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Physics

Strong Coupling Constant from Hadronic ² Decays within the Dispersive Treatment

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ABSTRACT. We extract numerical values for the strong coupling constant $\sim_s$ from the revised ALEPH ² decay data for the non-strange vector hadronic spectral function. The distinguished feature of our procedure is that we employ the global quark-hadron duality in the bounded below region, $s_c \leq s \leq m_\tau^2$, where s_c is the onset of the perturbative QCD continuum. On this interval, we construct the modified Finite Energy Sum Rules (FESRs) using the "spectral weights" $w_{kl}(s)$ (k,l=1,2..) associated with the spectral moments of the invariant mass distribution. These sum rules are used in conjunction with the dimension-two sum rule valid in the chiral limit. We have performed several determinations of the strong coupling constant $\sim_s$ and the continuum threshold s_c, combining different w_{kl} based FESRs with the dimension-two FESR. The numerical values for the strong coupling constant from different determinations are found to be in good agreement. We obtain, in the $\overline{\mathrm{MS}}$ scheme, the values $\alpha_s(m_\tau^2) = 0.322 \pm 0.011_{ex}$ using a dispersive modification of Contour Improved Perturbation Theory and $\alpha_s(m_\tau^2) = 0.298 \pm 0.012_{ex}$ using Fixed Order Perturbation Theory. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: Quantum Chromodynamics, renormalization group equation, perturbation theory

The inclusive hadronic decays of the τ-lepton provide an ideal system to study quantum chromodynamics (QCD) at low and moderate energies (see the pioneering work [1] and the literature therein). The accuracy of the experimental data for various observables of the τ-lepton system is steadily improving [2,3]. On the theoretical side, very accurate formulas are available. The relevant theoretical quantity, the Adler function, has been calculated in perturbation theory up to order $\sim_s^4$ [4]. The small non-perturbative corrections to the observables are under control within the Operator Product Expansion (OPE). Confronting the perturbative QCD (pQCD) predictions with the experimental data, one may determine the fundamental and non-perturbative parameters of the theory. In particular, the precise determination of the strong coupling constant, $\sim_s(m_\tau^2)$, from non-strange τ decays has been the subject of active investigations in recent years [1-5].

The central theoretical quantity of interest in these decays is the hadronic spectral function related with the two point correlator of hadronic currents. The correlator is approximated in perturbation theory supplemented with the OPE. The QCD predictions can be confronted with the time-like t decay data owing to the notion of the quark-hadron duality. An exhaustive review on the duality can be found in [5]. The duality is implemented via the Finite Energy Sum Rules (FESRs) [6]. The FESRs are derived from Cauchy's theorem using the analytical properties of the hadronic current-current correlator. Analyticity and the renormalization group (RG) invariance cannot be combined unambiguously. Usually, two approaches are used. They are referred to as Fixed Order Perturbation Theory (FOPT) [1] and Contour Improved Perturbation Theory (CIPT) [7,8]. It should be remarked that the predictions obtained within CIPT are deteriorated due to the non-physical "Landau singularities", which are present in the perturbative running coupling [9]. The dispersive or analytic modifications of pQCD, developed in past years [9,10], are free from this inconsistency.

The present work is the continuation of works [11,12], where we have analyzed the high precision hadronic τ-decay data from the ALEPH [2] collaboration using a new dispersive approach. This approach is based on the "semi-experimental" representation of the hadronic vector spectral function

$$\mathrm{v}_1(s)\,|_{s.ex} = \bar{}\,(s_c - s)\mathrm{v}_1(s)\,|_{ex} + \bar{}\,(s - s_c)\mathrm{v}_1(s)\,|_{pQCD}, \tag{1}$$

where $\mathrm{v}_1(s)\,|_{ex}$ denotes the spectral function measured on the experiment and $\mathrm{v}_1(s)\,|_{pQCD}$ is the corresponding pQCD expression. The parameter s_c is the so-called continuum threshold, the minimal value of the energy squared above, which we trust pQCD. This parameter should be bounded above $0 < s_c < m_²^2$ ($m_²$ being the mass of the τ lepton). Otherwise, s_c must be large enough for pQCD to be valid. From (1) one can derive modified FESRs which relates the weighted data integrals over the bounded below region $s_c < s < s_{\max} = m_²^2$ with the corresponding QCD expressions. In previous works [11,12], we employed a modified FESR involving the kinematic weight function. In the present work, we extend the same approach to the FESRs involving the weights determining spectral moments of the hadronic invariant mass distribution [13]

$$w_{kl}(s, s_{\max}) = \frac{1}{s_{\max}}\left(1 - \frac{s}{s_{\max}}\right)^{k+2}\left(\frac{s}{s_{\max}}\right)^{l}\left(1 + 2\frac{s}{s_{\max}}\right), \tag{2}$$

where $s_{\max} = m_²^2$ and $k, l = 1, 2..$ .The purpose of this work is to examine the compatibility of these FESRs with the dimension-two FESR, the sum rule which is fulfilled in the chiral limit because of the absence of the dimension d=2 operator in the OPE of the correlator [14].

The basic object of the theoretical calculation is the two point correlator of hadronic non-strange vector currents:

$$\Pi_{«¬}(q^2) = i\int \exp(iqx)\left\langle 0\,|\,T(V_«(x)V_¬(x))\,|\,0\right\rangle d^4x = (-g_{«¬}q^2 + q_«q_¬)\Pi^{(1)}(q^2) + q_«q_¬\Pi^{(0)}(q^2), \tag{3}$$

where $V_«(x) = \bar{u}(x)¥_«d(x)$ is the vector current, the superscripts (1) and (0) label the spin. For the non-strange vector-vector correlator the spin zero component $\Pi^{(0)}(q^2)$ is numerically negligible. The imaginary part of $\Pi^{(1)}(q^2)$ is connected with the measured vector spectral function $\mathrm{v}_1(s)$

$$\mathrm{v}_1(s) = 2®\,\mathrm{Im}\,\Pi^{(1)}(s + i0). \tag{4}$$

The exact non-perturbative correlation function is analytic in the whole complex s-plane except the cut running along the positive real axis. Using this fact one obtains from Cauchy's theorem the FESR:

$$\int_{s_{th}}^{s_0} w(s) \text{v}_1(s) ds = i® \oint_{|s|=s_0} w(s) \Pi^{(1)}(s) ds, \tag{5}$$

where s_{th} is the hadronic threshold and $w(s)$ is any analytic function in $|s| < M$ with $M > s_0$. The introduction of the Adler function

$$D(-q^2) = -4®^2 q^2 \frac{d}{q^2} \Pi^{(1)}(q^2) \tag{6}$$

allows us to rewrite the FESR as follows:

$$\int_{s_{th}}^{s_0} w(s) \text{v}_1(s) ds = -\frac{1}{4® i} \oint_{|z|=s_0} \frac{w_1(-z)}{z} D(z) dz, \tag{7}$$

where $z = Q^2 = -q^2$ and $w_1(z) = \int_{s_0}^{z} w(z) dz$.

For sufficiently large s_0 the right hand sides of (5) and (7) can be calculated in pQCD. The QCD expression for $\Pi^{(1)}(q^2)$ can be represented as:

$$\Pi^{(1)}(q^2)|_{QCD} = \Pi^{(1)}(q^2)|_{PT} + \Pi^{(1)}(q^2)|_{OPE} + \Pi^{(1)}(q^2)|_{DV}, \tag{8}$$

where $\Pi^{(1)}(q^2)|_{PT}$ stands for the pure perturbation theory approximation to the correlation function. The perturbation theory approximation to the Adler function reads

$$D(Q^2)|_{PT} = \sum_{n=0} d_n (Q^2/«^2) \left(\frac{\sim_s(«^2)}{®} \right)^n, \tag{9}$$

. . . . $\sim_s(«^2)$ is the running coupling parameter of QCD at scale « in the $\overline{\text{MS}}$ scheme. Since the Adler function is a renormalization group (RG) invariant quantity, the expansion (9) can be resummed with the choice $«^2 = Q^2$

$$D(Q^2)|_{\text{PT}} = \sum_{n=0} K_n \left(\frac{\sim_s(Q^2)}{®} \right)^n, \tag{10}$$

where $K_n = d_n(1)$. The known coefficients in the $\overline{\text{MS}}$ scheme, for three active quarks ($n_f = 3$), have the values [4]

$$K_0 = K_1 = 1, \; K_2 = 1.63982, \; K_3 = 6.37101, \; K_4 = 49.07570. \tag{11}$$

The second term in (8) corresponds to the higher-dimension ($\text{d} \geq 2$) OPE contribution to the correlation function

$$\Pi^{(1)}(s)|_{\text{OPE}} = \sum_{k=1,2...} \frac{C_{2k}(s)}{(-s)^k}, \tag{12}$$

where $C_{2k}(s)$, $k = 1, 2...$ stand for the effective condensate combinations. They depend on s via $\sim_s(s)$. Up to logarithmic corrections, proportional to $\sim_s^2$

$$C_{2k} = C_{2k}^{(0)} + C_{2k}^{(1)} \alpha_s(m_\tau^2)\,, \tag{13}$$

where $k = d/2 \geq 2$ (the dimension-two term is tiny and can safely be neglected). The coefficients $C_{2k}^{(i)}$ are scale invariant quantities. The third term in (8) stands for the non-perturbative contributions that are invisible in the OPE. They are referred to as "duality violations" [5]. No systematic method is known to calculate the duality violations. Recently, to describe this effect in τ decays a physically motivated model was suggested (see [15] and the literature therein).

Let us insert in the FESR (7) a spectral weight $w_{kl}(s)$. Taking into account the prescription (1) we can rewrite the FESR as follows

$$\int_{s_c}^{s_0} w_{kl}(s) v_1(s)|_{ex}\, ds = -\frac{1}{4\pi i}\left(\int_{|z|=s_0} \frac{w_{1,kl}(-z)}{z} D(z)|_{PT}\, dz - \int_{|z|=s_c} \frac{w_{1,kl}(-z)}{z} D(z)|_{PT}\, dz \right), \tag{14}$$

here we assume that $s_0 = m_\tau^2 > s_c$. Depending on the method used (FOPT or CIPT) we must use, on the right of (14), the series expansion (9) or (10) respectively. To distinguish the new approach from the conventional one, in this work we will employ the abbreviations FOPT$^+$ and CIPT$^+$. Note that, on the right hand side of eq. (14), we have neglected the OPE and duality violating contributions to the Adler function. Let us explain this point. It follows from eq. (13) that the integrated leading order power suppressed OPE contributions will cancel in the difference between two terms on the right of (14), while the integrated duality violating term (for a given w_{kl} weight) may become negligible at some special values of s_c referred to as duality points [16]. We will assume that s_c is a duality point. Further advantage of the representation (14) is that, in the case of CIPT$^+$, the non-physical contributions coming from the "Landau singularity" of the running coupling are also eliminated in the difference [11]. Note that each of the w_{kl} FESRs given in (14) connects the parameters s_c and Λ ($\Lambda = \Lambda_3$ being the QCD scale parameter in the $\overline{\text{MS}}$ scheme for $n_f = 3$ quark flavours). A further important constraint on the parameters is imposed by the dimension-two FESR [14]. In the vector channel the dimension-two FESR reads

$$\int_{s_{th}}^{s_c} \mathrm{v}_1(s)|_{ex}\, ds = \frac{1}{4\pi i} \oint_{|z|=s_c} \frac{z+s_c}{z} D(z) dz\,. \tag{15}$$

Let us point out that we identified the duality radius s_c in the FESR (15) with the continuum threshold entering in the FESR (14). In our analysis, we use the recently revised and updated ALEPH data for non-strange vector (V) spectral distribution measured in hadronic τ decays [3]. The updated and corrected data are publicly available (see literature in [3]). The input values from the ALEPH Collaboration are

$$\begin{aligned} m_\tau &= 1.77682 \pm 0.00016 \text{ GeV}, \\ \mathrm{B}_e &= 0.17818 \pm 0.00032, \\ S_{EW} &= 1.0198 \pm 0.0006, \\ |\mathrm{V}_{ud}| &= 0.97418 \pm 0.00019, \end{aligned}$$

here B_e denotes the electronic branching fraction, S_{EW} is an electroweak correction and $|\mathrm{V}_{ud}|$ represents the flavor mixing matrix element. The new ALEPH data for the invariant mass distributions are organized in bins with variable width. The bin, number k, is centered at sbin(k) and has the width dsbin(k). The highest bin is centered at sbin(80)=3.3375> m_τ^2 . The invariant mass distribution sfm2(k) is related to the vector spectral function by

$$\mathrm{v}_1(sbin(k)) = \frac{m_\tau^2 sfm2(k)}{6\,|\,\mathrm{V}_{ud}\,|^2\; S_{EW}\,100\mathrm{B}_e\, w_T\,(sbin(k))dsbin(k)},$$

where $w_T(s) = (1 - s/m_\tau^2)^2(1 + 2s/m_\tau^2)$ is the kinematic weight. The hadronic integrals on the left hand sides of FESRs (14) and (15) are computed in the standard fashion by using the rectangle rule:

$$I_{ex}^{w}(s_c, s_0) = \int_{s_c}^{s^0} w(s)\mathrm{v}_1(s)ds \approx \sum_{k_c}^{k_0} w(sbin(k))\mathrm{v}_1(sbin(k))dsbin(k).$$

A particular w_{kl} FESR from (14) will be compatible with the OPE condition (15) if the parameters satisfy the following system of equations

$$\begin{aligned} &\Phi_1^{kl}(s_c, \Lambda) = I_1^{kl}(s_c)\,|_{ex}, \\ &\Phi_2(s_c, \Lambda) = I_2(s_c)\,|_{ex}, \end{aligned} \tag{16}$$

where the functions $\Phi_1^{kl}(s_c, \Lambda)$ and $\Phi_2(s_c, \Lambda)$ stand for the QCD parts of the sum rules (14) and (15) respectively, while $I_1^{kl}(s_c)\,|_{ex}$ and $I_2(s_c)\,|_{ex}$ denote the corresponding weighted integrals over the hadronic data. In general the system of equations (16) has several solutions. To select admissable solutions, we impose the following constraints on the expected values of the parameters

$$1\ \mathrm{GeV}^2 \le s_c \le m_\tau^2, \tag{17}$$

$$0.280\ \ \mathrm{GeV} \le \Lambda \le 0.420\ \ \mathrm{GeV}, \tag{18}$$

with these constraints the system (16) has a unique solution. The bound (17) guarantees that pQCD can safely be used on the circle $s = |\,s_c\,|$, while the most of determinations of the strong coupling constant from the τ decays [17] satisfy the constraint (18). In numerical calculations, we employ the RG equation at four-loop order in the $\overline{\mathrm{MS}}$ scheme. As in [11,12] we use a very accurate analytic approximation to the four-loop order running coupling determined in terms of the Lambert W function (for details see [18]). We employ the $\mathrm{N^3LO}$ order approximation to the Adler function. The relevant coefficients are given in (11). To determine the errors on the parameters, we use the system of equations (16) together with the covariance matrices provided by ALEPH [3]. In this paper we omit the cumbersome details of the error calculations. Relevant formulas for the error analysis can be found in the Appendix of [11].

We have solved numerically the system (16) for several w_{kl} weight functions. The results obtained by using the FOPT$^+$ and CIPT$^+$ approaches are presented in Tables 1 and 2 respectively. Looking at the numbers in Tables 1 and 2 one sees that the FOPT$^+$ and CIPT$^+$ estimates for the continuum threshold s_c, for a particular w_{kl} FESR, are very close. In contrast to this, the FOPT$^+$ scheme predicts systematically smaller values for the strong coupling constant. This issue is well known. Different ways of performing the RG resummation lead to differing results [2,3,7,8,13-17]. The numerical results for α_s obtained from different w_{kl} FESRs (within the same resummation scheme) are in good agreement within the errors. The experimental uncertainties on the parameters, obtained within the two resummation schemes, are found to be almost equal.

Table 1. Numerical results for the parameters obtained from w_{kl} FESRs combined with the dimension-two OPE condition and using FOPT$^+$. The errors are given from the experimental uncertainties only

w_{kl}	s_c GeV^2	$\Lambda_{n_f=3}$ GeV	$\alpha_s(m_\tau^2)$
w_{00}	1.69 ± 0.03	0.303 ± 0.024	0.298 ± 0.012
w_{10}	1.72 ± 0.02	0.299 ± 0.022	0.296 ± 0.011
w_{11}	1.72 ± 0.03	0.299 ± 0.024	0.296 ± 0.012
w_{12}	1.69 ± 0.03	0.303 ± 0.024	0.298 ± 0.012
w_{13}	1.69 ± 0.03	0.306 ± 0.028	0.299 ± 0.014

Table 2. Numerical results for the parameters obtained from w_{kl} FESRs combined with the dimension-two OPE condition and using CIPT$^+$. The errors are given from the experimental uncertainties only

w_{kl}	s_c GeV^2	$\Lambda_{n_f=3}$ GeV	$\alpha_s(m_\tau^2)$
w_{00}	1.70 ± 0.03	0.349 ± 0.021	0.322 ± 0.011
w_{10}	1.73 ± 0.03	0.339 ± 0.019	0.316 ± 0.010
w_{11}	1.72 ± 0.03	0.344 ± 0.020	0.319 ± 0.011
w_{12}	1.70 ± 0.03	0.348 ± 0.022	0.321 ± 0.011
w_{13}	1.68 ± 0.04	0.358 ± 0.025	0.327 ± 0.013

As our best values for $\sim_s(m_z^2)$, we take the values obtained from the w_{00} FESR

$$\sim_s(m_z^2)\,|_{\mathrm{FOPT}^+} = 0.298 \pm 0.012\,|_{ex}, \tag{19}$$

$$\sim_s(m_z^2)\,|_{\mathrm{CIPT}^+} = 0.322 \pm 0.011\,|_{ex}\,. \tag{20}$$

Our determinations (19) and (20) are in good agreement with the $\mathrm{N^4LO}$ order FOPT- and CIPT determinations of [15] obtained from the same data using V channel $¡^2$ fits. Comparing our FOPT$^+$ result (19) with the FOPT result from Ref. [15], we see that the predicted central values for the coupling constant in the two determinations are practically the same. The errors quoted in [15] are also close to those given in (19)-(20).

To conclude, we have extracted numerical values for the strong coupling constant from the revised ALEPH non-strange vector data. We have extended the analysis method employed in our previous publications [11,12] to the FESRs involving the particular class of weights, the "spectral weights" $w_{kl}(s)$. A valuable feature of the new approach to the determination of the coupling is that it enables us to minimize the "contamination" of the extracted values of $\sim_s(m_z^2)$ from non-perturbative effects and the "Landau pole" contributions. We performed several independent determinations of $\sim_s(m_z^2)$ using different w_{kl} FESRs augmented with the dimension-two sum rule. The results obtained within the FOPT$^+$ and CIPT$^+$ resummation schemes are separately presented. Performing evolution of the $\sim_s$ values (19) and (20) to the Z^0-mass scale, we obtain:

$$\sim_s(M_z^2)\,|_{\mathrm{FOPT}^+} = 0.1158 \pm 0.0016\,|_{ex}\ \pm 0.0005\,|_{ev}, \tag{21}$$

$$\sim_s(M_z^2)\,|_{\mathrm{CIPT}^+} = 0.1189 \pm 0.0013\,|_{ex}\ \pm 0.0005\,|_{ev}\,. \tag{22}$$

We remark that the CIPT$^+$ value in (22) is in good agreement with the recent world summary of the determinations of the strong coupling constant [17].

ფიზიკა

ძლიერი ურთიერთმოქმედების მუდმივას განსაზღვრა ^{2} ლეპტონის ადრონული დაშლებიდან დისპერსიული მეთოდის გამოყენებით

ბ. მაღრაძე

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(წარმოდგენილია აკადემიის წევრის ა. კვინიხიძის მიერ)

ნაშრომში განსაზღვრულია ძლიერი ურთიერთმოქმედების $\sim_s$ მუდმივას რიცხვითი მნიშვნელობა ^{2} ლეპტონის ადრონული დაშლების შესწავლის საფუძველზე. გამოყენებულია ALEPH კოლაბორაციის განახლებული და შესწორებული მონაცემები ვექტორული არაუცნაური სპექტ-რალური ფუნქციისათვის. კვარკ-ადრონული დუალობა მოთხოვნილია ენერგიის კვადრატის ცვლა-დის ქვემოდან შემოსაზღვრულ არეში $s_c \le s \le m_2^2$, სადაც s_c პერტურბაციული კვანტური ქრომო-დინამიკის ზღურბლია. ამ ინტერვალში გამოყენებულ იქნა მოდიფიცირებული სასრულ ენერგიულ ჯამთა წესები, ჩაწერილი $w_{kl}(s)$ "სპექტრალური წონითი ფუნქციებით", რომლებიც განსაზღვრავენ მასის ინვარიანტული განაწილების ფუნქციის სპექტრალურ მომენტებს. მოთხოვნილია თითოეული ამ ჯამთა წესის თავსებადობა ჯამთა წესთან, რომელიც აღწერს კირალურ ზღვარში d=2 განზომილების ოპერატორის განულების პირობას. მიღებულ იქნა $\sim_s$ მუდმივას და s_c პარამეტრის დამოუკიდებელი შეფასებები სხვადასხვა w_{kl} ჯამთა წესების საფუძველზე. მიღებული რიცხვითი შეფასებები კარგად თავსებადია ერთმანეთთან. მიღებულია: $\alpha_s(m_\tau^2) = 0.322 \pm 0.011_{ex}$ კონტურით გაუმჯობესებულ შეშფოთების თეორიაში და $\alpha_s(m_\tau^2) = 0.298 \pm 0.012_{ex}$ ფიქსირებული რიგის შეშფოთების თეორიაში.

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