X})$ for the collection of the isomorphism classes of objects of $\mathscr{X}$. For any $x \in \mathscr{X}$, $[x]$ denotes the class of x. Clearly, for any functor $H : \mathscr{X} \to \mathscr{Y}$, the assignment $[x] \to [H(x)]$ yields a map $\pi_0(S) : \pi_0(\mathscr{X}) \to \pi_0(\mathscr{Y})$.

Proposition 2.9 at once yields the following

2.10. Corollary. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction and let b be an arbitrary fixed object of $\mathscr{B}$. Then the map*

$$\pi_0(P_{\langle b\rangle}) : \pi_0(\mathsf{Ps}(F, \iota_{\langle b\rangle})) \to \pi_0(\mathsf{P}(U^{\mathbb{G}}, i_{\langle b\rangle}))$$
$$[(a, f, b)] \to [(b, G(f) \cdot F(\eta_a) \cdot f^{-1})]$$

is a bijection with inverse

$$[(b, \theta] \longrightarrow [(R^{\mathbb{G}}(b, \theta), U^{\mathbb{G}}(\overline{\varepsilon}_{(b,\theta)}), b)].$$

3. Descent Cohomology Sets of Comonads

In this section, we introduce the zeroth and first descent cohomology (pointed) sets of a comonad and study their elementary properties. As is mentioned in the introduction, these sets should generalize those for a coring with values in a comodule introduced in [4]. To achieve this, we first recall the definitions from [4] and then present them in a form which makes it quite obvious how to transform them to general categories.

Let A be an algebra over a commutative base ring k, $\mathcal{C}$ be an A-coring. Given a right $\mathcal{C}$-comodule (M, ϱ), the *first descent cohomology set of $\mathcal{C}$ with coefficients in M* is defined as the set of equivalence classes of $\mathcal{C}$-comodule structures on M, where two $\mathcal{C}$-comodule structures are equivalent if they are isomorphic as $\mathcal{C}$-comodules. Since M comes already equipped with the right coaction ϱ, the set of equivalence classes of $\mathcal{C}$-comodule structures on M is a pointed set, with the distinguished point given by the equivalence class of (M, ϱ). The *zeroth descent cohomology group of $\mathcal{C}$ with coefficients in* (M, ϱ) is defined as the group $\mathsf{Aut}_{\mathbf{M}^{\mathcal{C}}}(M, \varrho)$ of $\mathcal{C}$-comodule automorphisms of (M, ϱ).

Since the assignment

$$(\mathcal{C}, \delta, \varepsilon) \longmapsto \mathbb{G}_{\mathcal{C}} = (- \otimes_{\mathsf{A}} \mathcal{C}, - \otimes_{\mathsf{A}} \delta, - \otimes_{\mathsf{A}} \varepsilon)$$

yields a bijective correspondence between A-corings and colimit preserving comonads on $\mathbf{M}_{\mathsf{A}}$ (e.g., [5]), and since the category $\mathbf{M}^{\mathcal{C}}$ of (right) $\mathcal{C}$-comodules can be identified with the Eilenberg-Moore category $\mathbf{M}^{\mathbb{G}_{\mathcal{C}}}$ of $\mathbb{G}_{\mathcal{C}}$-colagebras, it is easy to see that the basic structures of [4] can be defined for arbitrary categories $\mathscr{B}$, replacing $\mathbf{M}_{\mathsf{A}}$, and any comonad $\mathbb{G} : \mathscr{A} \to \mathscr{A}$, replacing $- \otimes_{\mathsf{A}} \mathcal{C} : \mathbf{M}_{\mathsf{A}} \to \mathbf{M}_{\mathsf{A}}$, and any $\mathbb{G}$-coalgebra (b, θ), replacing (M, ϱ). This leads to the following definitions.

3.1. Descent cohomology sets of comonads. Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $b \in \mathscr{B}$, the *first descent cohomology set of* $\mathbb{G}$ *with values in* b, denoted $\mathsf{Desc}^1(\mathbb{G}, b)$, is the set of equivalence classes of $\mathbb{G}$-coalgebra structures $\theta : b \to G(b)$ on b, where two $\mathbb{G}$-coalgebra structures $\theta_1 : b \to G(b)$ and $\theta_2 : b \to G(b)$ are equivalent if they are isomorphic as the objects of the category $\mathscr{B}^{\mathbb{G}}$, i.e., there exists an isomorphism $f : b \to b$ in $\mathscr{B}$ making the diagram

$$\begin{array}{ccc} b & \xrightarrow{\theta_1} & G(b) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)} \\ b & \xrightarrow[\theta_2]{} & G(b) \end{array}$$

commute. When b already carries a $\mathbb{G}$-coalgebra structure $\theta : b \to G(b)$, this structure makes $\mathsf{Desc}^1(\mathbb{G}, b)$ a pointed set, with the distinguished point given by the equivalence class of (b, θ). We shall indicate this by writing $\mathsf{Desc}^1(\mathbb{G}, (b, \theta))$ rather than $\mathsf{Desc}^1(\mathbb{G}, b)$. Moreover, in this special case, the 0-*descent cohomology group* $\mathsf{Desc}^0(\mathbb{G}, (b, \theta))$ *of* $\mathbb{G}$ *with coefficients in* (b, θ) can also be defined as the group of all automorphisms of (b, θ) in $\mathscr{B}^{\mathbb{G}}$:

$$\mathsf{Desc}^0(\mathbb{G}, (b, \theta)) := \mathsf{Aut}_{\mathscr{B}^{\mathbb{G}}}(b, \theta).$$

Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $b \in \mathscr{B}$, we write $\mathrm{Desc}\,(\mathbb{G}, b)$ for the category whose objects are the $\mathbb{G}$-coalgebras with underlying object b, and whose morphisms are those of $\mathscr{A}^{\mathbb{G}}$. It is easy to see that $\pi_0(\mathrm{Desc}\,(\mathbb{G}, b)) = \mathsf{Desc}^1(\mathbb{G}, b)$.

3.2. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $b \in \mathscr{B}$. Then*

$$\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b\rangle}) = \mathrm{Desc}\,(\mathbb{G}, b).$$

Proof. The objects of both categories are precisely those $\mathbb{G}$-coalgebras whose underlying object is b. A morphism from (b,θ) to (b,θ') in $\mathsf{P}(U_{\mathbb{G}},\iota_{\langle b\rangle})$ is a pair (α,β), where $\beta : b \to b$ is a morphism in $\mathscr{B}$ and $\alpha : (b,\theta) \to (b,\theta)$ is a morphism in $\mathscr{B}^{\mathbb{G}}$ such that $U^{\mathbb{G}}(\alpha) = \iota_{\langle a\rangle}(\beta)$. But since $U^{\mathbb{G}}(\alpha) = \alpha$ and $\iota_{\langle b\rangle}(\beta) = \beta$, it follows that the morphisms in $\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle})$ is just the morphism in $\mathscr{B}^{\mathbb{G}}$. Thus $\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle}) = \textsc{Desc}\,(\mathbb{G},b)$. □

Since $\pi_0(\textsc{Desc}\,(\mathbb{G},b)) = \mathsf{Desc}^1(\mathbb{G},b)$, we have immediately the following result.

3.3. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $b \in \mathscr{B}$. Then*

$$\pi_0(\mathsf{P}(U^{\mathbb{G}},\iota_{\langle b\rangle})) = \mathsf{Desc}^1(\mathbb{G},b).$$

We give now two examples of calculating Desc^0 and Desc^1 for some comonads.

3.4. Example. Recall (for example, from [3]) that a monad $\mathbb{G} = (G,\delta,\varepsilon)$ on a category $\mathscr{B}$ is said to be *idempotent* if it satisfies one (hence all) of the following equivalent conditions:

- (i) $\delta : G \to GG$ is a natural isomorphism;
- (ii) $G\varepsilon$ (or εG) is an isomorphism;
- (iii) the structure morphism of every object in $\mathscr{B}^{\mathbb{G}}$ is an isomorphism; i.e., for every $\mathbb{G}$-coalgebra (b,θ), the $\mathbb{G}$-coaction $\theta : b \to G(b)$ is an isomorphism;
- (iv) the forgetful functor $U^{\mathbb{G}} : \mathscr{B}^{\mathbb{G}} \to \mathscr{B}$ is full (and faithful).

3.5. Proposition. *Let $\mathbb{G} = (G,\delta,\varepsilon)$ be an idempotent comonad on a category $\mathscr{B}$ and b be an arbitrary object of $\mathscr{A}$. Then*

$$\mathsf{Desc}^1(\mathbb{G},b) = \begin{cases} \{[(b,\varepsilon_b^{-1})]\}, & \textit{if } \varepsilon_b \textit{ is an isomorphism;} \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Moreover, if ε_b is an isomorphism (and hence $(b,\varepsilon_b^{-1}) \in \mathscr{B}^{\mathbb{G}}$), one has

$$\mathsf{Desc}^0(\mathbb{G},(b,\varepsilon_a^{-1})) = \mathsf{Aut}_{\mathscr{B}}(b)$$

Proof. Since the comonad $\mathbb{G} = (G,\delta,\varepsilon)$ is assumed to be idempotent, the following are equivalent for an arbitrary object b of $\mathscr{B}$ (e.g., [7, Lemma 2.8]):

1. b carries a $\mathbb{G}$-coalgebra structure.
2. $\varepsilon_b : G(b) \to b$ is a split epimorphism.
3. $\varepsilon_b : G(b) \to b$ is an isomorphism.

Accordingly, there is at most one $\mathbb{G}$-algebra structure on b. It then follows that the objects of $\mathscr{B}$ admitting a $\mathbb{G}$-coalgebra structure are precisely those objects b for which the morphism $\varepsilon_b : G(b) \to b$ is an isomorphism; the $\mathbb{G}$-coalgebra structure on such an object b is then unique, being $\varepsilon_b^{-1} : b \to G(b)$. This proves the first part of the proposition.

As, by the very definition, $\mathsf{Desc}^0(\mathbb{G},(b,\varepsilon_b^{-1})) = \mathsf{Aut}_{\mathscr{B}^{\mathbb{G}}}(b,\varepsilon_b^{-1})$, the second part of the proposition follows at once from the fact that the forgetful functor $U^{\mathbb{G}} : \mathscr{B}^{\mathbb{G}} \to \mathbb{G}$ is full and faithful. □

3.6. Example. Write $\mathbf{1}$ for the category with just one object $*$ and with $\mathbf{1}(*,*) = 1_*$. Then an arbitrary functor $H : \mathbf{1} \to \mathscr{A}$ simply consists in the choice of an object $a := H(*)$ in $\mathscr{A}$. If $a \in \mathscr{A}$, we write $\ulcorner a \urcorner : \mathbf{1} \to \mathscr{A}$ for the corresponding functor with value a.

Let us fix now a category $\mathscr{A}$ and its object $a \in \mathscr{A}$ and consider the comma category $1_{\mathscr{A}} \downarrow \ulcorner a \urcorner$, which is usually denoted by $\mathscr{A} \downarrow a$. Recall from 2.2 that the objects of this category are the pairs (x,f), where $x \in \mathscr{A}$ and f is a morphism $x \to a$ in $\mathscr{A}$, and the morphisms between two objects $(x, f : x \to a)$ and $(y, g : y \to y)$ are morphisms $h : x \to y$ in $\mathscr{A}$ such that $gh = f$.

Now let $\mathscr{A}$ admit pullbacks. Given morphisms $f : x \to a$, $g : x \to b$, $p : a \to c$ and $q : b \to c$ in $\mathscr{A}$ with $pf = qg$, we write $\langle f, g\rangle : x \to a \times_c b$ for the unique morphism making the diagram

$$\begin{array}{ccc} x & \xrightarrow{\quad g \quad} & b \\ & \searrow^{\langle f,g\rangle} \quad \nearrow_{\pi_2} & \\ f\big\downarrow & a \times_c b & \big\downarrow q \\ & \swarrow_{\pi_1} & \\ a & \xrightarrow[\quad p \quad]{} & c \end{array}$$

commute. Note that for any morphism $k : y \to x$ in $\mathscr{A}$, one has

$$\langle f \cdot k, g \cdot k\rangle = \langle f, g\rangle \cdot k. \tag{3.1}$$

Moreover, if $p' : a' \to c$ and $q' : b' \to c$ are arbitrary morphisms, while $k_1 : a \to a'$ and $k_2 : b \to b'$ are morphisms with $p = p' \cdot k_1$ and $q = q' \cdot k_2$, then

$$(k_1 \times_C k_2) \cdot \langle f, g\rangle = \langle k_1 \cdot f, k_1 \cdot g\rangle. \tag{3.2}$$

Now let $p : e \to b$ be a fixed morphism in $\mathscr{A}$. Then the *change-of-base functor* $p^* : \mathscr{A}/b \to \mathscr{A}/e$ assigns to an object $(x, f : x \to b)$ of $\mathscr{A}/b$ the object $(e \times_b x, \pi_1 : e \times_b x \to e)$ of $\mathscr{A}/e$. It is well-known that p^* has a left adjoint $p_! : \mathscr{A}/e \to \mathscr{A}/b$ given by composition with p. The components of the unit η of this adjunction are given as

$$\eta_{(y,\, g\colon y \to e)} = \langle g, 1_g\rangle : (y, g) \to (e \times_b y, \pi_1)$$

and the components of the counit ε as

$$\varepsilon_{(x,\, f\colon x \to b)} = \pi_2 : (e \times_b x, p \cdot \pi_1) \to (x, f).$$

Let $\mathbb{G}_p$ be the comonad on $\mathscr{A}/b$ generated by the adjunction $p_! \dashv p^*$. Then the functor-part of $\mathbb{G}_p$ is the composite $p_! p^*$, so $\mathbb{G}_p$ takes $(x,\ f : x \to b)$ to the object $(e \times_b x, p \cdot \pi_1)$. The counit $p_! p^* \to 1$ of $\mathbb{G}_p$ is ε, while the comultiplication $\delta : p_! p^* \to p_! p^* p_! p^*$ is defined by

$$\delta_{(x,\, f\colon x \to b)} = p_!(\eta_{p^*(x,\, f)}) = \langle \pi_1, 1_{e \times_b x}\rangle : e \times_b x \to e \times_b e \times_b x.$$

A $\mathbb{G}_p$-coalgebra structure on $(x, f : x \to b) \in \mathscr{A}/b$ is a morphism $\theta : x \to e \times_b x$ in $\mathscr{A}$ making the diagrams

$$\begin{array}{ccc} x & \xrightarrow{\theta} & e \times_b x \\ f\big\downarrow & (1) & \big\downarrow p\cdot\pi_1 \\ b & = & b, \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\theta} & e \times_b x \\ 1_x\big\downarrow & (2) & \big\downarrow \pi_2 \\ x & = & x \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & \xrightarrow{\quad\theta\quad} & e \times_b x \\ \theta\big\downarrow & (3) & \big\downarrow \langle \pi_1, 1_{e\times_b x}\rangle \\ e \times_b x & \xrightarrow[e\times_b\theta]{} & e \times_b e \times_b x\,. \end{array} \tag{3.3}$$

commute. Note that the commutativity of diagram (3.3)(1) expresses the fact that θ is a morphism from $(x, f : x \to b)$ to $\mathbb{G}_p(x,\ f :\ x \to b) = (e \times_b x, p \cdot \pi_1)$ in $\mathscr{A}/b$.

One easily concludes from the commutativity of (3.3)(2) that $\theta = \langle \overline{\theta}, 1_X\rangle$, where $\overline{\theta} = \pi_1\theta : x \to e$. Then the commutativity of (3.3)(1) means that $p \cdot \overline{\theta} = f$, while diagram (3.3)(3) can be rewritten as

$$\begin{array}{ccc} x & \xrightarrow{\quad\langle \overline{\theta}, 1_x\rangle\quad} & e \times_b x \\ \langle \overline{\theta}, 1_X\rangle\big\downarrow & & \big\downarrow \langle \pi_1, 1_{e\times_b x}\rangle \\ e \times_b x & \xrightarrow[1_e \times_b \langle \overline{\theta}, 1_x\rangle]{} & e \times_b e \times_b x. \end{array}$$

But since

$$\langle \pi_1,\, 1_{e\times_b x}\rangle \cdot \langle \overline{\theta},\, 1_x\rangle \overset{(3.1)}{=} \langle \pi_1 \cdot \langle \overline{\theta},\, 1_x\rangle, 1_{e\times_b x} \cdot \langle \overline{\theta},\, 1_x\rangle\rangle = \langle \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle\rangle$$

and

$$(1_e \times_b \langle \overline{\theta},\, 1_x\rangle) \cdot \langle \overline{\theta},\, 1_x\rangle \overset{(3.2)}{=} \langle 1_e \cdot \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle \cdot 1_x\rangle = \langle \overline{\theta}, \langle \overline{\theta},\, 1_x\rangle\rangle,$$

it follows that diagram (3.3)(3) is always commutative. Thus, to give a $\mathbb{G}_p$-coalgebra structure on $(x, f : x \to b) \in \mathscr{A}/a$ is to give a morphism $\theta : x \to e$ in $\mathscr{A}$ making the diagram

$$\begin{array}{ccc} x & \xrightarrow{\theta} & e \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ b & = & b \end{array}$$

commute. It is easy then to see that two $\mathbb{G}_p$-coalgebra structures θ and ϑ on $(x, f : x \to b) \in \mathscr{A}/b$ are isomorphic if and only if there is an automorphism $h : f \to f$ in $\mathscr{A}/b$ (i.e., an automorphism $h : x \to x$ in $\mathscr{A}$ with $f = fh$) making the diagram

$$\begin{array}{ccc} x & \xrightarrow{\theta} & e \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_e} \\ x & \xrightarrow[\vartheta]{} & e \end{array}$$

commute. Putting $f = 1_b : b \to b$ and using the fact that there is only one automorphism of 1_b in $\mathscr{A}/b$, namely 1_b, we obtain

$$\mathsf{Desc}^1(\mathbb{G}_p, 1_b : b \to b) = \{\theta : b \to e \text{ in } \mathscr{A} \text{ with } p \cdot \theta = 1_b\}.$$

4. Twisted Forms

In [10] (see also [11]), we introduced the notion of a twisted form w.r.t. any adjunction and proved that when the adjunction is comonadic, these twisted forms can be described in terms of the induced command coactons. In this section, we introduce twisted forms of an object w.r.t. functors and show how to describe the firs descent cohomology sets by using them.

4.1. Twisted forms w.r.t. functors. Let $H : \mathscr{Y} \to \mathscr{X}$ be a fixed, chosen functor. For any object $x \in \mathscr{X}$, let $\textsc{Twist}(H, x)$ be the category whose objects are the pairs (y, f), where $y \in \mathscr{Y}$ and $f : H(y) \to x$ is an isomorphism in $\mathscr{X}$, and whose morphisms between two objects (y, f) and (y', f') are just morphisms $y \to y'$ in $\mathscr{Y}$. An object $(y, f) \in \textsc{Twist}(H, y)$ is called an *H-twisted form of x*. A straightforward calculation shows that the following propositopn is valid.

4.2. Proposition. *In the situation considered above, the assignments*

$$(y, f) \to (y, f, x)$$

and

$$\left(\alpha \colon \begin{smallmatrix} (y,f) \\ \downarrow \\ (y',f') \end{smallmatrix}\right) \longmapsto \left((\alpha, f' \cdot H(\alpha) \cdot f^{-1}) \colon \begin{smallmatrix} (y,f,x) \\ \downarrow \\ (y',f',x) \end{smallmatrix}\right)$$

yield a functor

$$K_x : \textsc{Twist}(H, x) \to \mathsf{Ps}(H, \iota_{\langle x \rangle}),$$

which is an isomorphism of categories and makes the diagram

$$\begin{array}{ccc} \textsc{Twist}(H, x) & \xrightarrow{K_x} & \mathsf{Ps}(H, \iota_{\langle x \rangle}) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle P_1} & \\ & \mathscr{A} & \end{array}$$

where $U : \textsc{Twist}(H, x) \to \mathscr{A}$ is the evident forgetful functor, commute. The inverse K_x^{-1} of K_x takes (y, f, x) to (y, f) and takes $(\beta : y \to y', \gamma : x \to x)$ to $\beta : y \to y'$.

Let $\mathsf{Twist}_H(x)$ be the collection of isomorphic classes of h-twisted forms of $x \in \mathscr{X}$. Thus, $\pi_0(\textsc{Twist}(H, x)) = \mathsf{Twist}_H(x)$. According to the definition of the category $\textsc{Twist}(H, x)$, two H-twisted forms (y, f) and (y', f') of b are isomorphic if there exists an isomorphism $y \simeq y'$ in $\mathscr{Y}$.

When $x = H(y)$ for some $y \in \mathscr{Y}$, then an H-twisted form of $H(y)$ is called an *H-twisted form of y*. In this case we simply write $\mathsf{Twist}_H(y)$ instead of $\mathsf{Twist}_H(H(y))$. $\mathsf{Twist}_H(y)$ is a pointed set, with a distinguished point given by the class of $(y, 1_{H(y)})$.

As an immediate consequence of Proposition 4.2, we have

4.3. Corollary. *Let $H : \mathscr{Y} \to \mathscr{X}$ be a functor and $x \in \mathscr{X}$. Then*

$$\pi_0(K_x) : \mathsf{Twist}_H(x) \to \pi_0(\mathsf{Ps}(H, \iota_{\langle x \rangle}))$$

is a bijection.

The next result shows that twisted forms can be used to classify the descent cohomology.

4.4. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be an adjunction and let $\mathbb{G} = (FU, \varepsilon, F\eta U)$ be the corresponding comonad on the category $\mathscr{B}$. If F is comonadic, then for any $a \in \mathscr{A}$, the assignment*

$$(x, f) \to ((F(a), F(\eta_a)), FU(f) \cdot F(\eta_x) \cdot f^{-1})$$

yields a bijection

$$\omega_a^{\mathbb{G}} : \mathsf{Twist}_F(a) \to \mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a)))$$

of pointed sets [1].

Proof. Since F is assumed to be comonadic, the map

$$\pi_0(P_{\langle F(a) \rangle}) : \pi_0(\mathsf{Ps}(F, \iota_{\langle F(a) \rangle})) \to \pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle F(a) \rangle}))$$

is bijective by Corollary 2.10. So, in the light of Proposition 3.2 and Corollary 4.3, the assignment

$$[(x, f)] \to [(F(a), FU(f) \cdot F(\eta_x) \cdot f^{-1})]$$

yields a bijection from $\mathsf{Twist}_F(a)$ to $\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a)))$. Moreover, it is easy to see that $K_{\langle F(a) \rangle}(a, 1_{F(a)}) = (F(a), F(\eta_a))$. Hence $\pi_0(K_{\langle F(a) \rangle})$ is an isomorphism of pointed sets. This completes the proof. □

Let $\mathscr{Z} \xrightarrow{G} \mathscr{Y} \xrightarrow{H} \mathscr{X}$ be the functors and y be a fixed object of $\mathscr{Y}$. For any G-twisted form (z, f) of y, the pair $(z, H(f))$ is an (HG)-twisted form of $H(y)$. By the very definition of twisted form, (z, f) and (z', f') are equivalent as G-twisted forms of y if and only if $(a, H(f))$ and $(a', H(f'))$ are equivalent as (HG)-twisted forms of $H(y)$. Thus the passage

$$[(z, f)] \to [(a, H(f))]$$

yields a map

$$S_y : \mathsf{Twist}_G(y) \to \mathsf{Twist}_{(HG)}(H(y)).$$

Quite obviously, S_y is injective. Moreover, it is easy to see that when $y = G(z)$ for some $z \in \mathscr{Z}$, then $S_{G(z)}$, which is a map from $\mathsf{Twist}_G(z)(= \mathsf{Twist}_G(G(z)))$ to $(\mathsf{Twist}_{(HG)}(H(G(z)) = \mathsf{Twist}_{(HG)}((HG)(z))$ $=)\mathsf{Twist}_{(HG)}(z)$, is a morphism of pointed sets.

It turns out that in some cases, this map is surjective (and hence bijective). In order to prove this, we need one preliminary result.

4.5. Lemma. *For any adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$, the diagram*

$$FUFUF \underset{\varepsilon FUF}{\overset{FU\varepsilon F}{\rightrightarrows}} FUF \xrightarrow{\varepsilon F} F \tag{4.1}$$

is a split coequalizer, a splitting being given by $F\eta$ and $FUF\eta$.

Proof. $\varepsilon F \cdot F\eta = 1$ by one of the triangular identities for the adjunction $F \dashv U$, and hence $FU\varepsilon F \cdot FUF\eta = 1$; and the remaining splitting condition follows from the naturality of $\varepsilon : FU \to 1$. □

[1] Recall that $(F(a), F(\eta_a))$ is a $\mathbb{G}$-coalgebra.

4.6. Proposition. *Let $\mathbb{T}$ be the monad on a category $\mathscr{A}$ generated by an adjoint pair $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$. For any $a \in \mathscr{A}$, the map*

$$S_{F(a)} : \mathsf{Twist}_F(a) \to \mathsf{Twist}_{F_{\mathbb{T}}}(a),$$

induced by the composition $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{K_{\mathbb{T}}} \mathscr{A}_{\mathbb{T}}$, is an isomorphism of pointed sets.

Proof. Since we have already observed that $S_{F(a)}$ is injective, it suffices to verify that $S_{F(a)}$ is surjective. To this end, consider an arbitrary $F_{\mathbb{T}}$-twisted form

$$(x, f : F_{\mathbb{T}}(x) \to F_{\mathbb{T}}(a))$$

of $F_{\mathbb{T}}(a)$. Since f is an (iso)morphism in $\mathscr{A}_{\mathbb{T}}$, it follows that $f : UF(x) \to UF(a)$ is an (iso)morphism in $\mathscr{A}$ and that the diagram

$$\begin{array}{ccc} UFUF(x) & \xrightarrow{UF(f)} & UFUF(a) \\ {\scriptstyle U(\varepsilon_{F(x)})}\downarrow & & \downarrow{\scriptstyle U(\varepsilon_{F(a)})} \\ UF(x) & \xrightarrow[f]{} & UF(a) \end{array} \tag{4.2}$$

commutes. Then since $F(f) \cdot \varepsilon_{FUF(x)} = \varepsilon_{FUF(a)} \cdot FUF(f)$ by naturality of ε, the diagram

$$\begin{array}{ccccc} FUFUF(x) & \overset{FU(\varepsilon_{F(x)})}{\underset{\varepsilon_{FUF(x)}}{\rightrightarrows}} & FUF(x) & \xrightarrow{\varepsilon_{F(x)}} & F(x) \\ {\scriptstyle FUF(f)}\downarrow & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle f'} \\ FUFUF(a) & \overset{FU(\varepsilon_{F(a)})}{\underset{\varepsilon_{FUF(a)}}{\rightrightarrows}} & FUF(a) & \xrightarrow[\varepsilon_{F(a)}]{} & F(a) \end{array} \tag{4.3}$$

is serially commutative. Since, by Lemma 4.5, each row of this diagram is a (split) coequalizer and since f is an isomorphism, it follows that there exists a unique isomorphism $f' : F(x) \to F(a)$ in $\mathscr{B}$ making the right square in Diagram (4.3) commute. Thus, in particular, $(x, f') \in \mathsf{Twist}_F(a)$ and $U(\varepsilon_{F(a)}) \cdot UF(f) = U(f') \cdot (\varepsilon_{F(x)})$. It then follows – since $U(\varepsilon_{F(x)})$ is a (split) epimorphism – from the commutativity of Diagram (4.2) that $U(f') = f$. Thus $K_{\mathbb{T}}(f) = U(f') = f$, and hence $S_{F(a)}([(x, f')]) = [(x, K_{\mathbb{T}}(f'))] = [(x, f]$. Therefore $S_{F(a)}$ is surjective. □

4.7. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be an adjunction, $\mathbb{T} = (UF, \eta, U\varepsilon F)$ be the monad on $\mathscr{A}$ generated by the adjunction, and $\mathbb{G}$ (resp., $\mathbb{G}_{\mathbb{T}}$) be the comonad on $\mathscr{B}$ (resp., $\mathscr{A}_{\mathbb{T}}$) corresponding to the adjunction $F \dashv U$ (resp. $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$). Suppose that idempotents split in $\mathscr{A}$[2]. If F is comonadic, then for any object $a \in \mathscr{A}$,*

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) = \mathsf{Desc}^1(\mathbb{G}_{\mathbb{T}}, (F_{\mathbb{T}}(a), F_{\mathbb{T}}(\eta_a)))$$

as pointed sets.

Proof. Since idempotents splits in $\mathscr{A}$, to say that the functor F is comonadic is to say that the functor $F_{\mathbb{T}}$ is comonadic (see Theorem 3.20 in [16]).The result now follows from Theorem 4.4 and Proposition 4.6. □

5. Subobjects and Descent Cohomology Sets

We continue supposing that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is an adjunction and that $\mathbb{G}$ is the comoanad on $\mathscr{B}$ generated by the adjunction.

In this section, we study the relationship between the set $\mathsf{Desc}^1(\mathbb{G}, b)$ for a given object $b \in \mathscr{B}$ and the set of some subobjects of $U(b)$. We will show that when the left adjoint functor is F comonadic, the aforementioned relationship takes the form of a bijection.

[2]It is said that idempotents split in $\mathscr{A}$ if whenever $a \in \mathscr{A}$, $e : a \to a$ with $e^2 = e$, then there exist an object $a' \in \mathscr{A}$ and morphisms $p : a \to a'$ and $\iota : a' \to a$ such that $\iota p = e$ and $p\iota = 1_{a'}$

5.1. The category of subobjects associated with adjunctions. Given an object $b \in \mathscr{B}$, write $\textsc{Sub}_F(U(b))$ for the category whose objects are the pairs (a, ι) for which $\iota : a \to U(b)$ is a regular monomorphism[3] in $\mathscr{A}$ and, moreover, its image $\widetilde{\iota} : F(a) \to b$ under the adjunction bijection

$$\alpha_{a,b} : \mathscr{B}(F(a), b) \simeq \mathscr{A}(a, U(b))$$

is an isomorphism.

Clearly, if $(a, \iota) \in \textsc{Sub}_F(U(b))$, then $(a, \widetilde{\iota}) \in \textsc{Twist}(F, b)$. Hence we can define a functor

$$S_b : \textsc{Sub}_F(U(b)) \to \textsc{Twist}(F, b)$$

by $S_b(a, \iota) = (a, \widetilde{\iota})$. The morphism assignment is given by the identity function. In other words, S_b is the identity on morphisms. It can be easily checked that S_b is indeed a functor.

The following result gives a criterion for determining when the functor S_b is an isomorphism of categories.

5.2. Proposition. *In the situation described above, if the adjunction $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is precomonadic, then the functor*

$$S_b : \textsc{Sub}_F(U(b)) \to \textsc{Twist}(F, b)$$

is an isomorphism of categories.

Proof. Suppose that the adjunction $\eta, \varepsilon : F \dashv U$ is precomonadic. Then (see, for example, [13, Theorem 2.4]) η_a is a regular monomorphism for all $a \in \mathscr{A}$.

Now, if $(a, f) \in \textsc{Twist}(F, b)$, then $f : F(a) \to b$ (and hence also $U(f)$) is an isomorphism, and hence $\widetilde{f} = \alpha_{a,b}(f) = U(f) \cdot \eta_a$ is a regular monomorphism. Thus, $(a, \widetilde{f}) \in \textsc{Sub}_F(U(b))$. It follows – since $\widetilde{\widetilde{f}} = f$ – that $S_b(a, \widetilde{f}) = (a, f)$. The result now follows by noting that (as is easily seen) the functor S_b is full and faithful. □

Combining Propositions 4.2 and 5.2, we have

5.3. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a precomonadic adjunction and $b \in \mathscr{B}$. Then the assignments*

$$(a, \iota) \longmapsto (a, \widetilde{\iota}, b)$$

and

$$(\alpha : (a, \iota) \to (a', \iota')) \longmapsto ((\alpha, \iota' \cdot F(f) \cdot \iota^{-1}) : (a, \widetilde{\iota}, b) \to (a', \widetilde{\iota'}, b))$$

yield an isomorphism of categories

$$\textsc{Sub}_F(U(b)) \simeq \mathsf{Ps}(F, \iota_{\langle b \rangle}).$$

Its inverse takes (a, f, b) into $(a, \widetilde{f})$ and $(\alpha, beta)$ into α.

We write $\mathsf{Sub}_{F \dashv U}(b)$ for $\pi_0(\textsc{Sub}_F(U(b)))$. Note that in case $F \dashv U : \mathscr{B} \to \mathscr{A}$ is a precomonadic adjunction and $b = F(a)$ for some $a \in \mathscr{A}$, we find that $\mathsf{Sub}_{F \dashv U}(F(a))$ is a pointed set with a base point of the class $\eta_a : a \to UF(a)$, which is a regular monomorphism because of the precomonadicity of the adjunction (see again [13, Theorem 2.4]).

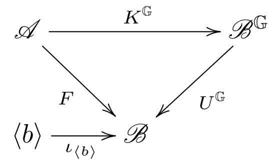
As an immediate consequence of Proposition 5.3, we have

5.4. Proposition. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a precomonadic adjunction and $b \in \mathscr{B}$. Then there is a bijection*

$$\mathbf{Sub}_{F \dashv U}(b) \simeq \pi_0(\mathsf{Ps}(F, \iota_{\langle b \rangle})).$$

[3]Recall that regular monomorphisms are morphisms occurring as equalizers of some pairs of parallel morphisms.

Suppose now that $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ is a comonadic adjunction. Write $\mathbb{G}$ for the induced comonad on $\mathscr{B}$. Then in the commutative diagram



$K^{\mathbb{G}}$ is an equivalence of categories and thus it induces (see 2.6) an equivalence of categories

$$\mathsf{Ps}(F, \iota_{\langle b \rangle}) \simeq \mathsf{Ps}(U_{\mathbb{G}}, \iota_{\langle b \rangle}).$$

In the light of Theorem 2.9, from Propositions 3.2, 5.2 and 5.4 follows

5.5. Theorem. *Let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ be a comonadic adjunction with the corresponding comonad $\mathbb{G}$ on $\mathscr{B}$ and $b \in \mathscr{B}$. Then there are the bijections*

$$\mathsf{Desc}^1(\mathbb{G}, b) \simeq \mathbf{Sub}_{F \dashv U}(b) \simeq \mathsf{Twist}_F(b).$$

Moreover, when $b = F(a)$ for some $a \in \mathscr{A}$, then these bijections are isomorphisms of pointed sets.

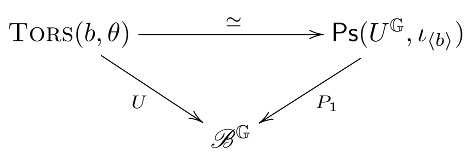
6. Torsors

Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, a (b,θ)-*torsor* is a triple (x, ϑ, α), where $(x, \vartheta) \in \mathscr{B}^{\mathbb{G}}$ and $\alpha : b \to x$ is an isomorphism in $\mathscr{B}$. Morphism between two (b,θ)-torsors (x, ϑ, α) and $(x', \vartheta', \alpha')$ are morphisms from (x, ϑ) to (x', ϑ') in $\mathscr{B}^{\mathbb{G}}$. The (b,θ)-torsors and their morphisms constitute a category $\textsc{Tors}(b, \theta)$. It is easy to see that this category is just the category $\textsc{Twist}(U^{\mathbb{G}}, (b, \theta))$. Therefore, applying Proposition 4.2, we obtain

6.1. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$. Then there is an isomorphism of categories*

$$\textsc{Tors}(b, \theta) \simeq \mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$$

making the diagram



where $U : \textsc{Tors}(b, \theta) \to \mathscr{B}^{\mathbb{G}}$ is the evident forgetful functor, commute.

6.2. The pointed set $\mathsf{Tors}(b, \theta)$. Given a comonad $\mathbb{G}$ on a category $\mathscr{B}$ and an object $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, denote by $\mathsf{Tors}(b, \theta)$ the set of isomorphic classes of (b, θ)-torsors. It is pointed with distinguished point of the class $(b, \theta, 1_b)$. It is clear that $\mathsf{Tors}(b, \theta) = \pi_0(\textsc{Tors}(b, \theta))$. From Proposition 6.1 follows

6.3. Proposition. *Let $\mathbb{G}$ be a comonad on a category $\mathscr{B}$ and $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$. Then*

$$\pi_0(\mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) = \mathsf{Tors}(b, \theta).$$

6.4. Theorem. *For any comonad $\mathbb{G}$ on $\mathscr{B}$ and any $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$, the assignment $(b, \varrho) \longmapsto ((b, \varrho), 1_b)$ yields an isomorphism of pointed sets*

$$\mathsf{Desc}^1(\mathbb{G}, (b, \theta)) \simeq \mathsf{Tors}(b, \theta),$$

whose inverse takes (x, ν, α) to $(b, G(\alpha^{-1}) \cdot \nu \cdot \alpha)$.

Proof. Since

- $\pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) = \mathsf{Desc}^1(\mathbb{G}, (b, \theta))$ by Proposition 3.2, and
- $K_{\mathsf{P}} : \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle}) \to \mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle})$ is an equivalence of categories by Proposition 2.5, and thus $\pi_0(K_{U^{\mathsf{P}}}) : \pi_0(\mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b \rangle})) \to \pi_0(\mathsf{Ps}(U^{\mathbb{G}}, \iota_{\langle b \rangle}))$ is an isomorphism of pointed sets,

the result follows from the previous proposition. $\square$

6.5. Galois comodule functors. Given a functor $F : \mathscr{A} \to \mathscr{B}$ and a comonad $\mathbb{H} = (H, \delta, \sigma)$ on $\mathscr{B}$, F is called a *left* $\mathbb{H}$*-comodule* (e.g., [19, Section 3]) if there exists a natural transformation $\kappa_F : F \to HF$ inducing commutativity of the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\kappa_F} & HF \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \sigma F} \\ & & F, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\kappa_F} & HF \\ {\scriptstyle \kappa_F}\downarrow & & \downarrow{\scriptstyle \delta F} \\ HF & \xrightarrow[H\kappa_F]{} & HHF. \end{array}$$

Suppose now that F has a right adjoint $U : \mathscr{B} \to \mathscr{A}$, with unit $\eta : 1_{\mathscr{A}} \to UF$ and counit $\varepsilon : FU \to 1_{\mathscr{B}}$. Write $\mathbb{G}$ for the comonad on $\mathscr{B}$ generated by this adjunction. Recall (e.g., from [14]) that there exist bijective correspondences between

(i) functors $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ with the commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{K} & \mathscr{B}^{\mathbb{H}} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle U^{\mathbb{H}}} \\ & & \mathscr{B}\,; \end{array} \tag{6.1}$$

(ii) left $\mathbb{H}$-comodule structures $\kappa_F : F \to HF$ on F;

(iii) comonad morphisms $t_K : \mathbb{G} \to \mathbb{H}$.

These bijections are constructed as follows: Given a functor K making Diagram (6.1) commute, then $K(a) = (F(a), \kappa_a)$ for some morphism $\kappa_a : F(a) \to HF(a)$ and the collection $\{\kappa_a,\, a \in \mathscr{A}\}$ constitutes a natural transformation $\kappa_F : F \to HF$ making F a $\mathbb{H}$-comodule. Conversely, if $(F, \kappa_F : F \to HF)$ is a $\mathbb{H}$-module, then $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ is defined by $K(a) = (F(a), (\kappa_F)_a)$. Next, for any (left) $\mathbb{H}$-comodule structure $\kappa_F : F \to HF$, the composite

$$t_K : FU \xrightarrow{\kappa_F U} HFU \xrightarrow{H\varepsilon} H$$

is a comonad morphism from the comonad $\mathbb{G}$ generated by the adjunction $F \dashv U$ to the comonad $\mathbb{H}$. On the other hand, for any comonad morphism $t : \mathbb{G} \to \mathbb{H}$, the composite

$$\kappa_F : F \xrightarrow{F\eta} FUF \xrightarrow{tF} HF$$

defines a left $\mathbb{H}$-comodule structure on F.

A left $\mathbb{H}$-comodule functor F is said to be $\mathbb{H}$-*Galois* provided t_K is an isomorphism (e.g. [18, Definition 1.3]).

For more details on the Galois comodule functors, see, e.g., [14, 17–21].

Now, let $\eta, \varepsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$, an adjunction with the corresponding comonad $\mathbb{G} = (FU, F\eta U, \varepsilon)$, $\mathbb{H} = (H, \delta, \sigma)$, be a comonad on $\mathscr{B}$ and $K : \mathscr{A} \to \mathscr{B}^{\mathbb{H}}$ be a functor making Diagram (6.1) commute.

6.6. Theorem. *In the situation described above, suppose that F is a comonadic functor and that the comonad morphism $t_K : \mathbb{G} \to \mathbb{H}$ induced by the triangle (6.1), is an isomorphism (i.e., F is an $\mathbb{H}$-Galois comodule functor). Then for any object $a \in \mathscr{A}$, there is an isomorphism of pointed sets*

$$\mathsf{Desc}^1(\mathbb{H}, K(a)) \simeq \mathsf{Twist}_F(a).$$

Proof. Since the functor $F : \mathscr{A} \to \mathscr{B}$ is assumed to be comonadic, it follows from Theorem 4.4 that

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) \simeq \mathsf{Twist}_F(a)$$

as pointed sets. Now, if, in addition, t_K is an isomorphism, then the functor $t_K^* : \mathscr{B}^{\mathbb{G}} \to \mathscr{B}^{\mathbb{H}}$ that takes $(b, \theta) \in \mathscr{B}^{\mathbb{G}}$ to $(b, (t_K)_b \cdot \theta) \in \mathscr{B}^{\mathbb{H}}$ is an isomorphism of categories. Moreover, the diagram

$$\begin{array}{ccc} \mathscr{B}^{\mathbb{G}} & \xrightarrow{t_K^*} & \mathscr{B}^{\mathbb{H}} \\ & {\scriptstyle U^{\mathbb{G}}}\searrow & \downarrow{\scriptstyle U^{\mathbb{H}}} \\ & & \mathscr{B}\,; \end{array} \tag{6.2}$$

commutes. Then the induced functor $\mathsf{P}(t_K^*, \iota_{\langle b\rangle}) : \mathsf{P}(U^{\mathbb{G}}, \iota_{\langle b\rangle}) \to \mathsf{P}(U^{\mathbb{H}}, \iota_{\langle b\rangle})$ is an isomorphism for all $b \in \mathscr{B}$. It then follows from Proposition 3.3 that the map

$$\pi_0(\mathsf{P}(t_K^*, \iota_{\langle b\rangle})) : \mathsf{Desc}^1(\mathbb{G}, b)) \simeq \mathsf{Desc}^1(\mathbb{H}, b),$$

taking $[(b, \theta)]$ to $[(b, (t_K)_b \cdot \theta)]$, is bijective for all $b \in \mathscr{B}$. In particular,

$$\pi_0(\mathsf{P}(t_K^*, \iota_{\langle F(a)\rangle})) : \mathsf{Desc}^1(\mathbb{G}, (F(a)) \simeq \mathsf{Desc}^1(\mathbb{H}, F(a))$$

is bijective. But since $t_K^*((F(a), F(\eta_a))) = K(a)$ by [17, Lemma 4.3.], it follows that $\pi_0(\mathsf{P}(t_K^*, \iota_{\langle F(a)\rangle}))$ is, in fact, an isomorphism of the pointed sets

$$\mathsf{Desc}^1(\mathbb{G}, (F(a), F(\eta_a))) \simeq \mathsf{Desc}^1(\mathbb{H}, K(a)).$$

Consequently, the pointed sets $\mathsf{Desc}^1(\mathbb{H}, K(a))$ and $\mathsf{Twist}_F(a)$ are isomorphic. □

6.7. Example. Let K be a commutative ring, A be a K-algebra and $\mathcal{C} = (\mathcal{C}, \Delta, \epsilon)$ be an A-coring. Given a right $\mathcal{C}$-comodule (X, θ), we write $\mathsf{Desc}^1(\mathcal{C}, (X, \theta))$ for the pointed set $\mathsf{Desc}^1(\mathbb{G}_{\mathcal{C}}, (X, \theta))$.

Let (Σ, ν) be a fixed, chosen right $\mathcal{C}$-comodule, and consider the K-algebra $B = \mathbf{M}^{\mathcal{C}}((\Sigma, \nu), (\Sigma, \nu))$. Then Σ has a canonical structure of (B, A)-bimodule and one has the adjunction

$$\mathbf{M}_B \underset{\mathbf{M}_A(\Sigma,-)}{\overset{-\otimes_B \Sigma}{\rightleftarrows}} \mathbf{M}_A$$

where $- \otimes_B \Sigma$ is left adjoint to $\mathbf{M}_A(\Sigma, -)$. Write $\mathbb{G}_\Sigma$ for the comonad on $\mathbf{M}_A$ generated by this adjunction.

The natural transformation

$$\mathrm{Can} : \mathbf{M}_A(\Sigma, -) \otimes_B \Sigma \xrightarrow{\mathbf{M}_A(\Sigma,-)\otimes_B \nu} \mathbf{M}_A(\Sigma, -) \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{\mathrm{ev}\otimes_A \mathcal{C}} - \otimes_A \mathcal{C},$$

where ev is the evaluation map, is a comonad morphism from the comonad $\mathbb{G}_\Sigma$ to the comonad $\mathbb{G}_{\mathcal{C}}$. (Σ, ν) is called a *Galois comodule* provided Σ_A is finitely generated and projective and the natural transformation Can is an isomorphism. For further details regarding the theory of Galois comodules we refer to [26].

Since Σ is a (B, A)–bimodule, $\Sigma^* = \mathbf{M}_A(\Sigma, A)$ is canonically endowed with a structure of (A, B)–bimodule, and $\Sigma^* \otimes_B \Sigma$ is an A–bimodule in a natural way. Assume that Σ_A is finitely generated and projective with a finite dual basis $\{(e_i^*, e_i)\} \subseteq \Sigma^* \otimes_B \Sigma$. Then A–bimodule $\Sigma^* \otimes_B \Sigma$ is an A-coring with comultiplication and counit defined, respectively, by

$$\Delta(f \otimes_B x) = \sum_i (f \otimes_B e_i \otimes_A e_i^* \otimes_B x) \ \text{ and } \ \epsilon(f \otimes_B x) = \mathrm{ev}(f \otimes_B x) = f(x).$$

Then Can may be written as the composite

$$- \otimes_A \Sigma^* \otimes_B \Sigma \xrightarrow{-\otimes_A \Sigma^* \otimes_B \nu} - \otimes_A \Sigma^* \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{-\otimes_A \mathrm{ev} \otimes_A \mathcal{C}} - \otimes_A \mathcal{C}.$$

Thus, Can is an isomorphism if and only if the composite

$$\mathrm{Can}_A : \Sigma^* \otimes_B \Sigma \xrightarrow{\Sigma^* \otimes_B \nu} \Sigma^* \otimes_B \Sigma \otimes_A \mathcal{C} \xrightarrow{\mathrm{ev}\otimes_A \mathcal{C}} \mathcal{C}$$

is an isomorphism.

For any $N \in \mathbf{M}_B$, we write $\mathsf{Twist}_B(\Sigma, N)$ for $\mathsf{Twist}_{-\otimes_B \Sigma}(N)$.

By applying Theorem 6.6 to the present situation, we obtain the following slight generalization of [4, Theorem 2.4].

6.8. Theorem. *Let $\mathcal{C}$ be an A-coring, A being an algebra over a commutative ring K, and let (Σ, ν) be a right $\mathcal{C}$-comodule. Write $B = \mathbf{M}^{\mathcal{C}}(\Sigma, \Sigma)$. If the functor*

$$- \otimes_B \Sigma : \mathbf{M}_B \to \mathbf{M}_A$$

is comonadic and Σ is a Galois $\mathcal{C}$-comodule, then for any right B-module N, there exists an isomorphism of pointed sets

$$\mathsf{Desc}^1(\mathcal{C}, N \otimes_B \Sigma) \simeq \mathsf{Twist}_B(\Sigma, N),$$

where $N \otimes_B \Sigma$ is a right $\mathcal{C}$-comodule with the induced coaction

$$N \otimes_B \nu : N \otimes_B \Sigma \to N \otimes_B \Sigma \otimes_B \mathcal{C}.$$

7. Descent Cohomology Sets of Monads

Dualizing the notions of descent cohomology sets of comonads leads to the descent cohomology sets of monads.

Given a monad $\mathbb{T}$ on a category $\mathscr{A}$ and an object $a \in \mathscr{A}$, the *first descent cohomology set of $\mathbb{T}$ with values in a*, denoted $\mathsf{Desc}^1(\mathbb{T}, a)$, is the set of equivalence classes of $\mathbb{T}$-algebra structures on a, where two $\mathbb{T}$-algebra structures are equivalent if they are isomorphic as the objects of the category $\mathscr{A}_{\mathbb{T}}$.

When a comes equipped with a $\mathbb{T}$-algebra structure $h : T(a) \to a$, $\mathsf{Desc}^1(\mathbb{T}, a)$ becomes a pointed set with a base point the equivalence class of (a, h), and to indicate this fact, we write $\mathsf{Desc}^1(\mathbb{T}, (a, h))$ in place of $\mathsf{Desc}^1(\mathbb{T}, a)$. Moreover, in this special case, the *zeroth descent cohomology group* $\mathsf{Desc}^0(\mathbb{T}, (a, h))$ *of $\mathbb{T}$ with coefficients in* (a, h) is also defined as the group of all automorphisms of (a, h) in $\mathscr{A}_{\mathbb{T}}$. Thus $\mathsf{Desc}^0(\mathbb{T}, (a, h)) = \mathsf{Aut}_{\mathscr{A}_{\mathbb{T}}}(a, h)$.

The rest of this section is devoted to describing descent cohomology sets for some monads.

7.1. example. Idempotent monads. Dualizing Proposition 3.5 gives the following result for the idempotent monads.

proposition. *Let $\mathbb{T} = (T, m, e)$ be an idempotent monad on a category $\mathscr{A}$ and a be an arbitrary object of $\mathscr{A}$. Then*

$$\mathsf{Desc}^1(\mathbb{T}, a) = \begin{cases} \{[(a, e_a^{-1})]\}, & \text{if } \; e_a \text{ is an isomorphism;} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Moreover, if e_a is an isomorphism (and hence $(a, e_a^{-1}) \in \mathscr{A}_{\mathbb{T}}$), one has

$$\mathsf{Desc}^0(\mathbb{T}, (a, e_a^{-1})) = \mathsf{Aut}_{\mathscr{A}}(a).$$

For the next two examples concerning monads on Set, we recall that a *variety of (finitary, one-sorted) algebras* is a class of algebras determined by finitary operations satisfying suitable identities (with morphisms preserving these operations). It is well known that every such a variety $\mathscr{V}$ is equivalent to $\mathsf{Set}_{\mathbb{T}}$ for some finitary (= filtered colimit preserving) monad $\mathbb{T}$ on Set. Therefore, any finitary variety of algebras $\mathscr{V}$ gives rise to a finitary monad $\mathbb{T}_{\mathscr{V}}$ on Set whose category of algebras is equivalent to the category defined by $\mathscr{V}$.

7.2. example. Left braces. A *left brace* is an abelian group $(A, +, 0)$ together with a multiplication $\cdot : A \times A \to A$ such that the following identities hold:

- $a \cdot (b + c) = a \cdot b + a \cdot c$;
- $(a \cdot b + a + b) \cdot c = a \cdot (b \cdot c) + a \cdot c + b \cdot c$;
- the map $x \mapsto a \cdot x + x$ is bijective for each $a \in A$.

Left braces and their homomorphisms (additive group homomorphisms that respect multiplication) form a variety denoted LBr.

proposition. *Let X be a set with cardinality $n = p^2 q$, where $2 < p < q$ are primes. Then*

$$|\mathsf{Desc}^1(\mathbb{T}_{\mathsf{LBr}}, X)| = \begin{cases} 4, & \text{if } \; p \nmid q - 1; \\ p + 8, & \text{if } \; p | q - 1, p^2 \nmid q - 1; \\ 2p + 8, & \text{if } \; p | q - 1. \end{cases}$$

Proof. The result follows from [8, Corollary 3], according to which

$$b(p^2q) = \begin{cases} 4, & \text{if} \quad p \nmid q-1; \\ p+8, & \text{if} \quad p|q-1, p^2 \nmid q-1; \\ 2p+8, & \text{if} \quad p|q-1. \end{cases}$$

for primes $2 < p < q$. Here $b(n)$, n being a positive integer, denotes the number of non-isomorphic left braces of fixed order n. □

7.3. EXAMPLE. Finite Abelian Groups. Let Ab be the variety of abelian groups and $\mathbb{T}_{\mathsf{Ab}}$ be Sthe corresponding finitary monad on Set.

Let N be a positive integer. Recall that a *partition* of N is a non-decreasing sequence $(n_1, n_2, \ldots, n_k)$ of positive integers with $n_1 + n_2 + \cdots + n_k = N$. The *partition function* $\pi(N)$ gives the number of partitions of N.

PROPOSITION. *Let X be a finite set with cardinality $|X| = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$, where $p_1, p_2, \ldots p_k$ are distinct primes. Then*

$$|\mathsf{Desc}^1(\mathbb{T}_{Ab}, X)| = \prod_{i=1}^{k} \pi(n_i).$$

Proof. The result follows from the fact (e.g., [22, p.129]) that there are $\prod_{i=1}^{k} \pi(n_i)$ isomorphism classes of abelian groups of order $\prod_{i=1}^{k} p^{n_i}$. □

For example, if $|X| = 16 = 2^4$, then $|\mathsf{Desc}^1(\mathbb{T}_{\mathsf{Ab}}, X)| = 4$ and

$$\{(\mathbb{Z}_2)^4, \mathbb{Z}_4 \times (\mathbb{Z}_2)^2, \mathbb{Z}_8 \times \mathbb{Z}_2, (\mathbb{Z}_4)^2, \mathbb{Z}_{16}\}$$

is a complete set of the representatives of the isomorphism classes of finite abelian groups of order 16.

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