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Original article

Connections between a system of forward–backward SDEs and backward stochastic PDEs related to the utility maximization problem

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Abstract

Connections between a system of Forward–Backward SDEs derived in Horst et al., (2014) and Backward Stochastic PDEs (Mania and Tevzadze, 2010) related to the utility maximization problem are established. Besides, we derive another version of Forward–Backward SDE of the same problem and prove the existence of solution.

Keywords: Utility maximization problem; Backward stochastic partial differential equation; Forward–backward stochastic differential equation

1. Introduction

We consider a financial market model, where the dynamics of asset prices is described by the continuous R^d-valued continuous semimartingale S defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $F = (F_t, t \in [0, T])$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and $T < \infty$. We work with discounted terms, i.e. the bond is assumed to be constant.

Let $U = U(x) : R \to R$ be a utility function taking finite values at all points of real line R such that U is continuously differentiable, increasing, strictly concave and satisfies the Inada conditions

$$U'(\infty) = \lim_{x\to\infty} U'(x) = 0, \quad U'(-\infty) = \lim_{x\to-\infty} U'(x) = \infty. \tag{1}$$

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We also assume that U satisfies the condition of reasonable asymptotic elasticity (see [1] and [2] for a detailed discussion of these conditions), i.e.

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \quad \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{2}$$

For the utility function U we denote by $\widetilde{U}$ its convex conjugate

$$\widetilde{U}(y) = \sup_x (U(x) - xy), \quad y > 0. \tag{3}$$

Denote by $\mathcal{M}^e$ (resp. $\mathcal{M}^a$) the set of probability measures Q equivalent (resp. absolutely continuous) with respect to P such that S is a local martingale under Q.

Let $\mathcal{M}_U^a$ (resp. $\mathcal{M}_U^e$) be the convex set of probability measures $Q \in \mathcal{M}^a$ (resp. $\mathcal{M}^e$) such that

$$E\widetilde{U}\Big(\frac{dQ_T}{dP_T}\Big) < \infty. \tag{4}$$

It follows from proposition 4.1 of [3] that (4) implies $E\widetilde{U}\big(y\frac{dQ_T}{dP_T}\big) < \infty$ for any $y > 0$.

Throughout the paper we assume that

$$\mathcal{M}_U^e \neq \emptyset. \tag{5}$$

The wealth process, determined by a self-financing trading strategy π and initial capital x, is defined as a stochastic integral

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u, \quad 0 \le t \le T.$$

We consider the utility maximization problem with random endowment H, where H is a liability that the agent must deliver at the terminal time T. H is an F_T-measurable random variable which for simplicity is assumed to be bounded (one can use also weaker assumption 1.6 from [4]). The value function $V(x)$ associated to the problem is defined by

$$V(x) = \sup_{\pi\in\Pi_x} E\Big[U\Big(x + \int_0^T \pi_u\, dS_u + H\Big)\Big], \tag{6}$$

where Π_x is a class of strategies which (following [2] and [4]) we define as the class of predictable S-integrable processes π such that $U(x + (\pi\cdot S)_T + H) \in L^1(P)$ and $\pi \cdot S$ is a supermartingale under each $Q \in \mathcal{M}_U^a$.

The dual problem to (6) is

$$\widetilde{V}(y) = \inf_{Q\in\mathcal{M}_U^e} E[\widetilde{U}(y\rho_T^Q) + y\rho_T^Q H], \quad y > 0, \tag{7}$$

where $\rho_t^Q = dQ_t/dP_t$ is the density process of the measure $Q \in \mathcal{M}^e$ relative to the basic measure P.

It was shown in [4] that under assumptions (2) and (5) an optimal strategy $\pi(x)$ in the class Π_x exists. There exists also an optimal martingale measure $Q(y)$ to the problem (7), called the minimax martingale measure and by $\rho^* = (\rho_t^*(y), t \in [0,T])$ we denote the density process of this measure relative to the measure P.

It follows also from [4] that under assumptions (2) and (5) optimal solutions $\pi^*(x) \in \Pi_x$ and $Q(y) \in \mathcal{M}_U^e$ are related as

$$U'\Big(x + \int_0^T \pi_u^*(x) dS_u + H\Big) = y\rho_T^*(y), \quad P\text{-}a.s. \tag{8}$$

The continuity of S and the existence of an equivalent martingale measure imply that the structure condition is satisfied, i.e. S admits the decomposition

$$S_t = M_t + \int_0^t d\langle M\rangle_s \lambda_s, \quad \int_0^t \lambda_s^T\, d\langle M\rangle_s \lambda_s < \infty$$

for all t P-a.s., where M is a continuous local martingale and λ is a predictable process. The sign ^{T} here denotes the transposition.

Let us introduce the dynamic value function of problem (6) defined as

$$V(t,x) = \operatorname*{ess\,sup}_{\pi \in \Pi_x} E\Big(U\Big(x + \int_t^T \pi_u \, dS_u + H\Big) \,\Big|\, F_t\Big). \tag{9}$$

It is well known that for any $x \in R$ the process $(V(t,x), t \in [0,T])$ is a supermartingale admitting an RCLL (right-continuous with left limits) modification.

Therefore, using the Galtchouk–Kunita–Watanabe (GKW) decomposition, the value function is represented as

$$V(t,x) = V(0,x) - A(t,x) + \int_0^t \psi(s,x)\, dM_s + L(t,x),$$

where for any $x \in R$ the process $A(t,x)$ is increasing and $L(t,x)$ is a local martingale orthogonal to M.

Definition 1. We shall say that $(V(t,x), t \in [0,T])$ is a regular family of semimartingales if

(a) $V(t,x)$ is two-times continuously differentiable at x P- a.s. for any $t \in [0,T]$,

(b) for any $x \in R$ the process $V(t,x)$ is a special semimartingale with bounded variation part absolutely continuous with respect to an increasing predictable process $(K_t, t \in [0,T])$, i.e.

$$A(t,x) = \int_0^t a(s,x)\, dK_s,$$

for some real-valued function $a(s,x)$ which is predictable and K-integrable for any $x \in R$,

(c) for any $x \in R$ the process $V'(t,x)$ is a special semimartingale with the decomposition

$$V'(t,x) = V'(0,x) - \int_0^t a'(s,x)\, dK_s + \int_0^t \psi'(s,x)\, dM_s + L'(t,x).$$

where a', φ' and L' are partial derivatives of a, φ and L respectively.

If $F(t,x)$ is a family of semimartingales then $\int_0^T F(ds, \xi_s)$ denotes a generalized stochastic integral, or a stochastic line integral (see [5], or [6]). If $F(t,x) = xG_t$, where G_t is a semimartingale then the stochastic line integral coincides with the usual stochastic integral denoted by $\int_0^T \xi_s dG_s$ or $(\xi \cdot G)_T$.

It was shown in [7–9] (see, e.g., Theorem 3.1 from [9]) that if the value function satisfies conditions (a)–(c) then it solves the following BSPDE

$$\begin{aligned} V(t,x) &= V(0,x) \\ &+ \frac{1}{2}\int_0^t \frac{1}{V''(s,x)} (\varphi'(s,x) + \lambda(s)V'(s,x))^T \, d\langle M\rangle_s (\varphi'(s,x) + \lambda(s)V'(s,x)) \\ &+ \int_0^t \varphi(s,x)\, dM_s + L(t,x), \qquad V(T,x) = U(x) \end{aligned} \tag{10}$$

and optimal wealth satisfies the SDE

$$X_t(x) = x - \int_0^t \frac{\varphi'(s, X_s(x)) + \lambda(s)V'(s, X_s(x))}{V''(s, X_s(x))} dS_s. \tag{11}$$

This assertion is a verification theorem since conditions are required directly on the value function $V(t,x)$ and not on the basic objects (on the asset price model and on the objective function U) only. In the case of complete markets [10] conditions on utility functions are given to ensure properties (a)–(c) and thus existence of a solution to the BSPDE (10), (11) is established. Note that the BSPDE (10), (11) is of the same form for random utility functions $U(\omega, x)$, for utility functions defined on half real line and properties (a)–(c) are also satisfied for standard (exponential, power and logarithmic) utility functions.

In the paper [11] a new approach was developed, where a characterization of optimal strategies to the problem (6) in terms of a system of Forward–Backward Stochastic Differential Equations (FBSDE) in the Brownian framework was given. The key observation was an existence of a stochastic process Y with $Y_T = H$ such that $U'(X_t + Y_t)$ is a martingale. The same approach was used in [12], where these results were generalized in semimartingale setting with continuous filtration rejecting also some technical conditions imposed in [11]. The FBSDE for the pair (X, Y) (where X is the optimal wealth and Y the process mentioned above) is of the form (see, [12])

$$Y_t = Y_0 + \int_0^t [\lambda_s^T \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} - \frac{1}{2}\lambda_s^T \frac{U'''(X_s+Y_s)U'(X_s+Y_s)^2}{U''(X_s+Y_s)^3} \tag{12}$$

$$+ Z_s^T] d\langle M\rangle_s \lambda_s - \frac{1}{2}\int_0^t \frac{U'''(X_s+Y_s)}{U''(X_s+Y_s)} d\langle N\rangle_s + \int_0^t Z_s dM_s + N_t, \quad Y_T = H.$$

$$X_t = x - \int_0^t \big(\lambda_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\big) dS_s, \tag{13}$$

where N is a local martingale orthogonal to M.

Note that in [11] and [12] an existence of a solution of FBSDE (12), (13) is not proved, since not all conditions of corresponding theorems are formulated in terms of basic objects. E.g., in both papers it is imposed that $E(U'(X_T^* + H))^2 < \infty$ and it is not clear if an optimal strategy satisfying this condition exists. Note that in [11] in the case of complete markets an existence of a solution of FBSDE (12), (13) is proved under certain regularity assumptions on the objective function U.

One of our goal is to derive another version of FBSDE (12), (13) and to prove the existence of a solution which will imply the existence of a solution of the system (12), (13) also.

The second goal is to establish relations between equations BSPDE (10), (11) and FBSDE (12), (13). Solutions of these equations give constructions of the optimal strategy of the same problem. BSPDE (29), (30) can be considered as a generalization of Hamilton–Jacobi–Bellman equation to the non Markovian case and FBSDE (12), (13) is linked with the stochastic maximum principle (see [11]), although Eqs. (12)–(13) is not obtained directly from the maximum principle. It is well known that the relation between Bellman's dynamic programming and the Pontryagin's maximum principle in optimal control is of the form $\psi_t = V'(t, X_t)$, where V is the value function, X an optimal solution and ψ is an adjoint process (see, e.g. [13,14]). Therefore, somewhat similar relation between above mentioned equations should be expected. In particular, it is shown in Theorem 2, that the first components of solutions of these equations are related by the equality

$$Y_t = -\widetilde{U}'(V'(t, X_t)) - X_t.$$

In addition, conditions are given when the existence of a solution of BSPDE (29), (30) imply the existence of a solution of the system (12)–(13) and vice versa.

2. Another version of the forward–backward system (12)–(13)

In this section we derive another version of the Forward–Backward system (12), (13) in which the backward component P_t is a process, such that $P_t + U'(X_t)$ is a martingale.

Theorem 1. *Let utility function U be three-times continuously differentiable and let the filtration F be continuous. Assume that conditions (2) and (5) are satisfied. Then there exists a quadruple (P, ψ, L, X), where P and X are continuous semimartingales, ψ is a predictable M-integrable process and L is a local martingale orthogonal to M, that satisfies the FBSDE*

$$X_t = x - \int_0^t \frac{\lambda_s P_s + \lambda_s U'(X_s) + \psi_s}{U''(X_s)} dS_s, \tag{14}$$

$$P_t = P_0 + \int_0^t \left[\lambda_s - \frac{1}{2}U'''(X_s)\frac{\big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big)}{U''(X_s)^2}\right]^T d\langle M\rangle_s \big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big)$$

$$+ \int_0^t \psi_s dM_s + L_t, \qquad P_T = U'(X_T + H) - U'(X_T). \tag{15}$$

In addition the optimal strategy is expressed as

$$\pi_t^* = -\frac{\lambda_t P_t + \lambda_t U'(X_t) + \psi_t}{U''(X_t)} \tag{16}$$

and the optimal wealth X^* coincides with X.

Proof. Define the process

$$P_t = E(U'(X_T^* + H)/F_t) - U'(X_t^*). \tag{17}$$

Note that the integrability of $U'(X_T^* + H)$ follows from the duality relation (8). It is evident that $P_T = U'(X_T^* + H) - U'(X_T^*)$.

Since U is three-times differentiable, $U'(X_t^*)$ is a continuous semimartingale and P_t admits the decomposition

$$P_t = P_0 + A_t + \int_0^t \psi_u dM_u + L_t, \tag{18}$$

where A is a predictable process of finite variations and L is a local martingale orthogonal to M.

Since ρ_t^* is the density of a martingale measure, it is of the form $\rho_t^* = \mathcal{E}_t(-\lambda \cdot M + R)$, $R \perp M$. Therefore, (8) and (17) imply that

$$E(U'(X_T^* + H)/F_t) = y\rho_t^* = y - \int_0^t \lambda_s y \rho_s^* dM_s + \tilde{R}_t$$

$$= y - \int_0^t \big(P_s + U'(X_s^*)\big)\lambda_s dM_s + \tilde{R}_t, \tag{19}$$

where $y = EU'(X_T^* + H)$ and $\tilde{R}$ is a local martingale orthogonal to M.

By definition of the process P_t, using the Itô formula for $U'(X_t^*)$ and taking decompositions (18), (19) in mind, we obtain

$$P_0 + A_t + \int_0^t \psi_s dM_s + L_t = y - \int_0^t \big(P_s + U'(X_s^*)\big)\lambda_s dM_s + \tilde{R}_t -$$

$$- U'(x) - \int_0^t U''(X_s^*)\pi_s^{*T} d\langle M\rangle_s \lambda_s - \frac{1}{2}\int_0^t U'''(X_s^*)\pi_s^{*T} d\langle M\rangle_s \pi_s^*$$

$$- \int_0^t U''(X_s^*)\pi_s^* dM_s. \tag{20}$$

Equalizing the integrands of stochastic integrals with respect to dM we have that $\mu^{\langle M\rangle}$-a.e.

$$\pi_t^* = -\frac{\lambda_t P_t + \lambda_t U'(X_t^*) + \psi_t}{U''(X_t^*)} \tag{21}$$

Equalizing the parts of finite variations in (20) we get

$$A_t = -\int_0^t \Big(U''(X_s^*)\lambda_s + \frac{1}{2}U'''(X_s^*)\pi_s^*\Big)^T d\langle M\rangle_s \pi_s^* \tag{22}$$

and from (21), substituting the expression for π^* in (22) we obtain that

$$A_t = \int_0^t \left[\lambda_s - \frac{1}{2}U'''(X_s)\frac{\big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big)}{U''(X_s)^2}\right]^T d\langle M\rangle_s \big(\lambda_s P_s + \lambda_s U'(X_s) + \psi_s\big) \tag{23}$$

Therefore, (23) and (18) imply that P_t satisfies Eq. (15). Integrating both parts of equality (21) with respect to dS and adding the initial capital we obtain Eq. (14) for the optimal wealth. $\square$

Corollary. Let conditions of Theorem 1 be satisfied. Then there exists a solution of FBSDE (12), (13). In particular, if the pair (X, P) is a solution of (14), (15), then the pair (X, Y), where

$$Y_t = -\tilde{U}'(P_t + U'(X_t)) - X_t,$$

satisfies the FBSDE (12), (13).

Conversely, if the pair (X, Y) solves the FBSDE (12), (13), then $(X_t, P_t = U'(X_t + Y_t) - U'(X_t))$ satisfies (14), (15).

3. Relations between BSPDE (10)–(11) and FBSDE (12)–(13)

To establish relations between equations BSPDE (10), (11) and FBSDE (12), (13) we need the following

Definition 2 (*[15]*). The function $u(t,x)$ is called a decoupling field of the FBSDE (12), (13) if

$$u(T,x)=H, \quad a.s. \tag{24}$$

and for any $x\in R$, $s,\tau\in R_+$ such that $0\le s<\tau\le T$ the FBSDE

$$\begin{aligned} Y_t &= u(s,x) \\ &+\int_s^t \Big(\lambda_r^T\frac{U'(X_r+Y_r)}{U''(X_r+Y_r)}-\frac{1}{2}\lambda_r^T\frac{U'''(X_r+Y_r)U'(X_r+Y_r)^2}{U''(X_r+Y_r)^3}+Z_r^T\Big)d\langle M\rangle_r\lambda_r \\ &-\frac{1}{2}\int_s^t\frac{U'''(X_r+Y_r)}{U''(X_r+Y_r)}d\langle N\rangle_r+\int_s^t Z_r dM_r+N_t-N_s, \quad Y_\tau=u(\tau,X_\tau), \end{aligned} \tag{25}$$

$$X_t=x-\int_s^t\Big(\lambda_r\frac{U'(X_r+Y_r)}{U''(X_r+Y_r)}+Z_r\Big)dS_r, \tag{26}$$

has a solution (Y,Z,N,X) satisfying

$$Y_t=u(t,X_t), \quad a.s. \tag{27}$$

for all $t\in[s,\tau]$. We mean that all integrals are well defined.

We shall say that $u(t,x)$ is a regular decoupling field if it is a regular family of semimartingales (in the sense of Definition 1).

If we differentiate equation BSPDE (10) at x (assuming that all derivatives involved exist), we obtain the BSPDE

$$\begin{aligned} V'(t,x) &= V'(0,x) \\ &+\frac{1}{2}\int_0^t\bigg(\frac{(\varphi'(s,x)+\lambda_s V'(s,x))^T}{V''(s,x)}d\langle M\rangle_s(\varphi'(s,x)+\lambda_s V'(s,x))\bigg)' \\ &+\int_0^t\varphi'(s,x)\,dM_s+L'(t,x), \qquad V'(T,x)=U'(x+H). \end{aligned} \tag{28}$$

Thus, we consider the following BSPDE

$$\begin{aligned} V'(t,x) &= V'(0,x)+\int_0^t\bigg[\frac{(V''(s,x)\lambda_s+\varphi''(s,x))^T}{V''(s,x)} \\ &-\frac{1}{2}V'''(s,x)\frac{(V'(s,x)\lambda_s+\varphi'(s,x))^T}{V''(s,x)}\bigg]d\langle M\rangle_s(V'(s,x)\lambda_s+\varphi'(s,x)) \\ &+\int_0^t\varphi'(s,x)\,dM_s+L'(t,x), \qquad V'(T,x)=U'(x+H), \end{aligned} \tag{29}$$

where the optimal wealth satisfies the same SDE

$$X_t(x)=x-\int_0^t\frac{\varphi'(s,X_s(x))+\lambda(s)V'(s,X_s(x))}{V''(s,X_s(x))}dS_s. \tag{30}$$

The FBSDE (12), (13) is equivalent, in some sense, to BSPDE (29), (30) and the following statement establishes a relation between these equations.

Theorem 2. *Let the utility function $U(x)$ be three-times continuously differentiable and let the filtration F be continuous.*

(a) If $V'(t,x)$ is a regular family of semimartingales and $(V'(t,x), \varphi'(t,x), L'(t,x), X_t)$ is a solution of BSPDE (29), (30), then the quadruple
(Y_t, Z_t, N_t, X_t), where

$$Y_t = -\widetilde{U}'(V'(t,X_t)) - X_t, \tag{31}$$

$$Z_t = \lambda_t \widetilde{U}'(V'(t,X_t)) + \frac{\varphi'(t,X_t) + \lambda_t V'(t,X_t)}{V''(t,X_t)}, \tag{32}$$

$$N_t = -\int_0^t \widetilde{U}''(V'(s,X_s)) d\Big(\int_0^s L'(dr, X_r)\Big), \tag{33}$$

will satisfy the FBSDE (12), (13). Moreover, the function $u(t,x) = -\widetilde{U}'(V'(t,x)) - x$ will be the decoupling field of this FBSDE.

(b) Let $u(t,x)$ be a regular decoupling field of FBSDE (12), (13) and let $(U'(X_t + Y_t), s \le t \le T)$ be a true martingale for every $s \in [0,T]$. Then $(V'(t,x), \varphi'(t,x), L'(t,x), X)$ will be a solution of BSPDE (29), (30) and following relations hold

$$V'(t,x) = U'(x + u(t,x)), \quad \textit{hence} \quad V'(t,X_t) = U'(X_t + Y_t), \tag{34}$$

$$\varphi'(t,X_t) = (Z_t + \lambda_s \frac{U'(X_t + Y_t)}{U''(X_t + Y_t)}) V''(t,X_t) - \lambda_t U'(X_t + Y_t), \tag{35}$$

$$\int_0^t L'(ds, X_s) = \int_0^t U''(X_s + Y_s) dN_s, \tag{36}$$

where $\int_0^t L'(ds, X_s)$ is a stochastic line integral with respect to the family $(L'(t,x), x \in R)$ along the process X.

Proof. (a) It follows from BSPDE (29), (30) and from the Itô – Ventzel formula that $V'(t,X_t)$ is a local martingale with the decomposition

$$V'(t,X_t) = V'(0,x) - \int_0^t \lambda_s V'(s,X_s) dM_s + \int_0^t L'(ds, X_s). \tag{37}$$

Let $Y_t = -\widetilde{U}'(V'(t,X_t)) - X_t$. Since U is three-times differentiable (hence so is $\tilde{U}$), Y_t will be a special semimartingale and by GKW decomposition

$$Y_t = Y_0 + A_t + \int_0^t Z_u dM_u + N_t, \tag{38}$$

where A is a predictable process of finite variations and N is a local martingale orthogonal to M.

The definition of the process Y, decompositions (37), (38) and the Itô formula for $\widetilde{U}'(V'(t,X_t))$ imply that

$$A_t + \int_0^t Z_s dM_s + N_t = \tag{39}$$

$$= \int_0^t \widetilde{U}''(V'(s,X_s)) V'(s,X_s) \lambda_s dM_s - \int_0^t \widetilde{U}''(V'(s,X_s)) d\Big(\int_0^s L'(dr, X_r)\Big)$$

$$- \frac{1}{2}\int_0^t \widetilde{U}'''(V'(s,X_s)) V'(s,X_s)^2 \lambda_s^T d\langle M\rangle_s \lambda_s - \frac{1}{2}\int_0^t \widetilde{U}'''(V'(s,X_s)) d\langle \int_0^{\cdot} L'(dr, X_r)\rangle_s$$

$$+ \int_0^t \frac{\lambda_s V'(s,X_s) + \varphi'(s,X_s)}{V''(s,X_s)} dM_s + \int_0^t \frac{\lambda_s^T V'(s,X_s) + \varphi'(s,X_s)^T}{V''(s,X_s)} d\langle M\rangle_s \lambda_s.$$

Equalizing the integrands of stochastic integrals with respect to dM in (39) we have that $\mu^{\langle M\rangle}$-a.e.

$$Z_s = \frac{\lambda_s V'(s,X_s)+\varphi'(s,X_s)}{V''(s,X_s)} + \widetilde{U}''(V'(s,X_s))V'(s,X_s)\lambda_s. \tag{40}$$

Equalizing the orthogonal martingale parts we get P-a.s.

$$N_t = -\int_0^t \widetilde{U}''(V'(s,X_s))d\Big(\int_0^s L'(dr,X_r)\Big). \tag{41}$$

Equalizing the parts of finite variations in (39) we have

$$A_t = \int_0^t \frac{\lambda_s^T V'(s,X_s)+\varphi'(s,X_s)^T}{V''(s,X_s)} d\langle M\rangle_s\lambda_s \tag{42}$$

$$-\frac{1}{2}\int_0^t \widetilde{U}'''(V'(s,X_s))V'(s,X_s)^2\lambda_s^T d\langle M\rangle_s\lambda_s - \frac{1}{2}\int_0^t \widetilde{U}'''(V'(s,X_s))d\langle\int_0^\cdot L'(dr,X_r)\rangle_s$$

and by equalities (40), (41) we obtain from (42) that

$$A_t = \int_0^t \left(Z_s - \widetilde{U}''(V'(s,X_s))V'(s,X_s)\lambda_s - \frac{1}{2}\widetilde{U}'''(V'(s,X_s))V'(s,X_s)^2\lambda_s\right)^T d\langle M\rangle_s\lambda_s$$

$$-\frac{1}{2}\int_0^t \frac{\widetilde{U}'''(V'(s,X_s))}{\widetilde{U}''(V'(s,X_s))^2} d\langle N\rangle_s. \tag{43}$$

Therefore, using the duality relations

$$V'(t,X_t) = U'(X_t+Y_t),$$

$$\widetilde{U}''(V'(t,X_t)) = -\frac{1}{U''(X_t+Y_t)},$$

$$\widetilde{U}'''(V'(t,X_t)) = -\frac{U'''(X_t+Y_t)}{(U''(X_t+Y_t))^3},$$

we obtain from (43) that

$$A_t = \int_0^t \left(\lambda_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} - \frac{1}{2}\lambda_s \frac{U'''(X_s+Y_s)U'(X_s+Y_s)^2}{U''(X_s+Y_s)^3} + Z_s\right)^T d\langle M\rangle_s\lambda_s$$

$$-\frac{1}{2}\int_0^t \frac{U'''(X_s+Y_s)}{U''(X_s+Y_s)} d\langle N\rangle_s \tag{44}$$

Thus, (38) and (44) imply that Y satisfies Eq. (12).

Since

$$\widetilde{U}''(V'(s,X_s))V'(s,X_s) = -\frac{1}{U''(X_s+Y_s)},$$

from (30) and (40) we obtain Eq. (13) for the optimal wealth.

The proof that the function $u(t,x) = -\widetilde{U}'(V'(t,x)) - x$ is the decoupling field of the FBSDE (12) is similar. One should take integrals from s to t and use the same arguments.

(b) Since the quadruple $(Y^{s,x}, Z^{s,x}, N^{s,x}, X^{s,x})$ satisfies the FBSDE (25), (26), it follows from the Itô formula that for any $t \geq s$

$$U'(X_t^{s,x}+Y_t^{s,x}) = U'(x+u(s,x)) - \int_s^t \lambda_r U'(X_r^{s,x}+Y_r^{s,x})dM_r \tag{45}$$

$$+\int_s^t U''(X_r^{s,x}+Y_r^{s,x})dN_r.$$

Thus $U'(X_t^{s,x}+Y_t^{s,x}), t\geq s$, is a local martingale and a true martingale by assumption. Therefore, it follows from (24) and (27) that

$$U'(X_t^{s,x}+Y_t^{s,x})=E(U'(X_T^{s,x}+H)/F_t)=V'(t,X_t^{s,x}), \tag{46}$$

where the last equality is proved similarly to [3]. For $t=s$ we obtain that

$$U'(x+u(s,x))=V'(s,x), \tag{47}$$

hence

$$u(t,x)=-\tilde{U}'(V'(t,x))-x. \tag{48}$$

Since $U(x)$ is three-times differentiable and $u(t,x)$ is a regular decoupling field, equality (47) implies that $V'(t,x)$ will be a regular family of semimartingales. Therefore, using the Itô – Ventzel formula for $V'(t,X_t^{s,x})$ and equalities (45), (46) we have

$$\begin{aligned}
&\int_s^t\big[\varphi'(r,X_r^{s,x})-V''(r,X_r^{s,x})(\lambda_s\frac{U'(X_r^{s,x}+Y_r^{s,x})}{U''(X_r^{s,x}+Y_r^{s,x})}+Z_r^{s,x})\big]dM_r \\
&+\int_s^t L'(dr,X_r)+\int_s^t a'(r,X_r^{s,x})dK_r \\
&-\int_s^t\big(\lambda_r\frac{U'(X_r^{s,x}+Y_r^{s,x})}{U''(X_r^{s,x}+Y_r^{s,x})}+Z_r^{s,x}\big)^T d\langle M\rangle_r(V''(r,X_r^{s,x})\lambda_r+\varphi''(r,X_r^{s,x})) \\
&-\frac{1}{2}\int_s^t(V'''(r,X_r^{s,x}))\big(\lambda_r\frac{U'(X_r^{s,x}+Y_r^{s,x})}{U''(X_r^{s,x}+Y_r^{s,x})}+Z_r^{s,x}\big)^T d\langle M\rangle_r\big(\lambda_r\frac{U'(X_r^{s,x}+Y_r^{s,x})}{U''(X_r^{s,x}+Y_r^{s,x})}+Z_r^{s,x}\big) \\
&=-\int_s^t\lambda_r U'(X_r^{s,x}+Y_r^{s,x})dM_r+\int_s^t U''(X_r^{s,x}+Y_r^{s,x})dN_r.
\end{aligned} \tag{49}$$

Equalizing the integrands of stochastic integrals with respect to dM in (49) we have that μ^K-a.e.

$$Z_r^{s,x}=\frac{\lambda_r V'(r,X_r^{s,x})+\varphi'(r,X_r^{s,x})}{V''(r,X_r^{s,x})}-\lambda_r\frac{U'(X_r^{s,x}+Y_r^{s,x})}{U''(X_r^{s,x}+Y_r^{s,x})}. \tag{50}$$

Equalizing the parts of finite variations in (49), taking (50) in mind, we get that for any $t>s$

$$\begin{aligned}
\int_s^t a'(r,X_r^{s,x})dK_r=&\int_s^t\big[\frac{(V''(r,X_r^{s,x})\lambda_r+\varphi''(r,X_r^{s,x}))}{V''(r,X_r^{s,x})} \\
&-\frac{1}{2}V'''(r,X_r^{s,x})\frac{(V'(r,X_r^{s,x})\lambda_r+\varphi'(r,X_r^{s,x}))}{V''(r,X_r^{s,x})^2}\big]^T d\langle M\rangle_r(V'(r,X_r^{s,x})\lambda_r+\varphi'(r,X_r^{s,x})).
\end{aligned} \tag{51}$$

Let $\tau_s(\varepsilon)=\inf\{t\geq s: K_t-K_s\geq\varepsilon\}$. Since $\langle M^i,M^j\rangle\ll\tilde{K}$ for any $1\leq i,j\leq d$, where $\tilde{K}=\sum_{i=1}^d\langle M^i\rangle$, taking an increasing process $K+\tilde{K}$ (which we denote again by K), without loss of generality we can assume that $\langle M\rangle\ll K$ and denote by C_t the matrix of Radon–Nikodym derivatives $C_t=\frac{d\langle M\rangle_t}{dK_t}$. Then from (51)

$$\begin{aligned}
&\int_s^{\tau_s(\varepsilon)}\Big[\frac{(V''(r,X_r^{s,x})\lambda_r+\varphi''(r,X_r^{s,x}))^T C_r(V'(r,X_r^{s,x})\lambda_r+\varphi'(r,X_r^{s,x}))}{V''(r,X_r^{s,x})} \\
&-\frac{1}{2}V'''(r,X_r^{s,x})\frac{(V'(r,X_r^{s,x})\lambda_r+\varphi'(r,X_r^{s,x}))^T C_r(V'(r,X_r^{s,x})\lambda_r+\varphi'(r,X_r^{s,x}))}{V''(r,X_r^{s,x})^2} \\
&-a'(r,X_r^{s,x})\Big]dK_r=0.
\end{aligned} \tag{52}$$

Since for any $x \in R$ the process $X_r^{s,x}$ is a continuous function on $\{(r,s), r \geq s\}$ with $X_s^{s,x} = x$ (as a solution of Eq. (26)) and $V'(t,x)$ is a regular family of semimartingales, dividing equality (52) by ε and passing to the limit as $\varepsilon \to 0$ from [7] (Proposition B1) we obtain that for each x

$$a'(s,x) = \frac{(V''(s,x)\lambda_s + \varphi''(s,x))^T C_s (V'(s,x)\lambda_s + \varphi'(s,x))}{V''(s,x)} \tag{53}$$

$$- \frac{1}{2} V'''(s,x) \frac{(V'(s,x)\lambda_s + \varphi'(s,x))^T C_s (V'(s,x)\lambda_s + \varphi'(s,x))}{V''(s,x)^2}$$

$$= \frac{1}{2}\left(\frac{(V'(s,x)\lambda_s + \varphi'(s,x))^T C_s (V'(s,x)\lambda_s + \varphi'(s,x))}{V''(s,x)}\right)', \quad \mu^K - a.e.,$$

which implies that $V'(t,x)$ satisfies the BSPDE

$$V'(t,x) = V'(0,x) + \frac{1}{2}\int_0^t \left(\frac{(V'(s,x)\lambda_s + \varphi'(s,x))^T C_s (V'(s,x)\lambda_s + \varphi'(s,x))}{V''(s,x)}\right)' dK_s$$

$$+ \int_0^t \varphi'(s,x)\, dM_s + L'(t,x), \qquad V'(T,x) = U'(x+H). \quad \square \tag{54}$$

Remark 1. In the proof of the part (a) of the theorem we need the condition that $V'(t,x)$ is a regular family of semimartingales only to show equality (37) and to obtain representation (33). Equality (37) can be proved without this assumption (replacing the stochastic line integral by a local martingale orthogonal to M) from the duality relation

$$V'(t, X_t(x)) = \rho_t(y), \quad y = V'(x),$$

where $\rho_t(y)/y$ is the density of the minimax martingale measure (see [2] and [4] for the version with random endowment). Since $\rho_t(y)/y$ is representable in the form $\mathcal{E}(-\lambda \cdot M + D)$, for a local martingale D orthogonal to M, using the Dolean Dade equation we have

$$V'(t, X_t) = \rho_t = y - \int_0^t \lambda_s \rho_s dM_s + \int_0^t \rho_s dD_s =$$

$$= 1 - \int_0^t \lambda_s V'(s, X_s) dM_s + R_t,$$

where $R_t \equiv (Z \cdot D)_t$ is a local martingale orthogonal to M. Further the proof will be the same if we always use a local martingale R_t instead of the stochastic line integral $\int_0^t (L'(ds, X_s)$. Hence the representation (33) will be of the form

$$N_t = -\int_0^t \widetilde{U}''(V'(s, X_s)) dR_t.$$

Remark 2. It follows from the proof of Theorem 2, that if a regular decoupling field for the FBSDE (12), (13) exists, then the second component of the solution Z is also of the form $Z_t = g(\omega, t, X_t)$ for some measurable function g and if we assume that any orthogonal to M local martingale L is represented as a stochastic integral with respect to the given continuous local martingale $M^\perp$, then the third component N of the solution will take the same form $N_t = \int_0^t g^\perp(s, X_s) dM_s^\perp$, for some measurable function $g^\perp$.

Remark 3. Similarly to Theorem 2(b) one can show that $u(t,x) = V'(t,x) - U'(x)$ is the decoupling field of (14), (15).

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