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A Bayesian-martingale approach to the general disorder problem

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Abstract

We consider a Bayesian-martingale approach to the general change-point detection problem. In our setting the change-point represents a random time of bifurcation of two probability measures given on the space of right-continuous functions. We derive a reflecting backward stochastic differential equation (RBSDE) for the value process related to the disorder problem and show that in classical cases of the Wiener and Poisson disorder problems this RBSDE is equivalent to free-boundary problems for parabolic differential and differential–difference operators respectively.

Keywords: Disorder problem; Change-point; Bayesian-martingale approach; Reflecting backward equation; Wiener process; Poisson process; Value process; Optimal stopping

1. Introduction

Classical disorder problems consider the detection of a change in the mean (or in other probabilistic characteristics) of a stochastic process X_t that occurs at a random time θ which is called the change-point. The Bayesian formulation of the problem, proposed in [16], assumes that the change-point θ admits a known prior distribution, although the variable θ itself is unknown to us, since it cannot be observed directly. A sequential change-point detection procedure

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is identified with a stopping time τ with respect to the filtration F_t^X of observable events (interpreted as the time at which the "alarm signal" is given), at which it is declared that a change has occurred. The aim of the problem is to find a stopping time τ, based on the observations X_t, which is "as close as possible" to the change-point θ. More exactly, the design of the quickest change-point detection procedures involves optimizing the trade-off between two kinds of performance measures, one being a measure of detection delay and the other being a measure of the frequency of false alarm.

Among all processes considered in the context of disorder problems, the Wiener process and the Poisson process take a central place; in these cases the problem can be solved explicitly. In [16] (see also [17]) an explicit solution of a Wiener disorder problem is derived, reducing the initial optimal stopping problem to a free-boundary problem for a parabolic differential operator. The Poisson disorder problem was first studied in [6], where the problem was solved in some particular cases. Their results have been extended in [2], where lesser restrictions on the model parameters were required. The complete solution of the Poisson disorder problem was given in [12] by reducing the initial optimal stopping problem to the free-boundary problem for a differential–difference operator. Note that in all these papers the case of infinite time horizon is considered.

In this paper we present a Bayesian-martingale approach to the general disorder problem with infinite time horizon where the change-point represents a random time of bifurcation of two probabilistic measures given on the space of right-continuous functions. The setting of the problem is discussed in Section 2.

In Section 3 we derive a martingale stochastic differential equation (SDE) for the a posteriori probability process π_t of the change-point θ, which plays, as is well known, a crucial role by reducing the disorder problem to an optimal stopping problem and determining the value process and the optimal stopping rule.

In Section 4 we introduce the value process of the related optimal stopping problem and show that this process uniquely solves a suitable reflecting backward stochastic differential equation (RBSDE). The value functions related to disorder problems (or to an optimal stopping problem in general) of Markov processes are usually solutions of suitable free-boundary problems. So the RBSDE for the value processes and the free-boundary problems for the value functions should be equivalent in some sense, at least in simple cases when the a posteriori probability process π_t is a sufficient statistic and the value process V_t of the problem is related to the value function $\rho(\pi)$ of the same problem by the equality $V_t = \rho(\pi_t)$. The problem is to deduce the differentiability properties and smooth fit conditions for the value functions, based on the properties of the process $\rho(\pi_t)$ being a solution of a RBSDE. We consider classical disorder problems for Wiener and Poisson processes and show that in these cases related RBSDEs for value processes and the corresponding free-boundary problems are equivalent.

In Section 5 we consider the disorder problem for a Wiener process. This problem was solved in [16], where an explicit expression for the value function $\rho(\pi)$ of the initial stopping problem was given and it was shown that this function (together with the optimal threshold A^*) uniquely solves the corresponding free-boundary problem. On the basis of results of Section 4, we give a probabilistic proof of this fact. We show that $\rho(\pi)$ is a solution of the free-boundary problem if and only if the process $\rho(\pi_t)$ is a solution of the corresponding RBSDE. The key step here is showing that if the value process $V_t = \rho(\pi_t)$ satisfies the RBSDE, then the function $\rho(\pi)$ is continuously differentiable on $(0, 1]$ and twice continuously differentiable on $(0, A^*)$, $0 < A^* < 1$. In particular this implies that the smooth fit condition is satisfied. Besides, we show that the smooth fit of the second derivative fails.

In Section 6 we consider the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ. As mentioned above, the closed form solution of this problem was given in [12], where the problem was reduced to a free-boundary differential–difference problem. We show that this free-boundary problem is also equivalent to the well posedness of the general RDSDE studied in Section 4. In particular, this shows that the unique solution of the free-boundary differential–difference problem coincides with the value function of the problem. Besides, we derive the smooth fit conditions for the value function (in cases when this condition is satisfied) and establish when the smooth fit condition breaks down directly from the RBSDE for the value process.

2. Bayesian statement of the disorder problem

In this section after some preliminaries we discuss the Bayesian statement of the problem for a general martingale model.

Let $(\Omega, \mathcal{F}, F = (F_t, t \geq 0))$ be a filtered measurable space with $\mathcal{F} = F_\infty$. Assume that P^0 and P^1 are two fixed locally equivalent probability measures ($P^1 \overset{\text{loc}}{\sim} P^0$) defined on $(\Omega, \mathcal{F})$ and let $\psi = \psi(x)$ be a distribution function of some non-negative random variable. Without loss of generality (e.g., taking $P = \frac{1}{2}(P^1 + P^0)$) one can assume that there is a probability measure P on $(\Omega, \mathcal{F})$ such that

$$P^1 \ll P, \qquad P^0 \ll P, \qquad P^1 \overset{\text{loc}}{\sim} P, \qquad P^0 \overset{\text{loc}}{\sim} P.$$

For simplicity let us assume that the restrictions of the measures P^0 and P^1 coincide on the σ-algebra $\mathcal{F}_0$.

Let $(Z_t^i = \frac{\mathrm{d}P_t^i}{\mathrm{d}P_t}, t \geq 0)$, $i = 0, 1$, be the density process of the measure P^i relative to P, which is an uniformly integrable P-martingale with $Z_t^i > 0 P$-a.s. for any $t \in [0, \infty[$. Then there exists a local martingale $M^i \in \mathcal{M}_{\text{loc}}(F, P)$ such that

$$Z^i = \mathcal{E}(M^i) = (\mathcal{E}_t(M^i), t \geq 0), \quad i = 0, 1,$$

where $\mathcal{E}(M)$, called the Doléans exponential of M, is the unique solution of the linear Stochastic Differential Equation (SDE)

$$Z_t = 1 + \int_0^t Z_{s-} \mathrm{d}M_s \tag{2.1}$$

(see, e.g., [8] or [9]).

For the statement of the problem in a general martingale setting let us extend the initial probability space as follows:

$\overline{\Omega} = \Omega \otimes R^+$, $\overline{F} = \mathcal{F} \otimes \mathcal{B}(R^+)$, $\overline{F}_t = F_t \otimes \mathcal{B}(R^+)$, where $\mathcal{B}(R^+)$ is the Borel σ-algebra on $R^+ = [0, \infty)$.

The measure $\overline{P}^\psi$ on $\mathcal{F} \otimes \mathcal{B}(R^+)$ is defined in a following way: let for every $A \in \mathcal{F}$ and $B \in \mathcal{B}(R^+)$

$$\overline{P}^\psi(A \times B) = \int_A \int_B \mathcal{E}_\infty(M^x) \psi(\mathrm{d}x) P(\mathrm{d}\omega), \tag{2.2}$$

where

$$M_t^x = \int_0^t I_{\{x \leq s\}} \mathrm{d}M_s^1 + \int_0^t I_{\{x > s\}} \mathrm{d}M_s^0. \tag{2.3}$$

Note that, since

$$E\mathcal{E}_\infty(M^x) = E\mathcal{E}_{x-}(M^0)\frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_{x-}(M^1)} = E\mathcal{E}_{x-}(M^0)E\left(\frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_{x-}(M^1)}\Big/ F_{x-}\right) = 1,$$

the Fubini theorem implies that $\overline{P}^\psi$ is a probability measure.

Let us denote by P^ψ the restriction of the measure $\overline{P}^\psi$ on the σ-algebra $\mathcal{F}\otimes R^+$.

For every $u < v$ and t we have

$$\begin{aligned}\int_{(u,v]}\mathcal{E}_t(M^x)\psi(\mathrm{d}x) &= \int_{(u,v]} I_{\{x>t\}}\mathcal{E}_t(M^0)\psi(\mathrm{d}x) + \int_{(u,v]} I_{\{x\le t\}}\mathcal{E}_t(M^x)\psi(\mathrm{d}x)\\ &= \mathcal{E}_t(M^0)(\psi(v\vee t) - \psi(u\vee t))\\ &\quad + \mathcal{E}_t(M^1)\int_{(u\wedge t, v\wedge t]}\frac{\mathcal{E}_{x-}(M^0)}{\mathcal{E}_{x-}(M^1)}\psi(\mathrm{d}x). \end{aligned} \tag{2.4}$$

So, we could define the measure $\overline{P}^\psi$ just by P^0, P^1 and ψ. For every $u < v$ and $A\in\mathcal{F}_t$

$$\overline{P}^\psi(A\times]u,v]) = (\psi(v\vee t)-\psi(u\vee t))P^0(A) + \int_A\int_{(u\wedge t,v\wedge t]}\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(\mathrm{d}s)\mathrm{d}P^1.$$

If we denote by P_t^ψ the restriction of the measure $\overline{P}_t^\psi$ on the σ-algebra $\mathcal{F}_t \equiv F_t\times R^+$, we will have for every $A\in F_t$

$$P_t^\psi(A) = \overline{P}_t^\psi(A\times R^+) = (1-\psi(t))P^0(A) + \int_A\int_{[0,t]}\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(\mathrm{d}s)\mathrm{d}P^1, \tag{2.5}$$

where we assume that $\frac{\mathcal{E}_{0-}(M^0)}{\mathcal{E}_{0-}(M^1)} = 1$;

Thus, the measures $\overline{P}_t^\psi$ (and P_t^ψ) do not depend on the choice of the dominating measure P. It is easy to see that $P^\psi \ll P$ and

$$Z_t^\psi \equiv \frac{\mathrm{d}P_t^\psi}{\mathrm{d}P_t} = (1-\psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1)\int_{[0,t]}\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(\mathrm{d}s). \tag{2.6}$$

According to (2.2), $Z_t^\psi = \int_{R+}\mathcal{E}_t(M^x)\psi(\mathrm{d}x)$.

Remark 2.1. Since $P^1 \overset{\text{loc}}{\sim} P^0$, we have that $P^\psi \overset{\text{loc}}{\sim} P^0$ and one can express the density process $\hat{Z}_t^\psi = \mathrm{d}P_t^\psi/\mathrm{d}P_t^0$ in the form

$$\hat{Z}_t^\psi = \frac{\mathrm{d}P_t^\psi}{\mathrm{d}P_t^0} = (1-\psi(t)) + \mathcal{E}_t(M)\int_{[0,t]}\mathcal{E}_{s-}^{-1}(M)\psi(\mathrm{d}s), \tag{2.7}$$

where $Z_t = (\mathcal{E}_t(M), t\ge 0)$ is the density process of P^1 relative to P^0.

Let us define on the space $(\overline{\Omega},\overline{F})$ the random variable

$$\theta = \theta(\overline{\omega}) = \theta(\omega,x) = x.$$

It is evident from (2.2) that

$$\overline{P}^\psi(\theta\le x) = \overline{P}^\psi(\Omega\times[0,x]) = \psi(x).$$

This means that the distribution function $\psi = \psi(x)$ by means of which we have defined the new measure $\overline{P}^{\psi}$ on the extended measurable space $(\overline{\Omega}, \overline{F})$ comes to be the a priori distribution function of the variable θ, associated with the random time of 'disorder'.

The aim of the problem is to find a stopping time τ with respect to the filtration F_t of observable events (interpreted as the time at which the "alarm signal" is given) which is "as close as possible" to the change-point θ. Following [16] we define the cost criterion by

$$V(\tau) = \bar{P}^{\psi}(\tau < \theta) + E^{\psi} \max(K_\tau - K_\theta, 0), \tag{2.8}$$

where $\bar{P}^{\psi}(\tau < \theta)$ is a probability of "false alarm" and $E^{\psi} \max(K_\tau - K_\theta, 0)$ is an average delay (measured by an F_t predictable increasing process K) of detecting the change-point correctly.

The stopping time τ^* is called optimal if

$$V(\tau^*) = \inf_{\tau} V(\tau), \tag{2.9}$$

where inf is taken over the class of all F-stopping times.

Introducing the a posteriori probability process π_t

$$\pi_t = \overline{P}^{\psi}(\theta \le t \mid \mathcal{F}_t),$$

similarly to [16] one can reduce problem (2.9) to the optimal stopping problem

$$V(\tau^*) = \inf_{\tau} E^{\psi}\left[(1 - \pi_\tau) + \int_0^\tau \pi_{s-} \mathrm{d}K_s\right], \tag{2.10}$$

since $\bar{P}^{\psi}(\tau < \theta) = E^{\psi}(1 - \pi_\tau)$ and

$$\bar{E}^{\psi} \max(K_\tau - K_\theta, 0) = \bar{E}^{\psi} \int_0^\tau I_{(\theta \le s)} \mathrm{d}K_s = E^{\psi} \int_0^\tau \pi_{s-} \mathrm{d}K_s$$

by the projection theorem.

Let us introduce the value process of the problem (2.10)

$$V_t = \operatorname*{ess\,inf}_{\tau \ge t} E^{\psi}\left[(1 - \pi_\tau) + \int_t^\tau \pi_{s-} \mathrm{d}K_s / F_t\right]. \tag{2.11}$$

It is well known that under some regularity conditions (see, e.g., [4]) the stopping time τ^* defined by

$$\tau^* = \inf\{t : V_t = 1 - \pi_t\} \tag{2.12}$$

is optimal for the problem (2.10). In the case of the Wiener disorder problem considered in [16] the optimal stopping time is of the following simple form:

$$\tau^* = \inf\{t : \pi_t \ge A^*\}, \tag{2.13}$$

where the constant A^* is a solution of a certain integral equation and the value function V is explicitly calculated as a function of $\psi(0) = \pi$ and A^*. Here the differential equation for the process π_t plays a crucial role.

In our general setting the process π_t is no longer sufficient for determining the optimal stopping rule; however the equation for π_t is essential to characterize the value process V_t as a solution of the corresponding RBSDE. Therefore, in the next section we focus our attention on the derivation of a stochastic differential equation for π_t.

3. Differential equation for the a posteriori distribution process

After giving some auxiliary facts and recalling properties of Girsanov's transform we derive the stochastic differential equation for the a posteriori distribution process of the change-point θ based on knowing its a priori distribution function ψ and the local martingales $M^i \in \mathcal{M}_{\text{loc}}(F, P), i = 0, 1$.

It follows from the generalized Bayes theorem (see, e.g., [10] or [18]) that

$$\pi_t = \frac{\int_{R^+} I_{(x \le t)} \mathcal{E}_t(M^x) \psi(\mathrm{d}x)}{Z_t^\psi}, \tag{3.1}$$

where

$$Z_t^\psi = \int_{R^+} \mathcal{E}_t(M^x) \psi(\mathrm{d}x). \tag{3.2}$$

Using (2.4) and (2.6) we get

$$\pi_t = \frac{\mathcal{E}_t(M^1) \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(\mathrm{d}s)}{(1 - \psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(\mathrm{d}s)}. \tag{3.3}$$

Dividing the numerator and the denominator of the right-hand side of (3.3) by $\mathcal{E}_t(M^0)$, one can write π_t also in a form not depending on the dominating measure P:

$$\pi_t = \frac{\mathcal{E}_t(M) \int_{[0,t]} \mathcal{E}_{s-}^{-1}(M) \psi(\mathrm{d}s)}{(1 - \psi(t)) + \mathcal{E}_t(M) \int_{[0,t]} \mathcal{E}_{s-}^{-1}(M) \psi(\mathrm{d}s)}, \tag{3.4}$$

where $\mathcal{E}_t(M) = \mathrm{d}P_t^1/\mathrm{d}P_t^0$ is the density process of P^1 relative to P^0.

Lemma 3.1. *The martingale Z_t^ψ is the Doléans exponential of the local martingale M^ψ (i.e., $Z_t^\psi = \mathcal{E}_t(M^\psi)$), where*

$$M_t^\psi = \int_0^t (1 - \pi_{s-}) \mathrm{d}M_s^0 + \int_0^t \pi_{s-} \mathrm{d}M_s^1 + \sum_{s \le t} (1 - \pi_{s-}) \frac{\Delta \psi_s}{1 - \psi(s-)} (\Delta M_s^1 - \Delta M_s^0). \tag{3.5}$$

Proof. Note that from (3.3) we have that

$$\pi_t Z_t^\psi = \mathcal{E}_t(M^1) \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(\mathrm{d}s), \tag{3.6}$$

$$(1 - \pi_t) Z_t^\psi = (1 - \psi(t)) \mathcal{E}_t(M^0). \tag{3.7}$$

Therefore, an application of Itô's rule to (2.6) yields

$$Z_t^\psi = 1 + \int_0^t (1 - \psi(s-)) \mathcal{E}_{s-}(M^0) \mathrm{d}M_s^0 + \int_0^t \int_{[0,s)} \frac{\mathcal{E}_{u-}(M^0)}{\mathcal{E}_{u-}(M^1)} \psi(\mathrm{d}u) \mathcal{E}_{s-}(M^1) \mathrm{d}M_1^s$$
$$+ \sum_{s \le t} \left[\triangle \mathcal{E}_s(M^1) \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} - \triangle \mathcal{E}_s(M^0) \right] \triangle \psi(s). \tag{3.8}$$

Since by Eq. (3.7)

$$\mathcal{E}_{t-}(M^0) = \frac{Z^{\psi}_{t-}(1-\pi_{t-})}{1-\psi(t-)}, \tag{3.9}$$

and

$$\Delta\mathcal{E}_t(M) = \mathcal{E}_{t-}(M)\Delta M, \tag{3.10}$$

we have that the last term of (3.8) equals

$$\sum_{s\le t}(1-\pi_{s-})Z^{\psi}_{s-}\frac{\Delta\psi_s}{1-\psi(s-)}(\Delta M^1_s - \Delta M^0_s). \tag{3.11}$$

Therefore, from (3.6)–(3.8) we obtain that $Z^{\psi}_t = \mathcal{E}_t(M^{\psi})$ satisfies

$$Z^{\psi}_t = 1 + \int_0^t Z^{\psi}_{s-}\left[(1-\pi_{s-})\mathrm{d}M^0_s + \pi_{s-}\mathrm{d}M^1_s + (1-\pi_{s-})\frac{\Delta\psi_s}{1-\psi(s-)}\mathrm{d}(M^1_s - M^0_s)\right] \tag{3.12}$$

and the assertion of lemma follows from the uniqueness of the solution of Eq. (2.1). □

Remark 3.1. Similarly to above, one can show that the density process $\hat{Z}^{\psi}_t$ defined by (2.7) admits the representation $\hat{Z}^{\psi}_t = \mathcal{E}_t(\hat{M}^{\psi})$, where

$$\hat{M}^{\psi}_t = \int_0^t \pi_{s-}\mathrm{d}M_s + \sum_{s\le t}(1-\pi_{s-})\frac{\Delta\psi_s}{1-\psi(s-)}\Delta M_s. \tag{3.13}$$

For two semimartingales X and Y, with $\Delta Y_t \neq -1$ for all t, let us denote by $L(X,Y)$ the Girsanov transform

$$L_t(X,Y) = X_t - \int_0^t \frac{1}{1+\Delta Y_s}\mathrm{d}[Y,X]_s.$$

Note that (see, e.g., [11])

$$\frac{\mathcal{E}_t(X)}{\mathcal{E}_t(Y)} = \mathcal{E}_t(L(X-Y,Y)). \tag{3.14}$$

Since for any X-integrable predictable process H

$$L(H\cdot X, Y) = H\cdot L(X,Y),$$

from (3.5)

$$L_t(M^1 - M^{\psi}, M^{\psi}) = \int_0^t (1-\pi_{s-})\mathrm{d}L_s \times \left(M^1 - M^0 - \sum_{u\le\cdot}\frac{\Delta\psi_u}{1-\psi(u-)}\Delta(M^1_u - M^0_u), M^{\psi}\right). \tag{3.15}$$

It is also evident that

$$\Delta L(X,Y) = \frac{\Delta X}{1+\Delta Y}$$

and, in particular,

$$\Delta L_t(M^1 - M^\psi, M^\psi) = (1-\pi_{t-})\frac{\Delta(M_t^1 - M_t^0)}{1+\Delta M_t^\psi}\frac{1-\psi(t)}{1-\psi(t-)}. \tag{3.16}$$

Theorem 3.1. *The a posteriori probability process π_t satisfies the following stochastic differential equation:*

$$\begin{aligned}\pi_t &= \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})\mathrm{d}L_s\left(M^1 - M^0 - \sum_{u\le\cdot}\frac{\Delta\psi_u}{1-\psi(u-)}\Delta(M_u^1 - M_u^0), M^\psi\right)\\ &\quad + \sum_{s\le t}(1-\pi_{s-})^2\frac{(1-\psi(s))}{(1-\psi(s-))^2}\frac{\Delta(M_s^1-M_s^0)}{1+\Delta M_s^\psi}\Delta\psi(s) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s).\end{aligned} \tag{3.17}$$

Proof. By virtue of (3.6) and (3.14)

$$\pi_t = \mathcal{E}_t(L(M^1 - M^\psi, M^\psi))\int_{[0,t]}\frac{\mathcal{E}_{x-}(M^0)}{\mathcal{E}_{x-}(M^1)}\psi(\mathrm{d}x). \tag{3.18}$$

From (3.18) using the Itò formula we have

$$\begin{aligned}\pi_t &= \pi_0 + \int_0^t\int_{[0,s)}\frac{\mathcal{E}_{x-}(M^o)}{\mathcal{E}_{x-}(M^1)}\psi(\mathrm{d}x)\mathcal{E}_{s-}(L(M^1-M^\psi, M^\psi))\mathrm{d}L_s(M^1-M^\psi, M^\psi)\\ &\quad + \int_{[0,t]}\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^\psi)}\psi(\mathrm{d}s) + \sum_{s\le t}\Delta\mathcal{E}_s(L(M^1-M^\psi, M^\psi))\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\Delta\psi(s).\end{aligned} \tag{3.19}$$

Eqs. (3.15) and (3.18) imply that the second term of the right-hand side of (3.19) is equal to

$$\int_0^t \pi_{s-}(1-\pi_{s-})\mathrm{d}L_s(M^1 - M^0 - \sum_{u\le s}\frac{\Delta\psi_u}{1-\psi(u-)}\Delta(M_u^1-M_u^0), M^\psi). \tag{3.20}$$

On the other hand, using successively (3.10), (3.14), (3.16) and (3.9) we obtain

$$\begin{aligned}&\sum_{s\le t}\Delta\mathcal{E}_s(L(M^1-M^\psi, M^\psi))\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\Delta\psi(s)\\ &\quad = \sum_{s\le t}\Delta L_s(M^1-M^\psi, M^\psi)\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^\psi)}\Delta\psi(s)\\ &\quad = \sum_{s\le t}(1-\pi_{s-})^2\frac{(1-\psi(s))}{(1-\psi(s-))^2}\frac{\Delta(M_s^1-M_s^0)}{1+\Delta M_s^\psi}\Delta\psi(s).\end{aligned} \tag{3.21}$$

Note that (3.9) also implies that the third term of the right-hand side of (3.19) is equal to

$$\int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s). \tag{3.22}$$

Therefore relations (3.19)–(3.22) imply that π_t satisfies the SDE (3.17). □

Remark 3.2. Sometimes it is more convenient to write Eq. (3.17) using the martingale $\hat{M}^{\psi}$ from Remark 3.1. Like for Theorem 3.1 one can show that π_t satisfies equation

$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})\mathrm{d}L_s\left(M - \sum_{u\le\cdot}\frac{\Delta\psi_u}{1-\psi(u-)}\Delta M_u, \hat{M}^{\psi}\right) + \sum_{s\le t}(1-\pi_{s-})^2\frac{(1-\psi(s))}{(1-\psi(s-))^2}\frac{\Delta M_s}{1+\Delta\hat{M}_s^{\psi}}\Delta\psi(s) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s). \quad (3.23)$$

In particular, if $\psi(t)$ is continuous, this equation for π_t takes the form

$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})\mathrm{d}L_s\left(M, \hat{M}^{\psi}\right) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s)}\psi(\mathrm{d}s).$$

Remark 3.3. Another form for the equation for the a posteriori distribution function $(\pi_t, t\ge 0)$ can be given by applying Itô's formula to the left-hand side of (3.7):

$$\pi_t = \pi_0 - \int_0^t (1-\pi_{s-})\frac{1-\psi(s)}{1-\psi(s-)}\mathrm{d}L_s\left(M^0 - M^{\psi}, M^{\psi}\right) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s). \quad (3.24)$$

4. Reflecting backward stochastic differential equation (RBSDE) for the value process

In this section we provide the reflecting BSDE for the value process of the optimal stopping problem (2.10).

Let us introduce the value process of the problem (2.10):

$$V_t = \operatorname*{essinf}_{\tau\ge t} E^{\psi}\left[(1-\pi_\tau) + \int_t^{\tau}\pi_{s-}\mathrm{d}K_s/F_t\right],$$

where E^{ψ} is an expectation w.r.t. the measure P^{ψ}, which we consider as a reference probability measure throughout this section.

It is well known that (see, e.g., [4]) V_t is a RCLL process such that

(i) $V_t \le 1-\pi_t$ for all t,
(ii) the process $V_t + \int_0^t \pi_{s-}\mathrm{d}K_s$ is a submartingale,
(iii) V_t is the largest process satisfying (i) and (ii).

Moreover for any $t\ge 0$ the stopping time τ^* defined by

$$\tau_t^* = \inf\{s\ge t : V_s = 1-\pi_s\}$$

is t-optimal (at least if K and ψ are continuous and F is quasi-left-continuous (see [4] or [7])), that is

$$V_t = E^{\psi}\left[(1-\pi_{\tau_t^*}) + \int_t^{\tau_t^*}\pi_{s-}\mathrm{d}K_s/F_t\right].$$

Hence V_t is a special semimartingale with canonical decomposition

$$V_t = V_0 - \int_0^t \pi_{s-}\mathrm{d}K_s + B_t + N_t, \quad (4.1)$$

where N is a martingale and B is a predictable increasing process with $B_0 = 0$.

It is also well known (see e.g. [4,7] and [15]) that increasing process B_t is growing only on the set $\{V_{t-} = 1 - \pi_{t-}\}$ (on the stop region) and $V_t + (\pi_- \cdot K)_t$ is a martingale on the go region $\{V_{t-} < 1 - \pi_{t-}\}$, i.e., the process B_t satisfies the relation

$$\int_0^T I_{\{V_{s-}<1-\pi_{s-}\}} \mathrm{d}B_s = 0, \tag{4.2}$$

which implies that the process

$$\int_0^t I_{\{V_{s-}<1-\pi_{s-}\}} \mathrm{d}\left(V_s + \int_0^s \pi_{u-} \mathrm{d}K_u\right) = \int_0^t I_{\{V_{s-}<1-\pi_{s-}\}} \mathrm{d}N_s$$

is a martingale.

Note that relation (4.2) guaranties the maximality of V and together with (i) and (ii) uniquely determines the value process. But the maximality of V, as well as condition (4.2), is difficult to verify and this leads to the necessity of giving a differential characterization of the value process. We shall combine the results of [1,7,5] and [15] to derive a reflecting BSDE for the process V in our case.

Denote by S^1 the class of semimartingales X with the decomposition

$$X_t = X_0 + A_t + M_t, \quad t \geq 0,$$

where M_t is a uniformly integrable martingale and A_t is a process of integrable variation on $[0, \infty]$.

We define a solution of RBSDE related to the disorder problem as a triple (Y_t, ν_t, L_t) of adapted processes satisfying:

(I) L_t is a uniformly integrable martingale,
(II) ν_t is a predictable process with $0 \leq \nu_t \leq 1$,
(III) Y_t is a semimartingale from S^1,
(IV) $Y_t \leq 1 - \pi_t$ for all $t \geq 0$,
(V) $\lim_{t\to\infty} Y_t = 0$, P^ψ-a.s.,
(VI)

$$\begin{aligned} Y_t = Y_0 &+ \int_0^t (1-\nu_s) I_{(Y_{s-}=1-\pi_{s-})} \mathrm{d}\left(\int_0^\cdot \pi_{u-} \mathrm{d}K_u - \int_0^\cdot \frac{1-\pi_{u-}}{1-\psi(u-)}\psi(\mathrm{d}u)\right)_s^+ \\ &- \int_0^t \pi_{s-} \mathrm{d}K_s + L_t. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Assume that:*

(A) *ψ is a distribution function concentrated on* $[0, \infty]$.

(B) *K is a predictable increasing process such that $EK_t < \infty$ for any $t \in [0, \infty)$.*

Then there exists a solution of RBSDE (4.3) satisfying (I)–(VI). *If a triple (Y_t, ν_t, L_t) satisfies conditions* (I)–(VI), *then $Y_t = V_t$ and L_t coincides with the martingale part of the value process V.*

Proof. Using Eq. (3.24) for π_t and the decomposition (4.1) we have

$$\begin{aligned} 1 - \pi_t - V_t = 1 - \pi_0 - V_0 &- \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s) + \int_0^t \pi_{s-} \mathrm{d}K_s \\ &- B_t + \int_0^t (1-\pi_{s-}) \frac{1-\psi(s)}{1-\psi(s-)} \mathrm{d}\tilde{M}_s - N_t, \end{aligned} \tag{4.4}$$

where by $\tilde{M}$ we denoted the P^{ψ}-martingale $\tilde{M}_t = L_t\left(M^0 - M^{\psi}, M^{\psi}\right)$.

By Tanaka's formula

$$(1-\pi_t - V_t)^+ = (1-\pi_0 - V_0)^+ + \int_0^t I_{\{1-\pi_{s-} > V_{s-}\}} \mathrm{d}(1-\pi_s - V_s) + \frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s - V_s), \tag{4.5}$$

where $\mathcal{L}_t^{1-\pi-V}(0)$ is the local time of the process $1-\pi_t - V_t$ at 0. Therefore, from (4.4) and (4.5)

$$\begin{aligned}(1-\pi_t - V_t)^+ &= (1-\pi_0 - V_0)^+ + \int_0^t I_{(1-\pi_{s-} > V_{s-})}\pi_{s-}\mathrm{d}K_s \\ &\quad - \int_0^t I_{(1-\pi_{s-} > V_{s-})}\frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s) - \int_0^t I_{(1-\pi_{s-} > V_{s-})}\mathrm{d}B_s \\ &\quad + \sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s - V_s) + \frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) \\ &\quad + \int_0^t I_{(1-\pi_{s-} > V_{s-})}[\pi_{s-}(1-\pi_{s-})\mathrm{d}\tilde{M}_s - \mathrm{d}N_s]. \end{aligned} \tag{4.6}$$

Since $V_t \le 1-\pi_t$ and $\int_0^t I_{(1-\pi_{s-} > V_{s-})}\mathrm{d}B_s = 0$, comparing the finite variation parts of the right-hand sides of (4.4) and (4.6) we obtain that

$$\begin{aligned}&\int_0^t I_{(1-\pi_{s-} = V_{s-})}\pi_{s-}\mathrm{d}K_s - \int_0^t I_{(1-\pi_{s-} = V_{s-})}\frac{1-\pi_{s-}}{1-\psi(s-)}\psi(\mathrm{d}s) \\ &\quad - \left(\frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \tilde{A}_t\right) = B_t \end{aligned} \tag{4.7}$$

where by $\tilde{A}_t$ we have denoted the compensator of the process $(\sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s - V_s), t \ge 0)$.

Since B, $\mathcal{L}(0)$ and $\tilde{A}$ are increasing processes, relation (4.7) implies that the measures $\mathrm{d}B_t$ and $\mathrm{d}(\mathcal{L}(0) + \tilde{A})_t$ are absolutely continuous w.r.t. the measure $\mathrm{d}K_t$. Moreover, from (4.7) we also have

$$\begin{aligned}&\int_0^t I_{(1-\pi_{s-} = V_{s-})}\mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\right)_s^+ - \left(\frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \tilde{A}_t\right) \\ &\quad = B_t + \int_0^t I_{(1-\pi_{s-} = V_{s-})}\mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\right)_s^- \end{aligned} \tag{4.8}$$

and the process $\int_0^t I_{(1-\pi_{s-} = V_{s-})}\mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\right)_s^+ - \left(\frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \tilde{A}_t\right)$ is increasing; hence there exists a predictable process μ_t such that $0 \le \mu_t \le 1$ and

$$\left(\frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \tilde{A}_t\right) = \int_0^t \mu_s I_{(1-\pi_{s-} = V_{s-})}\mathrm{d}\left(\int_0^{\cdot}\pi_{u-}\mathrm{d}K_u - \int_0^{\cdot}\frac{1-\pi_{u-}}{1-\psi(u-)}\psi(\mathrm{d}u)\right)_s^+, \tag{4.9}$$

where $A_t = A_t^+ - A_t^-$ is a unique decomposition of a process of finite variation A as a difference of two increasing processes such that the non-negative measures induced by A^+ and A^- on $[0, t]$ have disjoint supports. The total variation of such a process is given by $(\mathrm{Var}A)_t = A_t^+ + A_t^-$.

It follows from (4.8) and (4.9) that

$$\int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} \mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)_s^+ - \int_0^t I_{(1-\pi_{s-}=V_{s-})} \mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)_s^- = B_t \tag{4.10}$$

is an increasing process, which implies that

$$0 \le \mu_s \le I_{(1-\pi_{s-}=V_{s-})} \mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)_s^+ \text{-a.e.} \quad \text{and} \tag{4.11}$$

$$\{s : 1-\pi_{s-} = V_{s-}\} \subseteq \operatorname{supp}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)^+. \tag{4.12}$$

In particular, we have that

$$B_t = \int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} \mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)_s^+ = \int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} \mathrm{d}\left(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi\right)_s. \tag{4.13}$$

Therefore (4.13) and (4.1) imply that

$$V_t = V_0 + \int_0^t (1-\mu_s) I_{(V_{s-}=1-\pi_{s-})} \mathrm{d}\left(\int_0^\cdot \pi_{u-} \mathrm{d}K_u - \int_0^\cdot \frac{1-\pi_{u-}}{1-\psi(u-)} \psi(\mathrm{d}u)\right)_s^+ - \int_0^t \pi_{s-} \mathrm{d}K_s + N_t, \tag{4.14}$$

which means that the triple (V, μ, N) satisfies Eq. (4.3).

It follows from equality (4.13) that the value process satisfies also the equation

$$V_t = V_0 - \int_0^t (I_{(1-\pi_{s-}>V_{s-})} + \mu_s I_{(1-\pi_{s-}=V_{s-})}) \pi_{s-} \mathrm{d}K_s - \int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} \frac{1-\pi_{s-}}{1-\psi(s-)} \psi(\mathrm{d}s) + N_t, \tag{4.15}$$

which implies that V_t is a supermartingale. Since V is bounded, it is a supermartingale of class D, and by the uniqueness of the Doob–Meyer decomposition N is a uniformly integrable martingale and V is a semimartingale from class S^1.

Since $0 \le V_t \le 1 - \pi_t$ and $\lim_{t\to\infty} \pi_t = 1 P^\psi$-a.s. (the proof of this fact is the same as in [16]), we have that $\lim_{t\to\infty} V_t$ exists and is equal to zero.

Thus, the triple (V, μ, N) is a solution of (I)–(VI).

Uniqueness: Let a triple (Y_t, ν_t, L_t) be a solution of (I)–(VI). Then it follows from (4.3) and (II) that the process $Y_t + \int_0^t \pi_{s-} \mathrm{d}K_s$ is a submartingale. Since V_t is the largest process that satisfies (i) and (ii), we have $V_t \ge Y_t$.

Let us show that $Y_t \geq V_t$. Let

$$\sigma_t = \inf\{s \geq t : Y_s = 1 - \pi_s\}.$$

By condition (IV) we have $Y_s < 1 - \pi_s$ on the interval $[t; \sigma_t)$. Therefore, it follows from (4.3) that

$$Y_{\sigma_t} - Y_t = -\int_t^{\sigma_t} \pi_{s-} \mathrm{d}K_s + L_{\sigma_t} - L_t. \tag{4.16}$$

On the other hand condition (V) implies that $Y_{\sigma_t} = 1 - \pi_{\sigma_t}$. Therefore taking conditional expectations in (4.16) we obtain that

$$Y_t = E\left(1 - \pi_{\sigma_t} + \int_t^{\sigma_t} \pi_{s-} \mathrm{d}K_s / \mathcal{F}_t\right)$$

and by definition of the value process $Y_t \geq V_t$. Thus $Y_t = V_t$. It is evident that the martingale parts of V and Y are also indistinguishable. □

Remark 4.1. By (4.9), (4.12) and (4.15) we have that the value process also satisfies the following equation:

$$V_t = V_0 - \int_0^t I_{(1-\pi_{s-} > V_{s-})} \pi_{s-} \mathrm{d}K_s - \int_0^t I_{(1-\pi_{s-} = V_{s-})} \frac{1 - \pi_{s-}}{1 - \psi(s-)} \psi(\mathrm{d}s)$$
$$- \left(\frac{1}{2}\mathcal{L}_t^{1-\pi-V}(0) + \tilde{A}_t\right) + N_t. \tag{4.17}$$

Remark 4.2. Comparing the martingale parts of (4.4) and (4.5) we have that

$$\int_0^t I_{(1-\pi_{s-} = V_{s-})} (1 - \pi_{s-}) \frac{1 - \psi(s)}{1 - \psi(s-)} \mathrm{d}\tilde{M}_s = \int_0^t I_{(1-\pi_{s-} = V_{s-})} \mathrm{d}N_s$$
$$+ \left(\sum_{0<s\leq t} I_{(1-\pi_{s-} - V_{s-} = 0)} (1 - \pi_s - V_s) - \tilde{A}_t\right). \tag{4.18}$$

Let us write the a priori distribution functions in the form

$$\psi^\pi(t) = \pi\delta_0(t) + (1 - \pi)\varphi(t) \tag{4.19}$$

where $\delta_0(t)$ is a Dirac measure having a mass at 0, and $\varphi(t)$ is any fixed distribution function of some positive random variable. From now on taking expectation with respect to the measure $\bar{P}^{\psi^\pi}$ (resp. P^{ψ^π}) we will denote as $\bar{E}^\pi$ (resp. E^π) ($\bar{E}^{\psi^\pi} \rightarrow \bar{E}^\pi$). Hence the value V_0 can be rewritten as a function of π (π and ω in general):

$$V_0(\pi) = \inf_\tau E^\pi\left[(1 - \pi_\tau) + \int_0^\tau \pi_s \mathrm{d}K_s\right].$$

Now we shall prove the concavity of the value function $V_0(\pi)$, which will be essentially used in the sequel. For the value function corresponding to the classical disorder problems this fact was proved in [16].

Lemma 4.1. *The value function $V_0(\pi)$ is a concave function of π.*

Proof. We need to show that for any $\pi_1, \pi_2 \in [0,1]$ and $\alpha \in (0,1)$

$$V_0(\alpha\pi_1 + (1-\alpha)\pi_2) \geq \alpha V_0(\pi_1) + (1-\alpha)V_0(\pi_2),$$

Let $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$. By (4.19), $\psi^{\pi}(t) = \alpha\psi^{\pi_1}(t) + (1-\alpha)\psi^{\pi_2}(t)$ and $\bar{P}^{\psi^{\pi}} = \alpha\bar{P}^{\psi^{\pi_1}} + (1-\alpha)\bar{P}^{\psi^{\pi_2}}$ by the definition of the measure $\bar{P}^{\psi}$ (see (2.2)).

As $V_0(\pi) = \inf_\tau \bar{E}^{\pi}(I_{(\tau<\theta)} + (K_\tau - K_\theta)^+)$ the concavity of the function $V_0(\pi)$ is straightforward. □

5. Disorder problem for a Wiener process

In this section we consider the classical disorder problem of a Wiener process and show that in this case the RBSDE (4.3) is equivalent to the free-boundary problem considered in [16].

Let Ω be the space C of continuous functions $x = (x_t, t \geq 0)$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}(C)$ of C, $(\mathcal{B}_t(C), t \geq 0)$ the corresponding filtration.

Assume that P^0 is the measure on $(C, \mathcal{B}(C))$ such that $\frac{1}{\sigma}X_t$ is a standard Wiener process and P^1 is the measure on $(C, \mathcal{B}(C))$ such that the process

$$\frac{1}{\sigma}(X_t - rt)$$

is a Wiener process under P^1, where X_t is a coordinate process and r is some constant. Then $P^1 \overset{\text{loc}}{\sim} P^0$ and the density process of P^1 with respect to P^0 is of the form

$$Z_t = \frac{\mathrm{d}P_t^1}{\mathrm{d}P_t^0} = \exp\left\{\frac{r}{\sigma}X_t - \frac{r^2}{2\sigma^2}t\right\}.$$

Thus, $Z_t = \mathcal{E}_t(M)$, with $M_t = \frac{r}{\sigma}X_t$.

Let ψ be a distribution function such that

$$\psi(0) - \psi(0-) = \pi$$
$$1 - \psi(t) = (1-\pi)\exp\{-\lambda t\}, \quad t > 0, \tag{5.1}$$

where λ is a known strictly positive constant and $0 \leq \pi \leq 1$.

In this case $\hat{M}_t^{\psi} = \frac{r}{\sigma}\int_0^t \pi_{s-}\mathrm{d}X_s$ and

$$L_t(M, \hat{M}^{\psi}) = \frac{r}{\sigma}\left(X_t - \frac{r}{\sigma}\int_0^t \pi_{s-}\mathrm{d}s\right) \tag{5.2}$$

where $\bar{W}_t = X_t - \frac{r}{\sigma}\int_0^t \pi_{s-}\mathrm{d}s$ is a Wiener process with respect to the measure $\hat{P}^{\psi}$ which we shall denote hereafter by P^{π}. Note also that in this case $\frac{1}{1-\psi(s)}\psi(\mathrm{d}s) = \lambda\mathrm{d}s$.

Therefore, it follows from Eq. (3.23) (see Remark 3.2) that in this case the equation for π_t coincides with the equation derived in [16]:

$$\pi_t = \pi_0 + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\mathrm{d}\bar{W}_s + \lambda\int_0^t (1-\pi_s)\mathrm{d}s. \tag{5.3}$$

Lemma 5.1. *Let $a \leq \pi$, where $a, \pi \in [0,1)$. Then*

$$0 < \lambda(1-a)\int_0^{\infty} P^{\pi}(\pi_s \leq a)\mathrm{d}s \leq E^{\pi}\mathcal{L}_{\infty}^{\pi}(a) \leq 2(1-\pi). \tag{5.4}$$

Proof. By the Itô–Tanaka formula

$$|\pi_t - a| = |\pi - a| + \lambda \int_0^t (1-\pi_s)\mathrm{sign}(\pi_s - a)\mathrm{d}s + \mathcal{L}_t^{\pi}(a)$$
$$+ \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\mathrm{sign}(\pi_s - a)\mathrm{d}\tilde{W}_t. \tag{5.5}$$

Taking expectations with respect to the measure P^{π}, since the stochastic integral from (5.5) is a martingale, we have

$$E^{\pi}\mathcal{L}_t^{\pi}(a) = E^{\pi}|\pi_t - a| - |\pi - a| - \lambda E^{\pi}\int_0^t (1-\pi_s)\mathrm{sign}(\pi_s - a)\mathrm{d}s. \tag{5.6}$$

Since (3.7) and (5.1) imply that

$$E^{\pi}(1-\pi_t) = (1-\pi)\exp\{-\lambda t\}, \tag{5.7}$$

from (5.6) we obtain

$$E^{\pi}\mathcal{L}_t^{\pi}(a) \le E^{\pi}|\pi_t - a| - |\pi - a| + \lambda\int_0^t E^{\pi}(1-\pi_s)\mathrm{d}s$$
$$\le E^{\pi}|\pi_t - a| - |\pi - a| + (1-\pi)(1-\exp\{-\lambda t\}).$$

Therefore, the passage to the limit as $t \to \infty$ in the last inequality, keeping in mind that $\lim_{t\to\infty}\pi_t = 1$, gives the last inequality of (5.4):

$$E^{\pi}\mathcal{L}_{\infty}^{\pi}(a) \le 1 - a - (\pi - a) + (1-\pi) = 2(1-\pi).$$

On the other hand, from (5.6) (keeping in mind that $1-\pi_s > 1-a$ on the set $(\pi_s < a)$) we also have

$$E^{\pi}\mathcal{L}_t^{\pi}(a) = E^{\pi}|\pi_t - a| - |\pi - a|$$
$$- \lambda E^{\pi}\int_0^t (1-\pi_s)I_{(\pi_s > a)}\mathrm{d}s + \lambda E^{\pi}\int_0^t (1-\pi_s)I_{(\pi_s \le a)}\mathrm{d}s$$
$$\ge E^{\pi}|\pi_t - a| - |\pi - a|$$
$$- \lambda E^{\pi}\int_0^t (1-\pi_s)\mathrm{d}s + \lambda(1-a)E^{\pi}\int_0^t I_{(\pi_s \le a)}\mathrm{d}s. \tag{5.8}$$

It follows from (5.7) and relation $\lim_{t\to\infty}\pi_t = 1$ that for $\pi \ge a$

$$\lim_{t\to\infty}\left(E^{\pi}|\pi_t - a| - |\pi - a| - \lambda\int_0^t E^{\pi}(1-\pi_s)\mathrm{d}s\right) = 0.$$

Therefore, passing to the limit in (5.8) we obtain the validity of the inequality

$$E^{\pi}\mathcal{L}_{\infty}^{\pi}(a) \ge \lambda(1-a)\int_0^{\infty} P^{\pi}(\pi_s \le a)\mathrm{d}s.$$

Finally, since

$$\int_{(\pi-\varepsilon,\pi+\varepsilon)} \frac{1+\lambda(1-x)}{x^2(1-x)^2}\mathrm{d}x < \infty, \quad \text{for some } \varepsilon > 0,$$

at every $\pi \in (0, 1)$, the process π_t is regular in $(0, 1)$ (see, e.g., [3]). This means that π_t reaches a level x with positive probability starting at π, for every x and π from $(0, 1)$. Therefore $\int_0^\infty P^\pi(\pi_s \le a)\mathrm{d}s$ is strictly positive. □

Assume that $K_t = ct$. So, the cost criterion is of the same form as in [16]:

$$\rho_\tau(\pi) = P^\pi(\tau < \theta) + cE_\pi \max(\tau - \theta, 0), \tag{5.9}$$

and the value function of the optimal stopping problem (2.10) is

$$\rho(\pi) = \inf_\tau E_\pi \left(1 - \pi_\tau + c\int_0^\tau \pi_s \mathrm{d}s\right). \tag{5.10}$$

Since $(\pi_t, \mathcal{F}_t, P^\pi)$ is a time-homogeneous Markov process, we have that

$$V_t = \rho(\pi_t) \quad \text{a.s. for all } t \ge 0. \tag{5.11}$$

According to the general theory of optimal stopping the optimal stopping rule is

$$\tau^* = \inf\{t : \rho(\pi_t) = 1 - \pi_t\}. \tag{5.12}$$

Since $\rho(\pi)$ is concave by Lemma 4.1, $\rho(\pi) \le 1 - \pi$ and $\rho(\pi) = 1 - \pi$ if $\pi = 1$, we have that $\rho(\pi) = 1 - \pi$ for all $\pi \ge A^*$ and $\rho(\pi) < 1 - \pi$ if $\pi < A^*$, where

$$A^* = \inf\{A : \rho(A) = 1 - A\}.$$

Therefore, the optimal stopping time of (2.10) is in this case of the form

$$\tau^*(\pi) = \inf\{t : \pi_t \ge A^*\} \tag{5.13}$$

and the aim is to calculate $\rho(\pi)$ and the constant A^*. This was done in [16] by first solving a suitable free-boundary problem and then showing that the unique solution of this problem is the value function. Our main aim in this section is to show that since the value process $V_t = \rho(\pi_s)$ satisfies the RBSDE (4.3), the value function $\rho(\pi)$ will be the solution of the free-boundary problem considered in [16].

Theorem 5.1. *The value function $\rho(\pi)$ is a non-negative concave function and there is a constant $A^* \in (0, 1]$ such that:*
(1) *$\rho(\pi)$ is twice continuously differentiable on $(0, A^*)$ and satisfies the PDE*

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi) + \lambda(1-\pi)\rho'(\pi) = -c\pi, \quad \text{if } 0 \le \pi < A^*, \tag{5.14}$$

(2) *$\rho(\pi)$ is equal to $1 - \pi$ if $\pi \ge A^*$ and*
(3) *satisfies the smooth fit condition*

$$\rho'(A^*) = -1.$$

Besides, the value function satisfies the normal entrance condition:

$$\rho'(0+) = 0.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative concave function with second-order derivative satisfying (1)–(3) *for some $B^* \in (0, 1]$, then the triple $V_t = \tilde{\rho}(\pi_t), \nu_t = 0$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ satisfies the RBSDE* (I)–(VI). *In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.*

Proof. Let $D = \{\pi : \rho(\pi) < 1 - \pi\}$ and let ∂D be the boundary of this set. It is evident that $\rho(\pi) \leq 1 - \pi$ and $\rho(1) = 0$ (since $\pi_t = 1$ for all $t \geq 0$, if $\pi_0 = 1$). Therefore, the concavity of $\rho(\pi)$ implies that ∂D contains only one point (say A^*) and according to Theorem 6 from [7] we have $L^{1-\pi-V}(0) = 0$, which means that the process μ_t from (4.15) is zero.

Thus (5.11) and (4.15) imply that the value process $V_t = \rho(\pi_t)$ satisfies the equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\rho(\pi_s)<1-\pi_s)}\mathrm{d}s - \lambda\int_0^t (1-\pi_s) I_{(\rho(\pi_s)=1-\pi_s)}\mathrm{d}s + N_t. \tag{5.15}$$

Since $\rho(\pi)$ is concave (by Lemma 4.1), $\rho(\pi) \leq 1-\pi$ and $\rho(\pi) = 1-\pi$ if $\pi = 1$, we have that $\rho(\pi) = 1-\pi$ for all $\pi \geq A^*$ and $\rho(\pi) < 1-\pi$ if $\pi < A^*$, where

$$A^* = \inf\{A : \rho(A) = 1 - A\} = \partial D.$$

Besides, the optimal stopping rule is of the form (5.13) and

$$\{(\omega, s) : \rho(\pi_s) < 1 - \pi_s\} = \{(\omega, s) : \pi_s < A^*\},$$
$$\{(\omega, s) : \rho(\pi_s) = 1 - \pi_s\} = \{(\omega, s) : \pi_s \geq A^*\}.$$

Therefore, there exists $A^* \in (0, 1)$ such that $\rho(\pi_t)$ satisfies the equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s<A^*)}\mathrm{d}s - \lambda\int_0^t (1-\pi_s) I_{(\pi_s\geq A^*)}\mathrm{d}s + \int_0^t Z_s \mathrm{d}\tilde{W}_s, \tag{5.16}$$

where $N = Z \cdot \tilde{W}$ by the martingale representation theorem and $Z_s = -\frac{b}{\sigma}\pi_s(1-\pi_s)$ on the set $\{\pi_s \geq A^*\}$ $\mathrm{d}P \times \mathrm{d}s$ a.e. by Remark 4.2.

Since $\rho(\pi)$ is concave, by the Tanaka–Meyer formula

$$\rho(\pi_t) = \rho(\pi_0) + \lambda\int_0^t \rho'_-(\pi_s)(1-\pi_s)\mathrm{d}s + \frac{1}{2}\int_R \mathcal{L}_t^\pi(a)\nu''(\mathrm{d}a)$$
$$+ \frac{r}{\sigma}\int_0^t \rho'_-(\pi_s)\pi_s(1-\pi_s)\mathrm{d}\tilde{W}_s, \tag{5.17}$$

where $L_t^\pi(a)$ is the local time at the point a of the process π_t, ρ'_- is the left-hand derivative of $\rho(\pi)$ and ν'' is the measure of the second derivative of ρ.

Comparing the parts of finite variations of (5.17) and (5.16), keeping in mind that $\rho'_-(\pi_s) = -1$ on the set $\{\pi_s > A^*\}$, we have

$$\frac{1}{2}\int_R \mathcal{L}_t^\pi(a)\nu''(\mathrm{d}a) = -\int_0^t [c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)] I_{(\pi_s<A^*)}\mathrm{d}s. \tag{5.18}$$

Let $h(x)$, $x \in R$ be a bounded measurable function. Since the measure $\mathrm{d}\mathcal{L}_t^\pi(a)$ is a.s. carried by the set $\{t : \pi_t = a\}$, integrating the process $h(\pi_s)\pi_s^2(1-\pi_s)^2$ with respect to both parts of equality (5.18) and using Fubini's theorem we get

$$\frac{1}{2}\int_R \mathcal{L}_t^\pi(a)h(a)a^2(1-a)^2\nu''(\mathrm{d}a)$$
$$= -\int_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2[c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)] I_{(\pi_s<A^*)}\mathrm{d}s. \tag{5.19}$$

By the occupation formula (see, e.g., [13] or [14]),

$$\int_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s<A^*)}\mathrm{d}s$$
$$=\frac{\sigma^2}{r^2}\int_0^t h(\pi_s)[c\pi_s+\lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s<A^*)}\mathrm{d}\langle\pi\rangle_s$$
$$=\frac{\sigma^2}{r^2}\int_R \mathcal{L}_t^\pi(a)h(a)[ca+\lambda(1-a)\rho'_-(a)]I_{(a<A^*)}\mathrm{d}a. \tag{5.20}$$

Therefore,

$$\int_{[0,1]}\mathcal{L}_t^\pi(a)h(a)a^2(1-a)^2\nu''(\mathrm{d}a)$$
$$=-\frac{2\sigma^2}{r^2}\int_{[0,1]}\mathcal{L}_t^\pi(a)h(a)[ca+\lambda(1-a)\rho'_-(a)]I_{(a<A^*)}\mathrm{d}a. \tag{5.21}$$

Since $\rho(\pi)$ is concave and decreasing we have that $-1 \le \rho'_- \le 0$ and we may use Fubini's theorem and the Lebesgue theorem of monotone convergence, i.e., taking mathematical expectations with respect to the measure P^π (for some $\pi < 1$) and passing to the limit as $t\to\infty$ in the last equality, we obtain that

$$\int_R h(a)a^2(1-a)^2E^\pi\mathcal{L}_\infty^\pi(a)\nu''(\mathrm{d}a)$$
$$=-\frac{2\sigma^2}{r^2}\int_R h(a)[ca+\lambda(1-a)\rho'_-(a)]I_{(a<A^*)}E^\pi\mathcal{L}_\infty^\pi(a)\mathrm{d}a \tag{5.22}$$

for any bounded measurable function h.

Since by Lemma 5.1 we have $0 < E^\pi\mathcal{L}_\infty^\pi(a) < \infty$ for all a,π such that $0\le a\le\pi<1$, (5.22) and the arbitrariness of the function h imply that the measure $\nu''(\mathrm{d}a)$ is absolutely continuous with respect to the Lebesgue measure on (0, 1) and, hence, $\rho(\pi)$ admits a second-order generalized derivative. Therefore, by Sobolev's embedding theorem (see [19]) there exists the first derivative of $\rho(\pi)$ in the usual sense and this derivative is continuous.

If we denote by $\rho''(\pi)$ the second-order generalized derivative of ρ from (5.22) we have that a.e. with respect to the Lebesgue measure the value function $\rho(\pi)$ satisfies the PDE

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi)=-\lambda(1-\pi)\rho'(\pi)-c\pi \tag{5.23}$$

on the open interval $(0, A^*)$.

Since equality (5.23) is fulfilled on the set $(0, A^*)$ a.e. with respect to the Lebesgue measure and the right-hand side of (5.23) is continuous, then there exists a modification of $\rho''(\pi)$ (for convenience we denote this modification also by $\rho''(\pi)$) which is continuous on $(0, A^*)$. It is evident that the continuous modification of $\rho''(\pi)$ coincides with the ordinary second-order derivative of ρ and Eq. (5.23) is satisfied for all $\pi\in(0, A^*)$.

Since $\rho(\pi)=1-\pi$ for all $\pi\ge A^*$ and $\rho(\pi)$ admits a continuous derivative, we have that $\rho'(\pi)=-1$ for all $\pi\ge A^*$ and, therefore, the constant A^* can be calculated from the smooth fit condition

$$\rho'(A^*)=-1.$$

Let us show now that $\rho'(0)=0$. We shall first show that the value function $\rho(\pi)$ is a decreasing function. Let $\pi\le\pi'\le A^*$ and define $\sigma=\inf\{t:\pi_t^\pi\ge\pi'\}$. It is evident that

$\pi_\sigma^\pi = \pi'$ and it follows from Eq. (5.16) that

$$\rho(\pi_\sigma^\pi) = \rho(\pi) - c\int_0^\sigma \pi_s^\pi I_{(\pi_s^\pi < A^*)} \mathrm{d}s + \int_0^\sigma Z_s \mathrm{d}\bar{W}_s. \tag{5.24}$$

Since $Z \cdot \bar{W}$ is a martingale and $\rho(\pi_\sigma^\pi) = \rho(\pi')$, taking expectations in (5.24) we obtain that

$$\rho(\pi') - \rho(\pi) = -cE^\pi \int_0^\sigma \pi_s^\pi \mathrm{d}s \le 0.$$

Let $(\pi_n, n \ge 1)$ be a sequence such that $\pi_n \downarrow 0$. Then from (5.23)

$$\frac{r^2}{2\sigma^2}\pi_n^2(1-\pi_n)^2\rho''(\pi_n) = -\lambda(1-\pi_n)\rho'(\pi_n) - c\pi_n \tag{5.25}$$

for each $n \ge 1$. Since $\rho'(\pi)$ is continuous, the limit as $n \to \infty$ of the right-hand side exists and is equal to $-\lambda\rho'(0+)$. Therefore there exists the limit of the left-hand side and since $\rho(\pi)$ is concave, this limit is non-positive, i.e., $\rho'(0+) \ge 0$. Since the function $\rho(\pi)$ is decreasing, $\rho'(\pi_n)$ is non-positive and, hence, the limit of the right-hand side is non-negative, i.e., $\rho'(0+) \le 0$. Thus $\rho'(0+) = 0$ and Eq. (5.23) for $\pi = 0$ is also satisfied.

Thus, we have shown that the value function $\rho(\pi)$ is a concave function admitting the second-order derivative ($\rho''(\pi)$ can be discontinuous only at points $\pi = 0$ and $\pi = A^*$) and it satisfies the free-boundary problem (1)–(3).

Conversely, let $\tilde{\rho}(\pi)$ be a non-negative concave function satisfying (1)–(3). Then by Itô's formula

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) + \lambda\int_0^t \tilde{\rho}'(\pi_s)(1-\pi_s)\mathrm{d}s + \frac{r^2}{2\sigma^2}\int_0^t \pi_s^2(1-\pi_s)^2\tilde{\rho}''(\pi_s)\mathrm{d}s$$
$$+ \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)\mathrm{d}\tilde{W}_s. \tag{5.26}$$

Since $\tilde{\rho}''(\pi) = 0$ and $\tilde{\rho}'(\pi) = -1$ for all $\pi > B^*$, it follows from (5.14) and (5.26) that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - \lambda\int_0^t (1-\pi_s)I_{(\pi_s \ge B^*)}\mathrm{d}s - c\int_0^t \pi_s I_{(\pi_s < B^*)}\mathrm{d}s$$
$$+ \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)\mathrm{d}\tilde{W}_s. \tag{5.27}$$

Let $\tilde{A} = \inf\{A : \tilde{\rho}(A) = 1-A\}$. Since $\tilde{\rho}(\pi)$ is concave, the smooth fit condition $\tilde{\rho}'(B^*) = -1$ implies that $B^* \in [\tilde{A}, 1]$. On the other hand if $B^* > \tilde{A}$ then on the interval $(\tilde{A}, B^*)$ we shall have $\tilde{\rho}''(\pi) = 0$, $\tilde{\rho}'(\pi) = -1$ and for any $\pi \in (\tilde{A}, B^*)$ Eq. (5.14) will not be satisfied. Thus $B^* = \tilde{A}$ and

$$\{\pi_s < B^*\} = \{\tilde{\rho}(\pi_s) < 1-\pi_s\},$$
$$\{\pi_s \ge B^*\} = \{\tilde{\rho}(\pi_s) = 1-\pi_s\}. \tag{5.28}$$

From (5.27) and (5.27) we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - \lambda\int_0^t (1-\pi_s)I_{(\tilde{\rho}(\pi_s)=1-\pi_s)}\mathrm{d}s$$
$$- c\int_0^t \pi_s I_{(\tilde{\rho}(\pi_s)<1-\pi_s)}\mathrm{d}s + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)\mathrm{d}\tilde{W}_s. \tag{5.29}$$

We shall show now that $\frac{\lambda}{\lambda+c} \le B^*$. Indeed, passing to the limit in (5.23) as $\pi \uparrow B^*$ and using the smooth fit condition we have that

$$-\frac{r^2}{2\sigma^2}(B^*)^2(1-B^*)^2 \liminf_{\pi\uparrow B^*} \rho''(\pi) \le cB^* - \lambda(1-B^*). \tag{5.30}$$

From the concavity of the function $\rho(\pi)$ we have that the left-hand side of this inequality is non-negative and hence $\frac{\lambda}{\lambda+c} \le B^*$. This inequality implies that $c\pi_s - \lambda(1-\pi_s)$ is positive on the set $\pi_s \ge B^*$. Therefore, we can rewrite (5.27) in the following form:

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s \mathrm{d}s + \int_0^t (c\pi_s - \lambda(1-\pi_s))^+ I_{(\tilde{\rho}(\pi_s)=1-\pi_s)}\mathrm{d}s$$
$$+ \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)\mathrm{d}\tilde{W}_s, \tag{5.31}$$

which enables us to conclude that the triple $Y_t = \tilde{\rho}(\pi_t)$, $\nu_t = 0$, $L_t = \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)\mathrm{d}\tilde{W}_s$ satisfies the RBSDE (4.3). It is easy to see that this triple satisfies (I)–(V). Indeed, since $\tilde{\rho}(\pi)$ is concave, condition (2) implies that $\tilde{\rho}(\pi_t) \le 1-\pi_t$ for all $t \ge 0$ and $\lim_{t\to\infty}\tilde{\rho}(\pi_t) \le \lim_{t\to\infty}(1-\pi_t) = 0$. Besides, the positivity of $\tilde{\rho}(\pi)$ implies that $\lim_{t\to\infty}\tilde{\rho}(\pi_t) = 0$ and that $\tilde{\rho}(\pi_t)$ is bounded. Therefore, it follows from (5.27) that $\tilde{\rho}(\pi_t)$ is a supermartingale from the class S^1. Thus conditions (I)–(V) are satisfied and by Theorem 4.1 $\tilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (5.11) $\tilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\tilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0, 1]$. □

Thus we have proved that the RBSDE (I)–(VI) and the free-boundary problem (1)–(3) are equivalent. The solution of the free-boundary problem (1)–(3) is given [16]. Following Shiryaev [16], if we denote $\rho'(\pi)$ by $g(\pi)$ from (5.14) we have that

$$g'(\pi) = -\frac{2\lambda\sigma^2}{r^2\pi^2(1-\pi)}g(\pi) - \frac{2c\sigma^2}{r^2\pi(1-\pi)^2}.$$

Since $g(0) = 0$, we find that for $\pi < A^*$

$$g(\pi) = \rho'(\pi) = -\frac{2c\sigma^2}{r^2}\int_0^\pi \exp\left\{-\frac{2\lambda\sigma^2}{r^2}(H(\pi) - H(y))\right\}\frac{\mathrm{d}y}{y(1-y)^2}, \tag{5.32}$$

where $H(y) = \ln\frac{y}{1-y} - \frac{1}{y}$. If we define A^* as a unique solution of equation $g(A^*) = -1$, then the value function $\rho(\pi)$ coincides with

$$\rho(\pi) = \begin{cases} 1 - A^* - \displaystyle\int_\pi^{A^*} g(x)\mathrm{d}x, & 0 \le \pi \le A^* \\ 1-\pi, & A^* \le \pi \le 1. \end{cases} \tag{5.33}$$

Remark 5.1. Let us note that the smooth fit of the second derivative cannot be fulfilled and the second-order derivative of $\rho(\pi)$ is discontinuous at the point A^*. Indeed (1)–(3) imply that $\rho''(\pi)$ can be continuous only if $A^* = \frac{\lambda}{\lambda+c}$. On the other hand by partial integration

$$\int_0^\pi \frac{y}{1-y}\exp\left\{\frac{2\lambda\sigma^2}{r^2}H(y)\right\}\mathrm{d}H(y)$$
$$= \frac{r^2}{2\lambda\sigma^2}\frac{\pi}{1-\pi}\exp\left\{\frac{2\lambda\sigma^2}{r^2}H(\pi)\right\} - \frac{r^2}{2\lambda\sigma^2}\int_0^\pi \exp\left\{\frac{2\lambda\sigma^2}{r^2}H(y)\right\}\frac{\mathrm{d}y}{(1-y)^2} \tag{5.34}$$

and from (5.32) and (5.34) we obtain that

$$g(\pi) > -\frac{c}{\lambda}\frac{\pi}{1-\pi}. \tag{5.35}$$

Therefore on the set $\{\pi : \pi \le \frac{\lambda}{\lambda+c}\}$ we have

$$g(\pi) > -1. \tag{5.36}$$

In particular $\rho'(\frac{\lambda}{\lambda+c}) > -1$ and hence $A^* \ne \frac{\lambda}{\lambda+c}$. Thus, the second-order derivative of the value function is discontinuous at the point A^*.

6. Poisson disorder problem

In this section we consider the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ and show that in this case the RBSDE (4.3) is equivalent to a free-boundary differential–difference problem considered in [12]. Besides, we derive the smooth fit conditions for the value function (in cases when this condition is satisfied) and establish when the smooth fit condition breaks down directly from the RBSDE for the value process.

Let Ω be the space X of piecewise-constant functions $x = (x_t, t \ge 0)$ such that $x_0 = 0$ and $x_t = x_{t-} + (0 \text{ or } 1)$, $\mathcal{B} = \sigma\{x : x_s, s \ge 0\}$, $\mathcal{B}_t = \sigma\{x : x_s, s \le t\}$.

Note that for any $x = (x_t, t \ge 0) \in X$, x_t is expressed as

$$x_t = \sum_{i\ge 1} I_{\{\tau_i(x)\le t\}},$$

where

$$\tau_i(x) = \begin{cases} \inf\{s \ge 0 : x_s = i\} \\ \infty \quad \lim_{t\to\infty} x_t < i. \end{cases} \tag{6.1}$$

Let P^0 and P^1 be two Poisson measures on $(X, \mathcal{B})$ with parameters λ_0 and λ_1 respectively. This means that under the measure P^i the compensator of the coordinate process $X_t(x) = x_t$, $t \ge 0$, is equal to $A_i(t, x) = \lambda_i t$, $i = 1, 2$. (Note that the family of σ-algebras $(\mathcal{B}_t, t \ge 0)$, completed by P^0 and P^1, is right-continuous.)

As is known,

$$P^1 \overset{\text{loc}}{\sim} P^0 \quad \text{and} \quad \frac{dP_t^1}{dP_t^0} = \exp\left\{\ln\frac{\lambda_1}{\lambda_0}X_t - (\lambda_1-\lambda_0)t\right\}$$

(see [10]).

It is easy to see that $\frac{dP_t^1}{dP_t^0} = \mathcal{E}_t(M)$, where

$$M_t = \left(\frac{\lambda_1}{\lambda_0} - 1\right)(X_t - \lambda_0 t), \quad M \in \mathcal{M}_{\text{loc}}(F, P^0).$$

Let $\psi(0) - \psi(0-) = \pi$ and $1 - \psi(t) = (1-\pi)\exp\{-\lambda t\}$, where λ is a known constant and $0 \le \pi \le 1$.

By Lemma 3.1 (see Remark 3.1)

$$\hat{M}_t^{\psi} = \left(\frac{\lambda_1}{\lambda_0} - 1\right)\int_0^t \pi_{s-} d(X_s - \lambda_0 s) \tag{6.2}$$

and, hence,

$$L_t(M, \hat{M}^{\psi}) = \left(\frac{\lambda_1}{\lambda_0} - 1\right) \int_0^t \frac{\mathrm{d}X_s}{1 + \pi_{s-}\left(\frac{\lambda_1}{\lambda_0} - 1\right)} - (\lambda_1 - \lambda_0)t. \tag{6.3}$$

Since $\Delta\psi_t = 0$, it follows from Remark 3.2 that the a posteriori probability process π_t satisfies the equation

$$\mathrm{d}\pi_t = \lambda(1 - \pi_{t-})\mathrm{d}t + \frac{\pi_{t-}(1 - \pi_{t-})(\lambda_1 - \lambda_0)}{\lambda_1\pi_{t-} + \lambda_0(1 - \pi_{t-})}(\mathrm{d}X_t - (\lambda_1\pi_{t-} + \lambda_0(1 - \pi_{t-}))\mathrm{d}t), \tag{6.4}$$

which coincides with the equation derived in [12].

Remark 6.1. The process $(X_t - \int_0^t(\lambda_1\pi_{s-} + \lambda_0(1 - \pi_{s-}))\mathrm{d}s, \mathcal{B}_t), t \geq 0$, is a martingale under P^π and $(\pi_t, \mathcal{B}_t, P^\pi)$ is a time-homogeneous (strong) Markov process.

Assume that $K_t = ct$. So, the cost criterion is of the same form as in (5.9) and the value function of the optimal stopping problem (2.10) is as in (5.10).

Since $(\pi_t, \mathcal{F}_t, P^\pi)$ is a time-homogeneous(strong) Markov process, we have that

$$V_t = \rho(\pi_t) \quad \text{a.s. for all } t \geq 0. \tag{6.5}$$

Note that the cases $\lambda_1 < \lambda_0$ and $\lambda_1 > \lambda_0$ are quite different. e.g., a key difference between these cases is the fact that when $\lambda_1 < \lambda_0$, Eq. (6.6) has no singularity points, whereas $\hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$ is a singularity point of (6.6) whenever $\lambda < \lambda_1 - \lambda_0$ (see [12] for detailed analysis of these cases). Let us first consider the case $\lambda_1 > \lambda_0$.

Theorem 6.1. *Let $\lambda_1 > \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0, 1]$ such that:*
(1) *$\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ (perhaps except for the point $\hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$) and satisfies a differential–difference equation:*

$$(\lambda - \pi(\lambda_1 - \lambda_0))(1 - \pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1 - \pi)) \times \left[\rho\left(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1 - \pi)}\right) - \rho(\pi)\right] = -c\pi, \tag{6.6}$$

if $\pi < B^*$.
(2) *It is equal to $1 - \pi$, if $\pi \geq B^*$.*
(3) *It satisfies the continuous fit condition*

$$\rho(B^*-) = 1 - B^*.$$

Moreover, if $c > \lambda_1 - \lambda_0 - \lambda$, then
(3*) *The value function $\rho(\pi)$ satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative, concave function satisfying (1)–(3) in the case $c \leq \lambda_1 - \lambda_0 - \lambda$ and (1), (2), (3) in the case $c > \lambda_1 - \lambda_0 - \lambda$ for some $A^* \in (0, 1]$, then the triple $Y_t = \tilde{\rho}(\pi_t)$, $\nu_t = 0$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ satisfies the RBSDE (I)–(VI). In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.*

Proof. Similarly to the Wiener case,

$$\{(\omega, s) : \rho(\pi_s) < 1 - \pi_s\} = \{(\omega, s) : \pi_s < B^*\},$$
$$\{(\omega, s) : \rho(\pi_s) = 1 - \pi_s\} = \{(\omega, s) : \pi_s \geq B^*\},$$

where

$$B^* = \inf\{B : \rho(B) = 1 - B\}$$

and the optimal stopping rule is of the form

$$\tau^* = \inf\{t : \rho(\pi_t) = 1 - \pi_t\} = \inf\{t : \pi_t \geq B^*\}. \tag{6.7}$$

Taking into account the above facts it follows from Remark 4.1. (see (4.17)) that the value process $\rho(\pi_t)$ satisfies the following equation:

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s < B^*)}\mathrm{d}s - \lambda\int_0^t (1 - \pi_s) I_{(\pi_s \geq B^*)}\mathrm{d}s - \left(\frac{1}{2}\mathcal{L}_t^0(1 - \pi - V) + \tilde{A}_t\right) + N_t. \tag{6.8}$$

It is evident that the process $(\mathcal{L}_t^0(1 - \pi - V), t \geq 0)$ is indistinguishable from zero. Indeed, since both functions π_t and V_t have jumps at the discontinuity points of the process X_t and the number of these points for the process X_t on each interval $(0, t]$ is finite, we will have that the following condition is fulfilled:

$$\sum_{0<s\leq t} |\triangle(1 - \pi_s - V_s)| < \infty \quad \text{a.s., for each } t > 0.$$

Besides, as the processes π_t and V_t do not have continuous martingale parts, we have that $\mathcal{L}_t^0(1 - \pi - V) = 0$ (see, e.g., Corollary 3 of Theorem 56, Chapter 4 from [13]).

Recall that $\tilde{A}_t$ is the compensator of the process $\sum_{0<s\leq t} I_{(1-\pi_{s-}=\rho(\pi_{s-}))}(1 - \pi_s - \rho(\pi_s))$. Therefore $\tilde{A}_t$ can be written as follows:

$$-\tilde{A}_t = \int_0^t (\lambda_1\pi_s + \lambda_0(1 - \pi_s))\left[\rho\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s + \lambda_0(1 - \pi_s)}\right) - \rho(\pi_s)\right] I_{(\pi_s \geq B^*)}\mathrm{d}s + \int_0^t (\lambda_1 - \lambda_0)\pi_s(1 - \pi_s) I_{(\pi_s \geq B^*)}\mathrm{d}s. \tag{6.9}$$

Since $\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} > \pi$ and $\rho(\pi) = 1 - \pi$ for any $\pi \geq B^*$, then $\rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}) = 1 - \frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}$. Therefore it follows easily from (6.9) that $\tilde{A}_t = 0$. Hence the process μ_t from (4.15) is indistinguishable from zero.

Thus from (6.8) we have

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s < B^*)}\mathrm{d}s - \lambda\int_0^t (1 - \pi_s) I_{(\pi_s \geq B^*)}\mathrm{d}s + N_t. \tag{6.10}$$

Since the function $\rho(\pi)$ is concave and the martingale part of the process $(\pi_t, t \geq 0)$ is a pure-jump process and so $\mathcal{L}_t^a(\pi) = 0$, by the Tanaka–Meyer formula

$$\rho(\pi_t) = \rho(\pi_0) + \int_0^t \rho'_-(\pi_{s-})\mathrm{d}\pi_s + \sum_{s\leq t}(\rho(\pi_s) - \rho(\pi_{s-}) - \rho'_-(\pi_{s-})\triangle\pi_s), \tag{6.11}$$

where by $\rho'_-(\pi)$ we have denoted the left derivative of the function $\rho(\pi)$.

As the compensator $\hat{A}$ of the last summand of Eq. (6.11) is equal to

$$\hat{A}_t \equiv \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))\left[\rho\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}\right) - \rho(\pi_s)\right]\mathrm{d}s - (\lambda_1-\lambda_0) \times \int_0^t \pi_s(1-\pi_s)\rho'_-(\pi_s)\mathrm{d}s$$

keeping in mind that $\rho'_-(\pi) = -1$ for $\pi > B^*$, from (6.9) we obtain that

$$\hat{A}_t = \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))\left[\rho\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}\right) - \rho(\pi_s)\right] I_{(\pi_s<B^*)}\mathrm{d}s - \int_0^t (\lambda_1-\lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}\mathrm{d}s. \tag{6.12}$$

Therefore by (6.11) we have

$$\begin{aligned}\rho(\pi_t) &= \rho(\pi_0) + \lambda\int_0^t (1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}\mathrm{d}s - \lambda\int_0^t (1-\pi_s)I_{(\pi_s\geq B^*)}\mathrm{d}s \\ &\quad + \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))\left[\rho\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}\right) - \rho(\pi_s)\right] I_{(\pi_s<B^*)}\mathrm{d}s \\ &\quad - \int_0^t (\lambda_1-\lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}\mathrm{d}s + \tilde{M}_t,\end{aligned} \tag{6.13}$$

where $\tilde{M}_t$ is a martingale.

By the uniqueness of the canonical decomposition from (6.9), (6.10) and (6.13) we have that

$$\int_0^t (\lambda - (\lambda_1-\lambda_0)\pi_s)(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}\mathrm{d}s + \int_0^t c\pi_s I_{(\pi_s<B^*)}\mathrm{d}s + \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))\left(\rho\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}\right) - \rho(\pi_s)\right) I_{(\pi_s<B^*)}\mathrm{d}s = 0. \tag{6.14}$$

Further let us define

$$\hat{B} = \frac{\lambda}{\lambda_1-\lambda_0} \tag{6.15}$$

and observe that (6.4) can be rewritten in the following form:

$$\mathrm{d}\pi_t = (\lambda_1-\lambda_0)(\hat{B}-\pi_{t-})(1-\pi_{t-})\mathrm{d}t + \frac{\pi_{t-}(1-\pi_{t-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-})}\mathrm{d}X_t. \tag{6.16}$$

Hence, if $\pi < \hat{B}$, then $\pi_t \downarrow \pi$ as $t \to 0$ P^π-a.s. and if $\pi > \hat{B}$, then $\pi_t \uparrow \pi$ as $t \to 0$ P^π-a.s. More exactly for each $\omega \in N$, for some $N \subset \Omega$, with $P^\pi(N) = 1$, there exists $t_0 = t_0(\omega)$, such that $\pi_t \uparrow \pi$ for $t_0(\omega) \geq t \to 0$.

At the same time, since $\rho(\pi)$ is a concave function and $\rho'_-(\pi)$ is a non-increasing left-continuous function having right-hand-side limits, we have that

$$\lim_{t\to 0}\rho'_-(\pi_t) = \rho'_+(\pi)P^\pi\text{-a.s,} \quad \text{if } \pi > \hat{B} \quad \text{and}$$

$$\lim_{t\to 0}\rho'_-(\pi_t) = \rho'_-(\pi)P^\pi\text{-a.s,} \quad \text{if } \pi \leq \hat{B}.$$

Taking into consideration these facts by dividing the right-hand side of (6.14) by t, the passage to the limit as $t \to 0$ gives that the value function $\rho(\pi)$ satisfies the following differential–difference equation:

$$(\lambda - (\lambda_1 - \lambda_0)\pi)(1-\pi)\tilde{\rho}'(\pi)$$
$$= -\left((\lambda_1\pi + \lambda_0(1-\pi))\left[\rho\left(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}\right) - \rho(\pi)\right] + c\pi\right) \tag{6.17}$$

for all $\pi < B^*$, where $\tilde{\rho}'(\pi) = \rho'_-(\pi)$, if $\pi < \hat{B}$ and $\tilde{\rho}'(\pi) = \rho'_+(\pi)$, if $\pi > \hat{B}$.

Since the right-hand side of (6.17) is a continuous function, then $\tilde{\rho}'(\pi)$ is continuous except at the point $\pi = \hat{B}$. Since $\tilde{\rho}'(\pi)$ coincides with left or right derivatives of the function $\rho(\pi)$ and $\tilde{\rho}'(\pi)$ is continuous, we obtain that $\rho(\pi)$ admits a continuous derivative and $\tilde{\rho}'(\pi) = \rho'(\pi)$ for all $\pi \in (0, B^*)$ (perhaps except at the point $\pi = \hat{B}$). Therefore (6.17) implies that $\rho(\pi)$ satisfies Eq. (6.6) for all $\pi \in (0, B^*)$.

Going to the limit as $\pi \to 0+$ in (6.17) we obtain that $\rho(\pi)$ satisfies the normal entrance condition

$$\rho'(0+) = 0,$$

and hence $\rho(\pi)$ is a decreasing function.

Since $\rho(\pi)$ is continuous and $\rho(\pi) = 1 - \pi$ for $\pi > B^*$, the continuous fit condition

$$\rho(B^*-) = 1 - B^*$$

is fulfilled. Thus the value function $\rho(\pi)$ satisfies conditions (1)–(3).

We shall show now that in the case $c > \lambda_1 - \lambda_0 - \lambda$ the smooth fit condition (3*) is satisfied. Since $\{\rho(\pi_s) = 1 - \pi_s\} = \{\pi_s \geq B^*\}$, from (4.12) we have that $c\pi_s - \lambda(1-\pi_s) \geq 0$ on the set $\{\pi_s \geq B^*\}$ and hence

$$B^* \geq \frac{\lambda}{\lambda + c}. \tag{6.18}$$

Passing to the limit in (6.6), as $\pi \uparrow B^*$, we have

$$\tilde{\rho}'(B^*-) = \frac{B^*[(\lambda_1 - \lambda_0 - c) - (\lambda_1 - \lambda_0)B^*]}{(1-B^*)[\lambda - B^*(\lambda_1 - \lambda_0)]}, \tag{6.19}$$

if $B^* \neq \hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$.

Since $\rho(\pi)$ is concave and satisfies the normal entrance condition, we have that $\rho'(\pi) \in [-1, 0]$. By resolving the system

$$\begin{cases} \tilde{\rho}'(B^*-) \geq -1 \\ \tilde{\rho}'(B^*-) \leq 0 \end{cases} \tag{6.20}$$

we have $B^* \in [\max(\frac{\lambda_1 - \lambda_0 - c}{\lambda_1 - \lambda_0}, 0), \frac{\lambda}{\lambda + c}]$, which together with (6.18) implies that

$$B^* = \frac{\lambda}{\lambda + c};$$

Substituting $\frac{\lambda}{\lambda + c}$ instead of B^* in (6.19) we obtain that

$$\rho'(B^*-) = -1.$$

Hence condition (3*) is also satisfied and the first part of the theorem is proved. □

Conversely, let $\tilde{\rho}(\pi)$ be a non-negative, concave function satisfying (1)–(3) if $c \leq \lambda_1 - \lambda_0 - \lambda$ and conditions (1), (2), (3*) if $c > \lambda_1 - \lambda_0 - \lambda$. Then by the Tanaka–Meyer formula

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) + \int_0^t \tilde{\rho}'_-(\pi_{s-})\mathrm{d}\pi_s + \sum_{s \leq t}(\tilde{\rho}(\pi_s) - \tilde{\rho}(\pi_{s-}) - \tilde{\rho}'_-(\pi_{s-})\triangle\pi_s). \tag{6.21}$$

Since the compensator of the last summand is equal to

$$\int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))\left[\tilde{\rho}\left(\frac{\lambda_1\pi_s}{\lambda_1\pi_s + \lambda_0(1-\pi_s)}\right) - \tilde{\rho}(\pi_s)\right]\mathrm{d}s$$
$$- (\lambda_1 - \lambda_0)\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'_-(\pi_s)\mathrm{d}s,$$

keeping in mind the fact that $\tilde{\rho}'(\pi) = -1$ for all $\pi > A^*$, it follows from (6.6) that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s I_{(\pi_s < A^*)}\mathrm{d}s - \lambda\int_0^t (1-\pi_s) I_{(\pi_s \geq A^*)}\mathrm{d}s + \tilde{L}_t, \tag{6.22}$$

where

$$\tilde{L}_t \equiv \int_0^t \frac{\pi_{s-}(1-\pi_{s-})(\lambda_1 - \lambda_0)}{\lambda_1\pi_{s-} + \lambda_0(1-\pi_{s-})}\tilde{\rho}'_-(\pi_{s-})(\mathrm{d}X_s - (\lambda_1\pi_{s-} + \lambda_0(1-\pi_{s-})))\mathrm{d}s$$

is the martingale part of this decomposition.

Let $\tilde{B} = \inf\{B : \tilde{\rho}(B) = 1 - B\}$. Since $\tilde{\rho}(\pi)$ is concave, the continuous fit condition implies that $A^* \in [\tilde{B}, 1]$. On the other hand if $A^* > \tilde{B}$ then on the interval $(\tilde{B}, A^*)$ we shall have $\tilde{\rho}'(\pi) = -1$. Further $\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} > \pi$, which implies that $\rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}) = 1 - \frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}$ for any $\pi \in (\tilde{B}, A^*)$. Taking into consideration these facts we obtain that Eq. (6.6) in this case can be satisfied only at the point $\pi = \frac{\lambda}{\lambda+c}$, which gives a contradiction. Thus $\tilde{B} = A^*$ and

$$\{\pi_s < A^*\} = \{\tilde{\rho}(\pi_s) < 1 - \pi_s\},$$
$$\{\pi_s \geq A^*\} = \{\tilde{\rho}(\pi_s) = 1 - \pi_s\}. \tag{6.23}$$

Therefore from (6.22) and (6.22) we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s I_{(\tilde{\rho}(\pi_s) < 1-\pi_s)}\mathrm{d}s - \lambda\int_0^t (1-\pi_s) I_{(\tilde{\rho}(\pi_s) = 1-\pi_s)}\mathrm{d}s + \tilde{L}_t. \tag{6.24}$$

Since $\tilde{\rho}(\pi)$ is a bounded function, $\tilde{\rho}(\pi_t)$ is a supermartingale of the class D. Hence $\tilde{\rho}(\pi_t) \in S^1$ and $\tilde{L}_t$ is a uniformly integrable martingale.

We shall show now that $A^* \geq \frac{\lambda}{\lambda+c}$. Indeed passing to the limit in (6.6) when $\pi \uparrow A^*$ and using the continuous fit condition we obtain that

$$\tilde{\rho}'(A^*-) = \frac{A^*[(\lambda_1 - \lambda_0 - c) - (\lambda_1 - \lambda_0)A^*]}{(1-A^*)[\lambda - A^*(\lambda_1 - \lambda_0)]} \tag{6.25}$$

if $A^* \neq \hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$.

Since $\tilde{\rho}(\pi)$ is concave and satisfies the normal entrance condition, we have $\tilde{\rho}'(\pi) \in [-1, 0]$.

Consider the following three cases:

(a) If $c < \lambda_1 - \lambda_0 - \lambda$, by resolving the system

$$\begin{cases} \tilde{\rho}'(A^*-) \geq -1 \\ \tilde{\rho}'(A^*-) \leq 0 \end{cases} \tag{6.26}$$

we obtain that $A^* \in [\frac{\lambda}{\lambda+c}, \frac{\lambda_1-\lambda_0-c}{\lambda_1-\lambda_0}]$.

(b) If $c = \lambda_1 - \lambda_0 - \lambda$, and $A^* \neq \hat{B} = \frac{\lambda}{\lambda+c} = \frac{\lambda}{\lambda_1-\lambda_0}$, then by (6.25) $\tilde{\rho}'(A^*-) = \frac{A^*}{1-A^*} > 0$, which gives a contradiction, since $\tilde{\rho}'$ is a non-positive function. Thus, $A^* = \hat{B} = \frac{\lambda}{\lambda+c}$.

(c) If $c > \lambda_1 - \lambda_0 - \lambda$, then the smooth fit condition and (6.25) implies that

$$\frac{A^*[(\lambda_1 - \lambda_0 - c) - (\lambda_1 - \lambda_0)A^*]}{(1 - A^*)[\lambda - A^*(\lambda_1 - \lambda_0)]} = -1,$$

which gives

$$A^* = \frac{\lambda}{\lambda + c}.$$

Hence $A^* \geq \frac{\lambda}{\lambda+c}$ in all cases. This inequality implies that $c\pi_s - \lambda(1 - \pi_s)$ is positive on the set $\pi_s \geq A^*$. Therefore we can rewrite (6.24) in the following form:

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s \mathrm{d}s + \int_0^t (c\pi_s - \lambda(1 - \pi_s))^+ I_{(\tilde{\rho}(\pi_s)=1-\pi_s)} \mathrm{d}s + \tilde{L}_t. \tag{6.27}$$

This implies that the triple $Y_t = \tilde{\rho}(\pi_t)$, $\nu_t = 0$, $L_t = \int_0^t \frac{\pi_{s-}(1-\pi_{s-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-})} \tilde{\rho}'_-(\pi_{s-})(\mathrm{d}X_s - (\lambda_1\pi_{s-} + \lambda_0(1 - \pi_{s-})))\mathrm{d}s$ satisfies RBSDE (4.3). Besides, (IV) straightforwardly follows from the concavity of the function $\tilde{\rho}(\pi)$. Since $\lim_{t\to\infty} \pi_t = 1$, this gives together with (IV) that condition (V) is also satisfied. Therefore by Theorem 4.1 $\tilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (6.5) $\tilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\tilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0, 1]$. □

In the next theorem, we consider the case $\lambda_1 < \lambda_0$. Note that in this case, contrary to the case $\lambda_1 > \lambda_0$, the process μ_t is not equal to zero, which leads to additional technical difficulties. Due to the lack of the space we give this theorem without proof.

Theorem 6.2. *Let $\lambda_1 < \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0, 1]$ such that:*
(1) *$\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ and satisfies a differential–difference equation:*

$$(\lambda - \pi(\lambda_1 - \lambda_0))(1 - \pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1 - \pi)) \times \left[\rho\left(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1 - \pi)}\right) - \rho(\pi)\right] = -c\pi, \tag{6.28}$$

if $\pi < B^*$.
(2) *It is equal to $1 - \pi$, if $\pi \geq B^*$.*
(3*) *It satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative, concave function satisfying (1), (2), (3*) *for some $A^* \in (0, 1]$, then the triple $V_t = \tilde{\rho}(\pi_t), \nu_t = \frac{(\lambda+\pi_t(\lambda_1-\lambda_0))(1-S(\pi_t)-\rho(S(\pi_t)))}{c\pi_t-\lambda(1-\pi_t)} I_{(\tilde{\rho}(\pi_t)=1-\pi_t)}$, where $S(\pi) = \frac{\lambda_1\pi}{\lambda_0+(\lambda_1-\lambda_0)\pi}$ and L_t is equal to the martingale part of $\tilde{\rho}(\pi_t)$, satisfies the RBSDE* (I)–(VI). *In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.*

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