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THE CHANGE - POINT PROBLEM FOR CONTINUOUS MARTINGALES

T. KAVTARADZE, N. LAZRIEVA, AND M. MANIA

Abstract. We consider the change-point detection problem where the change-point is a random time of bifurcation of two probabilistic measures with densities represented as stochastic exponents of continuous martingales. We derive a reflecting backward stochastic differential equation (RBSDE) for the value process related to the disorder problem and show that in the classical case of the Wiener disorder problem this RBSDE is equivalent to a free-boundary problem for a parabolic differential operator.

რეზიუმე. განხილულია დარღვევის ამოცანა, სადაც შეცვლის წერტილი, უწყვეტი მარტინგალებით წარმოქმნილი, ორი ალბათური ზომის გადაბმის მომენტს წარმოადგენს. ამოცანის ფასის პროცესისთვის გამოყვანილია არეკლილი შექცეული სტოქასტური დიფერენციალური განტოლება და ნაჩვენებია, რომ ვინერის პროცესის დარღვევის კლასიკურ ამოცანაში ამ განტოლების ამოხსნა პარაბოლური დიფერენციალური ოპერატორისთვის თავისუფალსაზღვრიანი ამოცანის ამოხსნის ეკვივალენტურია.

1. Introduction

Classical disorder problems consider the detection of a change in the mean (or in other probabilistic characteristics) of a stochastic process X_t that occurs at a random time θ which is called the change-point. The Bayesian formulation of the problem, proposed by Shiryaev (1978), assumes that the change-point θ admits a known prior distribution, although the variable θ itself is unknown for us, since it cannot be observed directly. A sequential change-point detection procedure is identified with a stopping time τ with respect to the filtration F_t^X of observable events (interpreted as the time at which the "alarm signal" is given), at which it is declared that a change has occurred. The aim of the problem is to find a stopping time τ, based on the observations X_t, which is "as close as possible" to the change-point θ. More exactly, the design of the quickest change-point detection procedures

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involves optimizing the tradeoff between two kinds of performance measures, one being a measure of detection delay and the other being a measure of the frequency of false alarm.

Among all processes considered in the context of disorder problem, the Wiener process takes a central place. Shiryaev (1978) derived an explicit solution of a Wiener disorder problem, reducing the initial optimal stopping problem to a free-boundary problem for a parabolic differential operator.

In this paper we present a Bayesian-martingale approach to the disorder problem with infinite time horizon where the change-point represents a random time of bifurcation of two probabilistic measures with densities represented as stochastic exponents of continuous martingales, assuming that all local martingales are continuous. The setting of the problem is discussed in Section 2.

In Section 3 we derive a martingale stochastic differential equation for the a posteriori probability process π_t of the change-point θ, which plays, as it is well known, a crucial role by reducing the disorder problem to optimal stopping problem and to determine the value process and the optimal stopping rule.

In Section 4 we introduce the value process of the related optimal stopping problem and show that this process uniquely solves a suitable reflecting backward stochastic differential equation (RBSDE). The value functions related to disorder problems (or to an optimal stopping problem in general) of Markov processes are usually solutions of suitable free boundary problems. So the RBSDE for the value processes and the free boundary problems for the value functions should be equivalent in some sense, at least in simple cases when the a posteriori probability process π_t is a sufficient statistics and the value process V_t of the problem is related with the value function $\rho(\pi)$ of the same problem by the equality $V_t = \rho(\pi_t)$. The problem is to deduce the differentiability properties and smooth fit conditions for the value functions, based on the properties of the process $\rho(\pi_t)$ being a solution of a RBSDE.

In Section 5 we consider the disorder problem for a Wiener process and show that in this case the related RBSDE for the value process and the corresponding free boundary problem are equivalent. The disorder problem for a Wiener process was solved by Shiryaev (1978) who gave an explicit expression for the value function $\rho(\pi)$ of initial stopping problem, showing that this function (together with the optimal threshold A^*) uniquely solves the corresponding free-boundary problem. Based on results of section 4, we give a probabilistic proof of this fact. We show that $\rho(\pi)$ is a solution of the free-boundary problem if and only if the process $\rho(\pi_t)$ is a solution of corresponding RBSDE. The key step here is to show that if the value process $V_t = \rho(\pi_t)$ satisfies RBSDE, then the function $\rho(\pi)$ is continuously differentiable on $(0, 1]$ and twice continuously differentiable on $(0, A^*)$, $0 <$

$A^* < 1$. In particular this implies that the smooth fit condition is satisfied. Besides, we show that the smooth fit of the second derivative fails.

2. Bayesian Statement of the Disorder Problem

In this section after some preliminaries we discuss the Bayesian statement of the problem for a general martingale model.

Let $(\Omega, \mathcal{F}, F = (F_t, t \geq 0))$ be a measurable space endowed with the continuous filtration, where $\mathcal{F} = F_\infty$. Assume that P^0 and P^1 are two fixed locally equivalent probability measures ($P^1 \overset{\text{loc}}{\sim} P^0$) defined on $(\Omega, \mathcal{F})$ and let $\psi = \psi(x)$ be a distribution function of some non-negative random variable, which is continuous on the interval $(0, \infty)$. Without loss of generality (e.g., taking $P = \frac{1}{2}(P^1 + P^0)$) one can assume that there is a probability measure P on $(\Omega, \mathcal{F})$ such that

$$P^1 \ll P, \quad P^0 \ll P, \quad P^1 \overset{\text{loc}}{\sim} P, \quad P^0 \overset{\text{loc}}{\sim} P.$$

For simplicity let us assume that the σ-algebra $\mathcal{F}_0$ is trivial.

Throughout the paper we shall make the following assumption:

C) all $P-$local martingales are continuous.

condition C) means the continuity of the filtration F. It is satisfied if the filtration F is generated by a Brownian motion, or, more generally, if F admits the integral representation property relative to some vector-valued continuous martingale.

Let $(Z_t^i = \frac{dP_t^i}{dP_t}, t \geq 0)$,$i = 0, 1$, be the density process of the measure P^i relative to P, which is an uniformly integrable $P-$martingale with $Z_t^i > 0$ $P - a.s.$ for any $t \in [0, \infty[$. Then there exists a local martingale $M^i \in \mathcal{M}_{\text{loc}}(F, P)$ such that

$$Z^i = \mathcal{E}(M^i) = \big(\mathcal{E}_t(M^i), t \geq 0\big), \quad i = 0, 1,$$

where $\mathcal{E}(M)$, called the Dolean exponential of M, is the unique solution of the linear Stochastic Differential Equation (SDE)

$$Z_t = 1 + \int_0^t Z_s dM_s \tag{2.1}$$

(see, e.g., [7] or [6]).

For the statement of the problem in a general martingale setting let us extend the initial probability space as follows:

$\overline{\Omega} = \Omega \otimes R^+$, $\overline{F} = \mathcal{F} \otimes \mathcal{B}(R^+)$, $\overline{F}_t = F_t \otimes \mathcal{B}(R^+)$, where $\mathcal{B}(R^+)$ is the Borel σ-algebra on $R^+ = [0, \infty)$.

The measure $\overline{P}^{\,\psi}$ on $\mathcal{F}\otimes\mathcal{B}(R^+)$ is defined in a following way: let for every $A\in\mathcal{F}$ and $B\in\mathcal{B}(R^+)$

$$\overline{P}^{\,\psi}(A\times B)=\int\limits_A\int\limits_B\mathcal{E}_\infty(M^x)\psi(dx)P(d\omega), \tag{2.2}$$

where

$$M_t^x=\int\limits_0^t I_{\{x\le s\}}dM_s^1+\int\limits_0^t I_{\{x>s\}}dM_s^0. \tag{2.3}$$

Note that, since

$$E\mathcal{E}_\infty(M^x)=E\mathcal{E}_x(M^0)\frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_x(M^1)}=E\mathcal{E}_x(M^0)E\Big(\frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_x(M^1)}/F_x\Big)=1,$$

the Fubini theorem implies that $\overline{P}^{\psi}$ is a probability measure.

Let us denote by P^ψ the restriction of the measure $\overline{P}^{\,\psi}$ on the σ-algebra $\mathcal{F}\otimes R^+$.

For every $u<v$ and t we have

$$\int\limits_{(u,v]}\mathcal{E}_t(M^x)\psi(dx)=$$
$$=\int\limits_{(u,v]}I_{\{x>t\}}\mathcal{E}_t(M^0)\psi(dx)+\int\limits_{(u,v]}I_{\{x\le t\}}\mathcal{E}_t(M^x)\psi(dx)=$$
$$=\mathcal{E}_t(M^0)\big(\psi(v\vee t)-\psi(u\vee t)\big)+\mathcal{E}_t(M^1)\int\limits_{(u,v\wedge t]}\frac{\mathcal{E}_x(M^0)}{\mathcal{E}_x(M^1)}\psi(dx). \tag{2.4}$$

So, we could define the measure $\overline{P}^{\,\psi}$ just by P^0, P^1 and ψ. For every $u<v$ and $A\in\mathcal{F}_t$

$$\overline{P}^{\,\psi}\big(A\times]u,v]\big)=\big(\psi(v)-\psi(u\vee t)\big)P^0(A)+\int\limits_A\int\limits_{(u,v\wedge t]}\frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds)dP^1.$$

If we denote by P_t^ψ the restriction of the measure P^ψ on the σ-algebra $\mathcal{F}_t\equiv F_t\times R^+$, we will have for every $A\in F_t$

$$P_t^\psi(A)=P^\psi(A\times R^+)=$$
$$=\big(1-\psi(t)\big)P^0(A)+\int\limits_A\int\limits_{[0,t]}\frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds)dP^1. \tag{2.5}$$

Thus, the measures P_t^ψ do not depend on the choice of the dominating measure P. It is easy to see that $P^\psi \ll P$ and

$$Z_t^\psi \equiv \frac{dP_t^\psi}{dP_t} = \big(1-\psi(t)\big)\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds). \tag{2.6}$$

Note that according (2.2) $Z_t^\psi = \int\limits_{R^+} \mathcal{E}_t(M^x)\psi(dx)$.

Remark 2.1. Since $P^1 \overset{\text{loc}}{\sim} P^0$, we have that $P^\psi \overset{\text{loc}}{\sim} P^0$ and one can express the density process $\widehat{Z}_t^\psi = dP_t^\psi / dP_t^0$ in the form

$$\widehat{Z}_t^\psi = \frac{dP_t^\psi}{dP_t^0} = \big(1-\psi(t)\big) + \mathcal{E}_t(M) \int\limits_{[0,t]} \mathcal{E}_s^{-1}(M)\psi(ds), \tag{2.7}$$

where $Z_t = (\mathcal{E}_t(M), t \geq 0)$ is the density process of P^1 relative to P^0.

Let us define on the space $(\overline{\Omega}, \overline{F})$ the random variable

$$\theta = \theta(\overline{\omega}) = \theta(\omega, x) = x.$$

It is evident from (2.2) that

$$\overline{P}^{\,\psi}(\theta \leq x) = \overline{P}^{\,\psi}\big(\Omega \times [0,x]\big) = \psi(x).$$

This means that the distribution function $\psi = \psi(x)$ by means of which we have defined the new measure $\overline{P}^{\,\psi}$ on the extended measurable space $(\overline{\Omega}, \overline{F})$ comes to be the a priori distribution function of the variable θ, associated with the random time of "disorder".

The aim of the problem is to find a stopping time τ with respect to the filtration F_t of observable events (interpreted as the time at which the "alarm signal" is given) which is "as close as possible" to the change point θ. Following [11] we define the cost criterion by

$$V(\tau) = \overline{P}^{\,\psi}(\tau < \theta) + \overline{E}^{\,\psi} \max(K_\tau - K_\theta, 0), \tag{2.8}$$

where $\overline{P}^{\,\psi}(\tau < \theta)$ is a probability of "false alarm" and $\overline{E}^{\,\psi} \max(K_\tau - K_\theta, 0)$ is an average delay (measured by an F_t adapted continuous increasing process K) of detecting the change point correctly.

The stopping time τ^* is called optimal if

$$V(\tau^*) = \inf_\tau V(\tau), \tag{2.9}$$

where inf is taken over the class of all F- stopping times.

Introducing the a posteriori probability process π_t

$$\pi_t = \overline{P}^{\,\psi}\big(\theta \leq t \mid \mathcal{F}_t\big),$$

similarly to [11] one can reduce problem (2.9) to the optimal stopping problem

$$V(\tau^*) = \inf_{\tau} E^{\psi}\Big[(1-\pi_\tau) + \int_0^\tau \pi_s dK_s\Big], \tag{2.10}$$

since $\overline{P}^{\,\psi}(\tau < \theta) = E^{\psi}(1-\pi_\tau)$ and

$$\overline{E}^{\,\psi} \max(K_\tau - K_\theta, 0) = \overline{E}^{\,\psi} \int_0^\tau I_{(\theta\le s)} dK_s = E^{\psi} \int_0^\tau \pi_s dK_s$$

by the projection theorem.

Let us introduce the value process of the problem (2.10)

$$V_t = \operatorname*{ess\,inf}_{\tau\ge t} E^{\psi}\Big[(1-\pi_\tau) + \int_t^\tau \pi_s dK_s / F_t\Big]. \tag{2.11}$$

It is well known that under the present conditions (see e.g., [3]) the stopping time τ^* defined by

$$\tau^* = \inf\{t : V_t = 1 - \pi_t\} \tag{2.12}$$

is optimal for the problem (2.10). In the case of the Wiener disorder problem considered by [11] the optimal stopping time is of the following simple form

$$\tau^* = \inf\{t : \pi_t \ge A^*\}, \tag{2.13}$$

where the constant A^* is a solution of a certain integral equation and the value function V is explicitly calculated as a function of $\psi(0) = \pi$ and A^*. Here the differential equation for the process π_t plays a crucial role.

In our general setting the process π_t is no longer sufficient to determine the optimal stopping rule, however equation for π_t is essential to characterize the value process V_t as a solution of the corresponding RBSDE. Therefore, in the next section we focus our attention to derivation of a stochastic differential equation for π_t.

3. Differential equation for the a posteriori distribution process

After giving some auxiliary facts and recalling properties of Girsanov's transform we derive the stochastic differential equation for the a posteriori distribution process of the change-point θ based on knowing it's a priori distribution function ψ and the local martingales $M^i \in \mathcal{M}_{\rm loc}(F, P)$, $i = 0, 1$.

It follows from the generalized Bayes' Theorem (see, e.g., [12, pp. 230–233] that

$$\pi_t = \frac{\int_{R^+} I_{(x\le t)} \mathcal{E}_t(M^x)\psi(dx)}{Z_t^{\psi}}, \tag{3.1}$$

where

$$Z_t^\psi = \int\limits_{R^+} \mathcal{E}_t(M^x)\psi(dx). \tag{3.2}$$

Using (2.4) and (2.6) we get

$$\pi_t = \frac{\mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds)}{(1-\psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds)} . \tag{3.3}$$

Dividing the numerator and the denominator of the right hand side of (3.3) on $\mathcal{E}_t(M^0)$, one can write π_t also in the form not depending on the dominating measure P

$$\pi_t = \frac{\mathcal{E}_t(M) \int\limits_{[0,t]} \mathcal{E}_s^{-1}(M)\psi(ds)}{(1-\psi(t)) + \mathcal{E}_t(M) \int\limits_{[0,t]} \mathcal{E}_s^{-1}(M)\psi(ds)} , \tag{3.4}$$

where $\mathcal{E}_t(M) = dP_t^1/dP_t^0$ is the density process of P^1 relative to P^0. Note that the process π_t is continuous by condition C).

Lemma 3.1. *The martingale Z_t^ψ is the Dolean exponential of the local martingale M^ψ (i.e., $Z_t^\psi = \mathcal{E}_t(M^\psi)$), where*

$$M_t^\psi = \int\limits_0^t (1-\pi_s)dM_s^0 + \int\limits_0^t \pi_s dM_s^1. \tag{3.5}$$

Proof. Note that from (3.3) we have that

$$\pi_t Z_t^\psi = \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^1)}\psi(ds), \tag{3.6}$$

$$(1-\pi_t)Z_t^\psi = \big(1-\psi(t)\big)\mathcal{E}_t(M^0). \tag{3.7}$$

An application of Itô's rule to (2.6) yields

$$Z_t^\psi = 1 + \int\limits_0^t (1-\psi(s))\mathcal{E}_s(M^0)dM_s^0 + \int\limits_0^t \int\limits_{[0,s)} \frac{\mathcal{E}_u(M^0)}{\mathcal{E}_u(M^1)}\psi(du)\mathcal{E}_s(M^1)dM_s^1. \tag{3.8}$$

Hence by (3.6), (3.7) and (3.8) we obtain that $Z_t^\psi = \mathcal{E}_t(M^\psi)$ satisfies

$$Z_t^\psi = 1 + \int\limits_0^t Z_s^\psi \big[(1-\pi_s)dM_s^0 + \pi_s dM_s^1\big] \tag{3.9}$$

and the assertion of lemma follows from the uniqueness of the solution of equation (2.1). □

Remark 3.1. Similarly as above, one can show that the density process $\widehat{Z}_t^\psi$ defined by (2.7) admits the representation $\widehat{Z}_t^\psi = \mathcal{E}_t(\widehat{M^\psi}))$, where

$$\widehat{M}_t^\psi = \int_0^t \pi_s dM_s. \tag{3.10}$$

For two continuous semimartingales X and Y let us denote $L(X,Y)$ the Girsanov transform $L_t(X,Y) = X_t - [Y,X]_t$.

Note that (see [8])

$$\frac{\mathcal{E}_t(X)}{\mathcal{E}_t(Y)} = \mathcal{E}_t\big(L(X-Y,Y)\big). \tag{3.11}$$

Since for any X-integrable predictable process H

$$L(H\cdot X, Y) = H\cdot L(X,Y),$$

from (3.5)

$$L_t(M^1 - M^\psi, M^\psi) = \int_0^t (1-\pi_s) dL_s\big(M^1 - M^0, M^\psi\big). \tag{3.12}$$

Theorem 3.1. *The a posteriori probability process π_t satisfies the following stochastic differential equation*

$$\pi_t = \pi_0 + \int_0^t \pi_s(1-\pi_s) dL_s\big(M^1 - M^0, M^\psi\big) + \int_0^t \frac{1-\pi_s}{1-\psi(s)}\psi(ds). \tag{3.13}$$

Proof. By virtue of (3.6) and (3.11)

$$\pi_t = \mathcal{E}_t\big(L(M^1 - M^\psi, M^\psi)\big) \int_{[0,t]} \frac{\mathcal{E}_x(M^0)}{\mathcal{E}_x(M^1)}\psi(dx). \tag{3.14}$$

From (3.14) using the Itò formula we have

$$\pi_t = \pi_0 +$$

$$+ \int_0^t \int_{[0,s)} \frac{\mathcal{E}_x(M^o)}{\mathcal{E}_x(M^1)}\psi(dx)\mathcal{E}_s\big(L(M^1 - M^\psi, M^\psi)\big) dL(M^1 - M^\psi, M^\psi) +$$

$$+ \int_{[0,t]} \frac{\mathcal{E}_s(M^0)}{\mathcal{E}_s(M^\psi)}\psi(ds). \tag{3.15}$$

Equations (3.12) and (3.14) imply that the first term of the right-hand side of (3.15) is equal to

$$\pi_0 + \int_0^t \pi_s(1-\pi_s)dL_s(M^1 - M^0, M^\psi). \tag{3.16}$$

Note that (3.7) also implies that the second term of the right-hand-side of (3.15) is equal to

$$\int_0^t \frac{1-\pi_s}{1-\psi(s)}\psi(ds). \tag{3.17}$$

Therefore relations (3.15)–(3.17) imply that π_t satisfies the stochastic differential equation (3.13). □

Remark 3.2. Sometimes it is more convenient to write equation (3.13) using the martingale $\widehat{M}^\psi$ from Remark 2.2. Similarly to Theorem 1 one can show that π_t satisfies equation

$$\pi_t = \pi_0 + \int_0^t \pi_s(1-\pi_s)dL_s\big(M, \widehat{M}^\psi\big) + \int_0^t \frac{1-\pi_s}{1-\psi(s)}\psi(ds). \tag{3.18}$$

4. Reflecting Backward Stochastic Differential Equation (RBSDE) for the Value process

In this section we provide the reflecting BSDE for the value process of the optimal stopping problem (2.10).

Let us introduce the value process of the problem (2.10)

$$V_t = \operatorname*{ess\,inf}_{\tau \geq t} E^\psi\left[(1-\pi_\tau) + \int_t^\tau \pi_s dK_s / F_t\right],$$

where E^ψ is an expectation w.r.t. the measure P^ψ, which we consider as a reference probability measure throughout this section.

It is well known that (see, e.g., [3]) V_t is a RCLL process such that

i) $V_t \leq 1 - \pi_t$ for all t,

ii) the process $V_t + \int_0^t \pi_s dK_s$ is a submartingale,

iii) V_t is the largest process satisfying i) and ii).

Moreover for any $t \geq 0$ the stopping time τ^* defined by

$$\tau_t^* = \inf\{s \geq t : V_s = 1 - \pi_s\}$$

is t-optimal (see [3] or [5]), that is

$$V_t = E^{\psi}\left[(1-\pi_{\tau_t^*}) + \int_t^{\tau_t^*} \pi_s dK_s / F_t\right].$$

Hence V_t is a special semimartingale with the canonical decomposition

$$V_t = V_0 - \int_0^t \pi_s dK_s + B_t + N_t, \tag{4.1}$$

where N is a martingale and B is a predictable increasing process with $B_0 = 0$.

It is also well-known (see e.g.,[3] [5] or [10]) that increasing process B_t is growing only on the set $\{V_{t-} = 1 - \pi_t\}$(on the stop region) and $V_t + (\pi \cdot K)_t$ is a martingale on the go-region $\{V_{t-} < 1 - \pi_t\}$, i.e., the process B_t satisfies relation

$$\int_0^T I_{\{V_{s-}<1-\pi_s\}} dB_s = 0, \tag{4.2}$$

which implies that the process

$$\int_0^t I_{\{V_{s-}<1-\pi_s\}} d\left(V_s + \int_0^s \pi_u dK_u\right) = \int_0^t I_{\{V_s<1-\pi_s\}} dN_s$$

is a martingale.

Note that relation (4.2) guaranties the maximality of V and together with i) and ii) uniquely determines the value process. But the maximality of V as well, as condition (4.2) is difficult to verify and this leads to necessity to give a differential characterization of the value process. We shall combine the results of [1], [5], [10] and [4] to derive a reflecting BSDE for the process V in our case.

Note that, since the density process Z_t^{ψ} is continuous, condition C) and the Girsanov theorem imply that all P^{ψ}-local martingales are continuous.

Denote by S^1 the class of continuous semimartingales X with the decomposition

$$X_t = X_0 + A_t + M_t, \quad t \geq 0,$$

where M_t is a uniformly integrable martingale and A_t is a process of integrable variation on $[0, \infty]$.

We define a solution of RBSDE related to the disorder problem as a triple (Y_t, ν_t, L_t) of adapted processes satisfying:

I) L_t is a uniformly integrable martingale,

II) ν_t is a predictable process with $0 \leq \nu_t \leq 1$,

III) Y_t is a semimartingale from S^1,

IV) $Y_t \le 1-\pi_t$ for all $t \ge 0$,

V) $\lim_{t\to\infty} Y_t = 0$, P^ψ − a.s.

VI) $$Y_t = Y_0 + \int_0^t (1-\nu_s) I_{(Y_s=1-\pi_s)} d\Big(\int_0^\cdot \pi_u dK_u - \int_0^\cdot \frac{1-\pi_u}{1-\psi(u)}\psi(du)\Big)_s^+ - \\ - \int_0^t \pi_s dK_s + L_t. \qquad (4.3)$$

Theorem 4.1. *Assume that*

A) *ψ is a distribution function concentrated on $[0,\infty]$ and continuous on $(0,\infty)$.*

B) *K is a predictable increasing continuous process such that $EK_t < \infty$ for any $t \in [0,\infty)$.*

There exists a solution of RBSDE (4.3) *satisfying* I)–VI). *If a triple (Y_t, ν_t, L_t) satisfies conditions* I)–VI), *then $Y_t = V_t$ and L_t coincides with the martingale part of the value process V.*

Proof. Using equation (3.13) for π_t and decomposition (4.1) we have

$$1-\pi_t - V_t = 1-\pi_0 - V_0 - \int_0^t \frac{1-\pi_s}{1-\psi(s)}\psi(ds) + \\ + \int_0^t \pi_s dK_s - B_t + \int_0^t \pi_s(1-\pi_s)d\widetilde{M}_s - N_t, \qquad (4.4)$$

where by $\widetilde{M}$ we denoted the P^ψ−martingale $\widetilde{M}_t = L_t\big(M^1 - M^0, M^\psi\big)$.

By Tanaka's formula

$$(1-\pi_t - V_t)^+ = (1-\pi_0 - V_0)^+ + \int_0^t I_{\{1-\pi_s > V_{s-}\}} d(1-\pi_s - V_s) + \\ + \frac{1}{2}\mathcal{L}_t^0(1-\pi-V) + \sum_{s\le t}(1-\pi_s - V_s) I_{(1-\pi_s - V_{s-})}, \qquad (4.5)$$

where $\mathcal{L}_t^0(1-\pi-V)$ is the local time of the process $1-\pi_t-V_t$ at 0 (note that we apriori don't know that the processes B_t and V_t are continuous).

Therefore, from (4.4) and (4.5)

$$(1-\pi_t-V_t)^+=(1-\pi_0-V_0)^+ +\int_0^t I_{(1-\pi_s>V_s)}\pi_s dK_s-$$
$$-\int_0^t I_{(1-\pi_s>V_s)}\frac{1-\pi_s}{1-\psi(s)}\psi(ds)-\int_0^t I_{(1-\pi_s>V_{s-})}dB_s+\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)-$$
$$-\Big(\sum_{s\le t}(1-\pi_s-V_s)I_{(1-\pi_s-V_{s-})}\Big)^p+\text{martingale}, \tag{4.6}$$

where A^p denotes the dual predictable projection of a locally integrable increasing process A.

Since $V_t\le 1-\pi_t$ and $\int_0^t I_{(1-\pi_s>V_{s-})}dB_s=0$, comparing the finite variation parts of right-hand sides of (4.4) and(4.6) we obtain that

$$\int_0^t I_{(1-\pi_s=V_s)}\pi_s dK_s-\int_0^t I_{(1-\pi_s=V_s)}\frac{1-\pi_s}{1-\psi(s)}\psi(ds)-$$
$$-\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)-\Big(\sum_{s\le t}(1-\pi_s-V_s)I_{(1-\pi_s-V_{s-})}\Big)^p=B_t. \tag{4.7}$$

Since all processes in (4.7) are increasing processes, relation (4.7) implies that the measures dB_t and $d\mathcal{L}_t^0$ are absolutely continuous w.r.t. the measure dK_t. In particular this implies that the process V_t is continuous. Moreover, from (4.7) we also have

$$\int_0^t I_{(1-\pi_s=V_s)}d\Big(\pi\cdot K-\frac{1-\pi}{1-\psi}\cdot\psi\Big)_s^+-\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)=$$
$$=B_t+\int_0^t I_{(1-\pi_s=V_s)}d\Big(\pi\cdot K-\frac{1-\pi}{1-\psi}\cdot\psi\Big)_s^-\in\mathcal{A}_{\rm loc}^+ \tag{4.8}$$

and hence, there exists a predictable process μ_t such that

$$\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)=$$
$$=\int_0^t \mu_s I_{(1-\pi_s=V_s)}d\Big(\int_0^\cdot \pi_u dK_u-\int_0^\cdot\frac{1-\pi_u}{1-\psi(u)}\psi(du)\Big)_s^+, \tag{4.9}$$

where $A_t=A_t^+-A_t^-$ is a unique decomposition of a process of finite variation A as a difference of two increasing processes such that the non-negative measures induced by A^+ and A^- on $[0,t]$ have disjoint supports. The variation of such a process is given by $(VarA)_t = A^+_t + A^-_t$.

It follows from (4.8) and(4.9) that

$$\int_0^t (1-\mu_s) I_{(1-\pi_s=V_s)} d\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)^+_s -$$

$$-\int_0^t I_{(1-\pi_s=V_s)} d\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)^-_s = B_t \in \mathcal{A}^+, \tag{4.10}$$

which implies that

$$0 \le \mu_s \le I_{(1-\pi_s=V_s)} \quad d\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)^+_s - \text{a.e. and} \tag{4.11}$$

$$\big\{s : 1-\pi_s = V_s\big\} \subseteq \operatorname{supp}\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)^+. \tag{4.12}$$

In particular, we have that

$$B_t = \int_0^t (1-\mu_s) I_{(1-\pi_s=V_s)} d\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)^+_s =$$

$$= \int_0^t (1-\mu_s) I_{(1-\pi_s=V_s)} d\Big(\pi\cdot K - \frac{1-\pi}{1-\psi}\cdot\psi\Big)_s. \tag{4.13}$$

Therefore (4.13) and(4.1) imply that

$$V_t = V_0 + \int_0^t (1-\mu_s) I_{(V_s=1-\pi_s)} d\Big(\int_0^\cdot \pi_u dK_u - \int_0^\cdot \frac{1-\pi_u}{1-\psi(u)}\psi(du)\Big)^+_s -$$

$$-\int_0^t \pi_s dK_s + N_t, \tag{4.14}$$

which means that the triple (V,μ,N) satisfies equation (4.3).

It follows from equality (4.13) that the value process satisfies also equation

$$V_t = V_0 - \int_0^t \big(I_{(1-\pi_s>V_s)} + \mu_s I_{(1-\pi_s=V_s)}\big)\pi_s dK_s -$$

$$-\int_0^t (1-\mu_s) I_{(1-\pi_s=V_s)} \frac{1-\pi_s}{1-\psi(s)}\,\psi(ds) + N_t, \tag{4.15}$$

which implies that V_t is a supermartingale. Since V is bounded, it is a supermartingale of the class D and by the uniqueness of the Doob-Meyer decomposition N is a uniformly integrable martingale and V is a semimartingale from the class S^1.

Since $0 \le V_t \le 1-\pi_t$ and (the proof of this fact is same as in [11]) $\lim\limits_{t\to\infty} \pi_t = 1$ (P^{ψ}-a.s.), we have that $\lim_{t\to\infty} V_t$ exists and is equal to zero.

Thus, the triple (V,μ,N) is a solution of I)-VI).

Uniqueness: Let a triple (Y_t,ν_t,L_t) be a solution of I)-VI). Then it follows from (4.3) and II) that the process $Y_t+\int\limits_0^t \pi_s dK_s$ is a submartingale. Since V_t is the largest process that satisfies i) and ii), we have $V_t \ge Y_t$.

Let us show that $Y_t \ge V_t$. Let

$$\sigma_t = \inf\big\{s \ge t : Y_s = 1-\pi_s\big\}.$$

By condition IV) we have $Y_t < 1-\pi_t$ on the interval $[t;\sigma_t)$. Therefore, it follows from (4.3)

$$Y_{\sigma_t} - Y_t = -\int\limits_t^{\sigma_t} \pi_s dK_s + L_{\sigma_t} - L_t. \tag{4.16}$$

On the other hand condition V) implies that $Y_{\sigma_t} = 1-\pi_{\sigma_t}$. Therefore taking conditional expectations in (4.16) we obtain that

$$Y_t = E\bigg(1-\pi_{\sigma_t} + \int\limits_t^{\sigma_t} \pi_s dK_s/\mathcal{F}_t\bigg)$$

and by definition of the value process $Y_t \ge V_t$. Thus $Y_t = V_t$. It is evident that the martingale parts of V and Y are also indistinguishable. □

Remark 4.1. By (4.9), (4.12) and (4.15) we have that the value process also satisfies the following equation:

$$V_t = V_0 - \int\limits_0^t I_{(1-\pi_s>V_s)}\pi_s dK_s - \int\limits_0^t I_{(1-\pi_s=V_s)}\frac{1-\pi_s}{1-\psi(s)}\,\psi(ds) - \\ -\frac{1}{2}\,\mathcal{L}_t^0(1-\pi-V) + N_t. \tag{4.17}$$

Let us write the a priori distribution functions in the form:

$$\psi^{\pi}(t) = \pi\delta_0(t) + (1-\pi)\varphi(t) \tag{4.18}$$

where $\delta_0(t)$ is a dirac measure having a mass at 0, and $\varphi(t)$ is any fixed distribution function of some positive continuous random variable. From now on taking expectation with respect to the measure $\overline{P}^{\,\psi^{\pi}}$(resp. $P^{\psi^{\pi}}$)

we will denote as $\overline{E}^{\,\pi}$(resp. E^π) ($\overline{E}^{\,\psi^\pi} \to \overline{E}^{\,\pi}$). Hence the value V_0 can be rewritten as a function of π (π and ω in general):

$$V_0(\pi) = \inf_\tau E^\pi \left[(1 - \pi_\tau) + \int_0^\tau \pi_s dK_s \right].$$

Now we shall prove the concavity of the value function $V_0(\pi)$, which will be essentially used in the sequel. For the value function corresponding to the classical disorder problems this fact was proved in [11].

Lemma 4.1. *The value function $V_0(\pi)$ is a concave function of π.*

Proof. We need to show that for any π_1, $\pi_2 \in [0,1]$ and $\alpha \in (0,1)$

$$V_0\big(\alpha\pi_1 + (1-\alpha)\pi_2\big) \geq \alpha V_0(\pi_1) + (1-\alpha)V_0(\pi_2).$$

Let $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$. By (4.18)

$$\psi^\pi(t) = \alpha\psi^{\pi_1}(t) + (1-\alpha)\psi^{\pi_2}(t)$$

and

$$\overline{P}^{\,\psi^\pi} = \alpha\overline{P}^{\,\psi^{\pi_1}} + (1-\alpha)\overline{P}^{\,\psi^{\pi_2}}$$

by the definition of the measure $\overline{P}^{\,\psi}$ (see 2.2).

As

$$V_0(\pi) = \inf_\tau \overline{E}^{\,\pi}(I_{(\tau<\theta)} + (K_\tau - K_\theta)^+)$$

the concavity of the function $V_0(\pi)$ is straightforward. □

5. Disorder Problem for a Wiener Process

In this section we consider the classical disorder problem of a Wiener process and show that in this case the RBSDE (4.3) is equivalent to the free boundary problem considered by [11].

Let Ω be the space C of continuous functions $x = (x_t)_{t\geq 0}$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}(C)$ of C, $(\mathcal{B}_t(C), t \geq 0)$ the corresponding filtration.

Assume that P^0 is the measure on $(C, \mathcal{B}(C))$ such that $\frac{1}{\sigma}X_t$ is a standard Wiener process and P^1 is the measure on $(C, \mathcal{B}(C))$ such that the process

$$\frac{1}{\sigma}(X_t - rt)$$

is a Wiener process under P^1, where X_t is a coordinate process and r is some constant. Then $P^1 \overset{\text{loc}}{\sim} P^0$ and the density process of P^1 with respect to P^0 is of the form

$$Z_t = Z_t(x) = \frac{dP_t^1}{dP_t^0}(x) = \exp\left\{\frac{r}{\sigma}x_t - \frac{r^2}{2\sigma^2}t\right\}.$$

Thus, $Z_t = \mathcal{E}_t(M)$, with $M_t = \frac{r}{\sigma}X_t$.

Let ψ be a distribution function such that

$$\psi(0)-\psi(0-)=\pi,$$
$$1-\psi(t)=(1-\pi)\exp\{-\lambda t\},\quad t>0, \tag{5.1}$$

where λ is a known strictly positive constant and $0\le\pi\le 1$.

In this case $\widehat{M}^{\psi}_t=\frac{r}{\sigma}\int\limits_0^t\pi_s dX_s$ and

$$L_t(M,\widehat{M}^{\psi})=\frac{r}{\sigma}\Big(X_t-\frac{r}{\sigma}\int\limits_0^t\pi_s ds\Big), \tag{5.2}$$

where $\overline{W}_t=X_t-\frac{r}{\sigma}\int\limits_0^t\pi_s ds$ is a Wiener process with respect to the measure $\widehat{P}^{\psi}$ which we shall denote hereafter by P^{π}. Note also that in this case $\dfrac{1}{1-\psi(s)}\psi(ds)=\lambda ds$.

Therefore, it follows from equation (3.18) (see Remark 3.2) that in this case the equation for π_t coincides with the equation derived in [11]

$$\pi_t=\pi_0+\frac{r}{\sigma}\int\limits_0^t\pi_s\big(1-\pi_s\big)d\overline{W}_s+\lambda\int\limits_0^t(1-\pi_s)ds. \tag{5.3}$$

Lemma 5.1. *Let $a\le\pi$, where $a,\pi\in[0,1)$. Then*

$$0<\lambda(1-a)\int\limits_0^\infty P^{\pi}(\pi_s\le a)ds\le E^{\pi}\mathcal{L}^a_\infty(\pi)\le 2(1-\pi). \tag{5.4}$$

Proof. By Itô-Tanaka formula

$$|\pi_t-a|=|\pi-a|+\lambda\int\limits_0^t(1-\pi_s)\operatorname{sign}(\pi_s-a)ds+\mathcal{L}^a_t(\pi)+$$
$$+\frac{r}{\sigma}\int\limits_0^t\pi_s(1-\pi_s)\operatorname{sign}(\pi_s-a)d\widetilde{W}_t. \tag{5.5}$$

Taking expectations with respect to the measure P^{π}, since the stochastic integral from (5.5) is a martingale, we have

$$E^{\pi}\mathcal{L}^a_t(\pi)=E^{\pi}|\pi_t-a|-|\pi-a|-\lambda E^{\pi}\int\limits_0^t(1-\pi_s)\operatorname{sign}(\pi_s-a)ds. \tag{5.6}$$

Since (3.7) and (5.1) imply that

$$E^{\pi}(1-\pi_s)=(1-\pi)\exp\{-\lambda t\}, \tag{5.7}$$

from (5.6) we obtain

$$E^\pi \mathcal{L}_t^a(\pi) \le E^\pi|\pi_t - a| - |\pi - a| + \lambda \int_0^t E^\pi(1-\pi_s)ds \le$$
$$\le E^\pi|\pi_t - a| - |\pi - a| + (1-\pi)\big(1 - \exp\{-\lambda t\}\big).$$

Therefore, the passage to the limit as $t \to \infty$ in the last inequality, taking in mind that $\lim_{t\to\infty} \pi_t = 1$, gives the second inequality of (5.4)

$$E^\pi \mathcal{L}_\infty^a(\pi) \le 1 - a - (\pi - a) + (1-\pi) = 2(1-\pi).$$

On the other hand, from (5.6) we also have

$$E^\pi \mathcal{L}_t^a(\pi) = E^\pi|\pi_t - a| - |\pi - a| - \lambda E^\pi \int_0^t (1-\pi_s) I_{(\pi_s > a)} ds +$$
$$+\lambda E^\pi \int_0^t (1-\pi_s) I_{(\pi_s \le a)} ds \ge E^\pi|\pi_t - a| - |\pi - a| -$$
$$-\lambda E^\pi \int_0^t E^\pi(1-\pi_s)ds + \lambda(1-a)E^\pi \int_0^t I_{(\pi_s \le a)} ds. \tag{5.8}$$

It follows from (5.7) and relation $\lim\limits_{t\to\infty} \pi_t = 1$ that for $\pi \ge a$

$$\lim_{t\to\infty}\left(E^\pi|\pi_t - a| - |\pi - a| - \lambda \int_0^t E^\pi(1-\pi_s)ds\right) = 0.$$

Therefore, passing to the limit in (5.8) we obtain the validity of the inequality

$$E^\pi \mathcal{L}_\infty^a(\pi) \ge \lambda(1-a) \int_0^\infty P^\pi(\pi_s \le a) ds.$$

Finally, since

$$\int_{(\pi-\varepsilon,\pi+\varepsilon)} \frac{1+\lambda(1-x)}{x^2(1-x)^2} dx < \infty \text{ for some } \varepsilon > 0,$$

at every $\pi \in (0,1)$, the process π_t is regular in $(0,1)$ (see e.g., [2]). This means that π_t reaches a level x with positive probability starting at π, for every x and π from $(0,1)$. Therefore $\int_0^\infty P^\pi(\pi_s \le a)ds$ is strictly positive. □

Assume that $K_t = ct$. So, the cost criterion is of the same form as in [11]

$$\rho_\tau(\pi) = P^\pi(\tau < \theta) + cE_\pi \max(\tau - \theta, 0), \tag{5.9}$$

and the value function of the optimal stopping problem (2.10) is

$$\rho(\pi)=\inf_{\tau} E_{\pi}\left(1-\pi_{\tau}+c\int_{0}^{\tau}\pi_s ds\right). \tag{5.10}$$

Since $(\pi_t,\mathcal{F}_t,P^{\pi})$ is a time-homogeneous Markov process, we have that

$$V_t=\rho(\pi_t)\ \text{ a.s. for all }\ t\geq 0. \tag{5.11}$$

According to the general theory of optimal stopping the optimal stopping rule is

$$\tau^{*}=\inf\big\{t:\rho(\pi_t)=1-\pi_t\big\}. \tag{5.12}$$

Since $\rho(\pi)$ is concave by Lemma 4.1, $\rho(\pi)\leq 1-\pi$ and $\rho(\pi)=1-\pi$, if $\pi=1$, we have that $\rho(\pi)=1-\pi$ for all $\pi\geq A^*$ and $\rho(\pi)<1-\pi$, if $\pi<A^*$, where

$$A^{*}=\inf\big\{A:\rho(A)=1-A\big\}.$$

Therefore, the optimal stopping time of (2.10) is in this case of the form

$$\tau^{*}(\pi)=\inf\big\{t:\pi_t\geq A^{*}\big\} \tag{5.13}$$

and the aim is to calculate $\rho(\pi)$ and the constant A^*. This was done in [11] first solving a suitable free boundary problem and then showing that the unique solution of this problem is the value function. Our main aim in this section is to show that since the value process $V_t=\rho(\pi_s)$ satisfies RBSDE (4.3), the value function $\rho(\pi)$ will be the solution of the free boundary problem considered by Shiryaev.

Theorem 5.1. *The value function $\rho(\pi)$ is a non-negative continuously differentiable concave function on $(0,1]$ and there is a constant $A^*\in(0,1]$ such that:*

1) *$\rho(\pi)$ is twice continuously differentiable on $(0,A^*)$ and satisfies the PDE*

$$\frac{r^2}{2\sigma^2}\,\pi^2(1-\pi)^2\rho''(\pi)+\lambda(1-\pi)\rho'(\pi)=-c\pi,\quad \textit{if}\quad 0\leq\pi<A^*, \tag{5.14}$$

2) *$\rho(\pi)$ is equal to $1-\pi$ if $\pi\geq A^*$ and*

3) *satisfies the smooth fit condition*

$$\rho'(A^*)=-1;$$

Besides the value function satisfies the normal entrance condition:

$$\rho'(0+)=0.$$

Conversely, if $\widetilde{\rho}(\pi)$ is a non-negative concave function satisfying 1), 2), 3) for some $B^*\in(0,1]$, then the triple $Y_t=\widetilde{\rho}(\pi_t)$, $\nu_t=0$ and L_t equal to the martingale part of $\widetilde{\rho}(\pi_t)$ satisfies the RBSDE I)-VI). In particular this implies that $\widetilde{\rho}(\pi)=\rho(\pi)$ and $A^*=B^*$.

Proof. Let $D = \{\pi : \rho(\pi) < 1 - \pi\}$ and let ∂D be the boundary of this set. It is evident that $\rho(\pi) \le 1 - \pi$ and $\rho(1) = 0$ (since $\pi_t = 1$ for all $t \ge 0$, if $\pi_0 = 1$). Therefore, the concavity of $\rho(\pi)$ implies that ∂D contains only one point (say A^*) and according to Theorem 6 of [5] $L^0(1 - \pi - V) = 0$, which means that the process μ_t from (4.15) is indistinguishable from zero.

Thus (5.11) and (4.15) imply that the value process $V_t = \rho(\pi_t)$ satisfies equation

$$\rho(\pi_t) = \rho(\pi_0) -$$
$$-c \int_0^t \pi_s I_{(\rho(\pi_s) < 1 - \pi_s)} ds - \lambda \int_0^t (1 - \pi_s) I_{(\rho(\pi_s) = 1 - \pi_s)} ds + N_t. \tag{5.15}$$

Since $\rho(\pi)$ is concave, $\rho(\pi) \le 1 - \pi$ and $\rho(\pi) = 1 - \pi$ if $\pi = 1$, we have that $\rho(\pi) = 1 - \pi$ for all $\pi \ge A^*$ and $\rho(\pi) < 1 - \pi$ if $\pi < A^*$, where

$$A^* = \inf \big\{ A : \rho(A) = 1 - A \big\} = \partial D.$$

Besides, the optimal stopping rule is of the form (5.13) and

$$\big\{(\omega, s) : \rho(\pi_s) < 1 - \pi_s\big\} = \big\{(\omega, s) : \pi_s < A^*\big\},$$
$$\big\{(\omega, s) : \rho(\pi_s) = 1 - \pi_s\big\} = \big\{(\omega, s) : \pi_s \ge A^*\big\}.$$

Therefore, there exists $A^* \in (0, 1)$ such that $\rho(\pi_t)$ satisfies equation

$$\rho(\pi_t) = \rho(\pi_0) - c \int_0^t \pi_s I_{(\pi_s < A^*)} ds -$$
$$- \lambda \int_0^t (1 - \pi_s) I_{(\pi_s \ge A^*)} ds + \int_0^t Z_s d\widetilde{W}_s, \tag{5.16}$$

where $N = Z \cdot \widetilde{W}$ by integral representation theorem.

Since $\rho(\pi)$ is concave , by Tanaka–Meyer's formula

$$\rho(\pi_t) = \rho(\pi_0) + \lambda \int_0^t \rho'_-(\pi_s)(1 - \pi_s) ds + \frac{1}{2} \int_R \mathcal{L}_t^a(\pi) \nu''(da) +$$
$$+ \frac{r}{\sigma} \int_0^t \rho'_-(\pi_s) \pi_s (1 - \pi_s) d\widetilde{W}_s, \tag{5.17}$$

where $L_t^a(\pi)$ is the local time at the point a of the process π_t, ρ'_- is the left-hand derivative of $\rho(\pi)$ and ν'' is the measure of the second derivative of ρ.

Comparing the parts of finite variations of (5.17) and (5.16), taking in mind that $\rho'_-(\pi_s)=-1$ on the set $\{\pi_s>A^*\}$, we have

$$\frac{1}{2}\int\limits_R \mathcal{L}^a_t(\pi)\nu''(da)=-\int\limits_0^t \big[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)\big]I_{(\pi_s<A^*)}ds. \tag{5.18}$$

Let $h(x), x\in R$ be a bounded measurable function. Since the measure $d\mathcal{L}^a_t(\pi)$ is a.s. carried by the set $\{t:\pi_t=a\}$, integrating the process $h(\pi_s)\pi_s^2(1-\pi_s)^2$ with respect to the both parts of equality (5.18) and using Fubini's theorem we get

$$\int\limits_R \mathcal{L}^a_t(\pi)h(a)a^2(1-a)^2\nu''(da)=$$

$$=-\int\limits_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2\big[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)\big]I_{(\pi_s<A^*)}ds. \tag{5.19}$$

By the occupation formula (see e.g., [9])

$$\int\limits_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2\big[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)\big]I_{(\pi_s<A^*)}ds=$$

$$=\frac{\sigma^2}{r^2}\int\limits_0^t h(\pi_s)\big[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)\big]I_{(\pi_s<A^*)}d\langle\pi\rangle_s=$$

$$=\frac{\sigma^2}{r^2}\int\limits_R \mathcal{L}^a_t(\pi)h(a)\big[ca+\lambda(1-a)\rho'_-(a)\big]I_{(a<A^*)}da. \tag{5.20}$$

Therefore,

$$\int\limits_{[0,1]} \mathcal{L}^a_t(\pi)h(a)a^2(1-a)^2\nu''(da)=$$

$$=-\frac{2\sigma^2}{r^2}\int\limits_{[0,1]} \mathcal{L}^a_t(\pi)h(a)\big[ca+\lambda(1-a)\rho'_-(a)\big]I_{(a<A^*)}da. \tag{5.21}$$

Since $\rho(\pi)$ is concave and decreasing we have that $-1\le\rho'_-\le 0$ and we may use Fubini's theorem and the Lebesgue theorem of monotone convergence, i.e., taking mathematical expectations with respect to the measure P^π (for some $\pi<1$) and passing to the limit as $t\to\infty$ in the last equality, we obtain that

$$\int\limits_R h(a)a^2(1-a)^2E^\pi\mathcal{L}^a_\infty(\pi)\nu''(da)=$$

$$= -\frac{2\sigma^2}{r^2} \int\limits_R h(a)\big[ca + \lambda(1-a)\rho'_-(a)\big] I_{(a<A^*)} E^\pi \mathcal{L}^a_\infty(\pi) da \tag{5.22}$$

for any bounded measurable function h.

Since by Lemma 5.1 we have $0 < E^\pi \mathcal{L}^a_\infty(\pi) < \infty$ for all a, π such that $0 \le a \le \pi < 1$, (5.22) and the arbitrariness of the function h imply that the measure $\nu''(da)$ is absolutely continuous with respect to the Lebesgue measure on $(0,1)$ and, hence, $\rho(\pi)$ admits a second order generalized derivative. Therefore, by embedding theorem (see [13]) there exists the first derivative of $\rho(\pi)$ in the usual sense and this derivative is continuous.

If we denote by $\rho''(\pi)$ the second order generalized derivative of ρ from (5.22) we have that a.e. with respect to the Lebesgue measure the value function $\rho(\pi)$ satisfies the differential equation

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi) = -\lambda(1-\pi)\rho'(\pi) - c\pi \tag{5.23}$$

on the open interval $(0, A^*)$.

Since equality (5.23) is fulfilled on the set $(0, A^*)$ a.e. with respect to the Lebesgue measure and the right-hand-side of (5.23) is continuous, then there exists a modification of $\rho''(\pi)$ (for convenience we denote this modification also by $\rho''(\pi)$) which is continuous on $(0, A^*)$). It is evident that the continuous modification of $\rho''(\pi)$ coincides with the ordinary second order derivative of ρ and equation (5.23) is satisfied for all $\pi \in (0, A^*)$.

Since $\rho(\pi) = 1-\pi$ for all $\pi \ge A^*$ and $\rho(\pi)$ admits a continuous derivative, we have that $\rho'(\pi) = -1$ for all $\pi \ge A^*$ and, therefore, the constant A^* one can calculate from the smooth fit condition

$$\rho'(A^*) = -1.$$

Let us show now that $\rho'(0) = 0$. We shall first show that the value function $\rho(\pi)$ is a decreasing function. Let $\pi \le \pi' \le A^*$ and define $\sigma = \inf\{t : \pi^\pi_t \ge \pi'\}$. It is evident that $\pi^\pi_\sigma = \pi'$ and it follows from equation (5.16) that

$$\rho(\pi^\pi_\sigma) = \rho(\pi) - c\int\limits_0^\sigma \pi^\pi_s I_{(\pi^\pi_s < A^*)} ds + \int\limits_0^\sigma Z_s d\overline{W}_s. \tag{5.24}$$

Since $Z \cdot \overline{W}$ is a martingale and $\rho(\pi^\pi_\sigma) = \rho(\pi')$, taking expectations in (5.24) we obtain that

$$\rho(\pi') - \rho(\pi) = -cE^\pi \int\limits_0^\sigma \pi^\pi_s ds \le 0.$$

Let $(\pi_n, n \ge 1)$ be a sequence such that $\pi_n \downarrow 0$. Then from (5.23)

$$\frac{r^2}{2\sigma^2}\pi_n^2(1-\pi_n)^2\rho''(\pi_n) = -\lambda(1-\pi_n)\rho'(\pi_n) - c\pi_n \tag{5.25}$$

for each $n \geq 1$. Since $\rho'(\pi)$ is continuous, the limit as $n \to \infty$ of the right-hand side exists and is equal to $-\lambda\rho'(0+)$. Therefore there exists the limit of the left-hand side and since $\rho(\pi)$ is concave, this limit is non-positive, i.e., $\rho'(0+) \geq 0$. But since $\rho(\pi)$ is decreasing, $\rho'(\pi_n)$ is non-positive and, hence, the limit of the right-hand side is non-negative, i.e., $\rho'(0+) \leq 0$. Thus $\rho'(0+) = 0$ and equation (5.23) for $\pi = 0$ is also satisfied.

Thus, we have showed that the value function $\rho(\pi)$ is a concave function admitting the second order derivative ($\rho''(\pi)$ can be discontinuous only at points $\pi = 0$ and $\pi = A^*$) and it satisfies the free boundary problem 1), 2), 3).

Conversely, let $\widetilde{\rho}(\pi)$ be a non-negative concave function satisfying 1), 2), 3) for some $B^* \in (0, 1]$. Then by Itô's formula

$$\widetilde{\rho}(\pi_t) = \widetilde{\rho}(\pi_0) + \lambda \int_0^t \widetilde{\rho}\,'(\pi_s)(1-\pi_s)ds +$$
$$+\frac{r^2}{2\sigma^2}\int_0^t \pi_s^2(1-\pi_s)^2\widetilde{\rho}\,''(\pi_s)ds + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\widetilde{\rho}\,'(\pi_s)d\widetilde{W}_s. \tag{5.26}$$

Since $\widetilde{\rho}\,''(\pi) = 0$ and $\widetilde{\rho}\,'(\pi) = -1$ for all $\pi > B^*$, it follows from (5.14) and (5.26) that

$$\widetilde{\rho}(\pi_t) = \widetilde{\rho}(\pi_0) - \lambda \int_0^t (1-\pi_s)I_{(\pi_s \geq B^*)}ds - c\int_0^t \pi_s I_{(\pi_s < B^*)}ds +$$
$$+\frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\widetilde{\rho}\,'(\pi_s)d\widetilde{W}_s. \tag{5.27}$$

Let $\widetilde{A} = \inf\{A : \widetilde{\rho}(A) = 1 - A\}$. Since $\widetilde{\rho}(\pi)$ is concave, the smooth fit condition $\widetilde{\rho}\,'(B^*) = -1$ implies that $B^* \in [\widetilde{A}, 1]$. On the other hand if $B^* > \widetilde{A}$ then on the interval $(\widetilde{A}, B^*)$ we shall have $\widetilde{\rho}\,''(\pi) = 0, \widetilde{\rho}\,'(\pi) = -1$ and for any $\pi \in (\widetilde{A}, B^*)$ equation (5.14) will not be satisfied. Thus $B^* = \widetilde{A}$ and

$$\begin{aligned} \{\pi_s < B^*\} &= \{\widetilde{\rho}(\pi_s) < 1 - \pi_s\}, \\ \{\pi_s \geq B^*\} &= \{\widetilde{\rho}(\pi_s) = 1 - \pi_s\}. \end{aligned} \tag{5.28}$$

From (5.27) and (5.28) we obtain that

$$\widetilde{\rho}(\pi_t) = \widetilde{\rho}(\pi_0) - \lambda \int_0^t (1-\pi_s)I_{(\widetilde{\rho}(\pi_s) = 1 - \pi_s)}ds -$$

$$- c \int_0^t \pi_s I_{(\widetilde{\rho}(\pi_s)<1-\pi_s)} ds + \frac{r}{\sigma} \int_0^t \pi_s(1-\pi_s)\widetilde{\rho}'(\pi_s) d\widetilde{W}_s. \tag{5.29}$$

We shall show now that $\frac{\lambda}{\lambda+c} \le B^*$. Indeed, passing to the limit in (5.23) as $\pi \uparrow B^*$ and using the smooth fit condition we have that

$$-\frac{r^2}{2\sigma^2}(B^*)^2(1-B^*)^2 \liminf_{\pi\uparrow B^*} \rho''(\pi) \le cB^* - \lambda(1-B^*). \tag{5.30}$$

From the concavity of the function $\rho(\pi)$ we have that the left-hand side of this inequality is non-negative and hence $\frac{\lambda}{\lambda+c} \le B^*$. This inequality implies that $c\pi_s - \lambda(1-\pi_s)$ is positive on the set $\pi_s \ge B^*$. Therefore, we can rewrite (5.27) in the following form:

$$\widetilde{\rho}(\pi_t) = \widetilde{\rho}(\pi_0) - c\int_0^t \pi_s ds + \int_0^t (c\pi_s - \lambda(1-\pi_s))^+ I_{(\widetilde{\rho}(\pi_s)=1-\pi_s)} ds +$$
$$+ \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\widetilde{\rho}'(\pi_s) d\widetilde{W}_s, \tag{5.31}$$

which enables us to conclude that the triple

$$Y_t = \widetilde{\rho}(\pi_t), \quad \nu_t = 0, \quad L_t = \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\widetilde{\rho}'(\pi_s) d\widetilde{W}_s$$

satisfies the RBSDE (4.3). It is easy to see that this triple satisfies I)–V). Indeed, since $\widetilde{\rho}(\pi)$ is concave, condition 2) implies that $\widetilde{\rho}(\pi_t) \le 1 - \pi_t$ for all $t \ge 0$ and $\lim_{t\to\infty} \widetilde{\rho}(\pi_t) \le \lim_{t\to\infty}(1-\pi_t) = 0$. Besides, the positivity of $\widetilde{\rho}(\pi)$ implies that $\lim_{t\to\infty}\widetilde{\rho}(\pi_t) = 0$ and that $\widetilde{\rho}(\pi_t)$ is bounded. Therefore, it follows from (5.27) that $\widetilde{\rho}(\pi_t)$ is a supermartingale from the class S^1. Thus condition I)-V) are satisfied and by Theorem 4.1 $\widetilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (5.11) $\widetilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\widetilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0,1]$. □

Thus we have proved that the RBSDE I)-VI) and the free boundary problem 1)-3) are equivalent. The solution of the free boundary problem 1)-3) is given in [11]. Following Shiryaev, if we denote $\rho'(\pi)$ by $g(\pi)$ from (5.14) we have that

$$g'(\pi) = -\frac{2\lambda\sigma^2}{r^2\pi^2(1-\pi)} g(\pi) - \frac{2c\sigma^2}{r^2\pi(1-\pi)^2}.$$

Since $g(0)=0$, we find that for $\pi < A^*$

$$g(\pi)=\rho'(\pi)=$$

$$=-\frac{2c\sigma^2}{r^2}\int_0^\pi \exp\Big\{-\frac{2\lambda\sigma^2}{r^2}(H(\pi)-H(y))\Big\}\frac{dy}{y(1-y)^2}, \tag{5.32}$$

where $H(y)=\ln\frac{y}{1-y}-\frac{1}{y}$. If we define A^* as a unique solution of equation $g(A^*)=-1$, then the value function $\rho(\pi)$ coincides with

$$\rho(\pi)=\begin{cases}1-A^*-\int_\pi^{A^*} g(x)dx, & 0\le\pi\le A^* \\ 1-\pi, & A^*\le\pi\le 1.\end{cases} \tag{5.33}$$

Remark 5.1. Let us note that the smooth fit of the second derivative can not be fulfilled and the second order derivative of $\rho(\pi)$ is discontinuous at the point A^*. Indeed 1)-3) implies that $\rho''(\pi)$ can be continuous only if $A^*=\frac{\lambda}{\lambda+c}$. On the other hand using the formula of integration by parts

$$\int_0^\pi \frac{y}{1-y}\exp\Big\{\frac{2\lambda\sigma^2}{r^2}H(y)\Big\}dH(y)=\frac{r^2}{2\lambda\sigma^2}\frac{\pi}{1-\pi}\exp\Big\{\frac{2\lambda\sigma^2}{r^2}H(\pi)\Big\}-$$

$$-\frac{r^2}{2\lambda\sigma^2}\int_0^\pi \exp\Big\{\frac{2\lambda\sigma^2}{r^2}H(y)\Big\}\frac{dy}{(1-y)^2} \tag{5.34}$$

and from (5.32) and (5.34) we obtain that

$$g(\pi)>-\frac{c}{\lambda}\frac{\pi}{1-\pi}. \tag{5.35}$$

Therefore on the set $\{\pi:\pi\le\frac{\lambda}{\lambda+c}\}$ we have

$$g(\pi)>-1. \tag{5.36}$$

In particular $\rho'(\frac{\lambda}{\lambda+c})>-1$ and hence $A^*\ne\frac{\lambda}{\lambda+c}$. Thus, the second order derivative of the value function is discontinuous at the point A^*.

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Authors' addresses:

T. Kavtaradze
Istituto di Metodi Quantitativi,
University Commerciale "Luigi Bocconi"

N. Lazrieva and M. Mania
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 0193
Georgia