Theoretical and Mathematical Physics, **135**(1): 531–540 (2003)

EXPLICIT EXPRESSIONS FOR TIMELIKE AND SPACELIKE OBSERVABLES OF QUANTUM CHROMODYNAMICS IN ANALYTIC PERTURBATION THEORY

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We study the possibility of expressing the invariant QCD coupling function (i.e., the effective coupling constant) in an explicit analytic form in two- and three-loop approximations as well as in the case of the Padé-transformed β-function. Both the timelike and spacelike domains are investigated. Technical aspects of the Shirkov–Solovtsov analytic perturbation theory are considered. Explicit expressions for the two- and three-loop effective coupling functions in the timelike domain are obtained. In the last case, we apply a new method of expanding functions represented in an arbitrary loop order of perturbation theory in powers of the two-loop function. The comparison with numerical data in the infrared region shows that the obtained explicit expressions for the three-loop functions agree fully with the exact numerical results.

Keywords: quantum chromodynamics, perturbation theory, renormalization group equation, running coupling constant, renormalization schemes

1. Introduction

A new "renormalization invariant analytic formulation" of the results of calculations in quantum chromodynamics was recently proposed [1]. In this scheme, the effective coupling function[1] and the matrix elements have no unphysical singularities like ghost poles, and a "further advance" into the infrared region becomes possible. A special version of the analytic approach proposed in [4], called analytic perturbation theory (APT), was successfully used to describe some physical processes [3]. In [2], the mathematical properties of APT nonpower asymptotic series were studied; the scheme stability of the APT results was explained based on this [5].

Let $D(Q^2)$ be the Adler function defined on the spacelike domain for some physical process. It is usually written as the power series[2]

$$D_{\mathrm{pt}}(Q^2) = D_0\biggl(1 + \sum_{n=1}^{\infty} d_n \alpha_s^n(Q^2, f)\biggr), \tag{1}$$

where D_0 is a process-dependent constant and f is the number of active quarks for the transferred momentum $Q \equiv \sqrt{Q^2}$. In APT, $D(Q^2)$ is represented by the nonpower asymptotic expansion

$$D_{\mathrm{an}}(Q^2) = D_0\biggl(1 + \sum_{n=1}^{\infty} d_n \mathcal{A}_n(Q^2, f)\biggr), \tag{2}$$

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[1]Following [2], [3], we use the term "effective coupling function" instead of "effective coupling constant."

[2]Here $Q^2 = -q^2$ and $Q^2 > 0$ in the Euclidean domain.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 135, No. 1, pp. 95–106, April, 2003. Original article submitted January 25, 2002; revised July 3, 2002.

where $\mathcal{A}_n(Q^2,f) = \{\alpha_s^n(Q^2,f)\}_{\rm an}$ is the "analyticized power" of the coupling function in the Euclidean domain. Let $R(s)$ be the physical quantity corresponding to $D(Q^2)$ in the timelike domain. In APT, it has the representation [3], [6]

$$R(s) = R_0\bigl(1 + r(s)\bigr), \qquad r(s) = \sum_{n=1}^{\infty} d_n \mathfrak{A}_n(s,f), \tag{3}$$

where the functions $\mathfrak{A}_n$ are related to $\mathcal{A}_n$ by the integral transformation [3]

$$\mathfrak{A}_n(s,f) = -\frac{1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z}\,\mathcal{A}_n(-z,f) \tag{4}$$

introduced in [7], [8] (see also [9]). The inverse transformation has the form

$$\mathcal{A}_n(Q^2,f) = Q^2\int_0^{\infty} \frac{ds}{(s+Q^2)^2}\,\mathfrak{A}_n(s,f). \tag{5}$$

In [2], [6], the universal functions $\mathcal{A}_n$ and $\mathfrak{A}_n$ were studied at the one-loop level. It turned out that they reveal an oscillating behavior in the infrared region. To compute these functions in higher orders, either the iterative solutions of the renormalization group (RG) equation were used [6], [10] or this equation was solved in the complex domain [11].

In [12], [13], the second-order RG equation was solved analytically. The solution was expressed in terms of the Lambert W-function. In the case of the Padé-transformed β-function, the third-order solution can be also expressed through the Lambert function [13]. The form of the obtained solutions is convenient for analytic continuation into the complex domain, which is especially important if APT is applied [14].

In [15], the higher-order perturbation theory solutions of the RG equation (in the class of massless renormalization schemes) were expanded in powers of the two-loop (scheme-independent) solution. Based on this, a new method for relaxing the scheme dependence of the observables in QCD was proposed. Analogous expansions for the observables (obtained from other considerations) were discussed in [16].

In the present paper, we propose concrete formulas for calculating the sets of the functions $\{\mathcal{A}_n(Q^2,f)\}$ and $\{\mathfrak{A}_n(s,f)\}$. In Sec. 2, we derive the general equations for both sets of functions. In Secs. 3 and 4, we calculate the APT functions in the respective second and third orders. In Sec. 5, we describe the general method for calculating these functions in the higher orders of the perturbation theory. In Sec. 6, we consider the global functions taking the quark thresholds into account (these functions were introduced in [6]). In Sec. 7, numerical results are given.

2. General results

The QCD coupling function satisfies the RG equation

$$Q^2\frac{\partial\alpha_s(Q^2,f)}{\partial Q^2} = \beta^f\bigl(\alpha_s(Q^2,f)\bigr) = -\sum_{n=0}^{\infty}\beta_n^f\alpha_s^{n+2}(Q^2,f) \tag{6}$$

with the condition $\alpha_s(\mu^2,f) = g^2/(4\pi)$, where μ is the normalization point. In massless schemes, β_0^f and β_1^f have the universal values

$$\beta_0^f = \frac{1}{4\pi}\left(11 - \frac{2}{3}f\right), \qquad \beta_1^f = \frac{1}{(4\pi)^2}\left(102 - \frac{38}{3}f\right), \tag{7}$$

and in the MS ($\overline{MS}$) scheme, β_2^f is equal [17] to

$$\beta_2^f = \frac{1}{(4\pi)^3}\left(\frac{2857}{2} - \frac{5033f}{18} + \frac{325f^2}{54}\right). \tag{8}$$

The "Euclidean functions," i.e., the functions of the Euclidean argument, are defined by the spectral representation

$$\mathcal{A}_n(Q^2,f) = \frac{1}{\pi}\int_0^\infty \frac{\rho_n(\sigma,f)}{\sigma+Q^2}\,d\sigma = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^t}{(e^t+Q^2\Lambda^{-2})}\tilde{\rho}_n(t,f)\,dt; \tag{9}$$

the spectral function is defined by the formula $\rho_n(\sigma,f) = \rho_n(t,f) = \operatorname{Im}\bigl\{\alpha_s(-\sigma-i0)\bigr\}^n$, where $t = \log(\sigma/\Lambda^2)$ and Λ is the QCD parameter. The "timelike functions" (of the timelike argument) $\bigl\{\mathfrak{A}_n(s,f)\bigr\}$ are defined by the representation

$$\mathfrak{A}_n(s,f) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho_n(\sigma,f) \tag{10}$$

obtained in [18]. For numerical analysis, it is convenient to represent (9) in a regularized form. By (9) and (10), we obtain

$$\mathcal{A}_n(Q^2,f) = \mathcal{A}_n(Q^2,f,T) + \mathfrak{A}_n(\Lambda^2 e^T,f) + \begin{cases} O(\overline{Q}^2 e^{-T}T^{-(1+n)}), & \overline{Q}^2 > 1,\\ O(\overline{Q}^{-2} e^{-T}T^{-(1+n)}), & \overline{Q}^2 < 1,\end{cases} \tag{11}$$

where $\overline{Q}^2 = Q^2/\Lambda^2$ and $\mathcal{A}_n(Q^2,f,T)$ denotes integral (9) taken over the finite interval $-T \le t \le T$. For T sufficiently large (when $(Q^2/\Lambda^2)e^{-T} \ll 1$), the contributions of the order e^{-T} can be dropped, and the second term in the right-hand side of (11) therefore compensates the leading term of the error arising from the regularization of the integral. Formula (11) allows achieving a good accuracy even for moderate values of T.

Below, we obtain equations for the functions $\mathcal{A}_n(Q^2,f)$, $\mathfrak{A}_n(s,f)$, and $\rho_n(\sigma,f)$. In the kth order, the results are given by

$$\frac{\partial\mathcal{A}_n^{(k)}(Q^2,f)}{\partial\log Q^2} = -n\sum_{N=0}^{k-1}\beta_N^f\mathcal{A}_{n+N+1}^{(k)}(Q^2,f), \quad n = 1,2,\dots, \tag{12}$$

$$\frac{\partial\mathfrak{A}_n^{(k)}(s,f)}{\partial\log s} = -n\sum_{N=0}^{k-1}\beta_N^f\mathfrak{A}_{n+N+1}^{(k)}(s,f), \qquad n = 1,2,\dots, \tag{13}$$

$$\frac{\partial\rho_n^{(k)}(\sigma,f)}{\partial\log\sigma} = -n\sum_{N=0}^{k-1}\beta_N^f\rho_{n+N+1}^{(k)}(\sigma,f), \qquad n = 1,2,\dots. \tag{14}$$

In the first order, these equations were obtained in [6]. We write the kth-order RG equation in the form

$$\frac{\partial\alpha_s^n(Q^2,f)}{\partial\log Q^2} = -n\sum_{N=0}^{k-1}\beta_N^f\alpha_s^{n+N+1}(Q^2,f), \tag{15}$$

where $n \ge 1$ is an integer. The analyticized version of Eq. (15) is given by

$$\left\{\frac{\partial\alpha_s^n(Q^2,f)}{\partial\log Q^2}\right\}_{\text{an}} = -n\sum_{N=0}^{k-1}\beta_N^f\mathcal{A}_{n+N+1}(Q^2,f). \tag{16}$$

On the other hand, we have

$$\left\{\frac{\partial\alpha_s^n(Q^2,f)}{\partial\log Q^2}\right\}_{\rm an} = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma+Q^2}\,\mathrm{Im}\,\frac{\partial\alpha_s^n(-\sigma-i0,f)}{\partial\log(-\sigma-i0)}. \tag{17}$$

From the identity $\log(-\sigma-i0)=\log\sigma-i\pi$, it follows that

$$\mathrm{Im}\,\frac{\partial\alpha_s^n(-\sigma-i0,f)}{\partial\log(-\sigma-i0)} = \frac{\partial\,\mathrm{Im}\,\alpha_s^n(-\sigma-i0,f)}{\partial\log\sigma} = \frac{\partial\rho_n(\sigma,f)}{\partial\log\sigma}, \tag{18}$$

and hence

$$\left\{\frac{\partial\alpha_s^n(Q^2,f)}{\partial\log Q^2}\right\}_{\rm an} = +\frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma+Q^2}\frac{\partial\rho_n(\sigma,f)}{\partial\log\sigma}. \tag{19}$$

Integrating (19) by parts and using the consequence of the asymptotic freedom condition for the spectral density,

$$\rho_n(\sigma\to\infty,f)\simeq\frac{1}{(\log\sigma)^2} \quad\Rightarrow\quad \frac{\sigma}{\sigma+Q^2}\rho_n(\sigma,f)\Big|_0^\infty=0, \tag{20}$$

we find that

$$\left\{\frac{\partial\alpha_s^n(Q^2,f)}{\partial\log Q^2}\right\}_{\rm an} = \frac{1}{\pi}\frac{\partial}{\partial\log Q^2}\int_0^\infty\frac{d\sigma}{\sigma+Q^2}\rho_n(\sigma,f) = \frac{\partial\mathcal{A}_n(Q^2,f)}{\partial\log Q^2}. \tag{21}$$

This relation and formula (16) imply system of equations (12). For $n=1$, Eqs. (12) and (13) are analogues of the RG equation: they can be obtained from the RG equation if α_s^n is respectively replaced by $\mathfrak{A}_n$ and $\mathcal{A}_n$.

3. Two-loop approximation

The exact solution of equation (6) in the second order has the form [12], [13]

$$\alpha_s^{(2)}(Q^2,f) = -\frac{\beta_0}{\beta_1}\frac{1}{1+W_{-1}(\zeta)}, \qquad \zeta = -\frac{1}{eb_1}\left(\frac{Q^2}{\Lambda^2}\right)^{-1/b_1}, \tag{22}$$

where $b_1=\beta_1/\beta_0^2$, $\Lambda\equiv\Lambda_{\overline{MS}}$, and $W_{-1}(\zeta)$ denotes one of the branches of the Lambert W-function [19]. The Lambert function is the multivalued solution of the equation $\zeta=W(\zeta)e^{W(\zeta)}$; $W_k(\zeta)$, $k=0,\pm1,\ldots$, are the branches of this function. Continuing solution (22) into the complex Q^2-plane (see [12]–[14]), we find the spectral functions $\rho_n^{(2)}(\sigma,f)\equiv\tilde\rho_n^{(2)}(t,f)$, $n=1,2\ldots$. For $0\le f\le 6$, we have[3]

$$\tilde\rho_n^{(2)}(t,f) = \left(\frac{\beta_0}{\beta_1}\right)^n \mathrm{Im}\left(-\frac{1}{1+W_1\big(z(t)\big)}\right)^n, \tag{23}$$

where $z(t)=\exp\big(-t/b_1+i(1/b_1-1)\pi\big)/eb_1$. Substituting (23) in formulas (10) and (11), we find the functions $\mathfrak{A}_n^{(2)}(s,f)$ and $\mathcal{A}_n^{(2)}(Q^2,f)$. Integrals (10) can be taken analytically:[4]

$$\mathfrak{A}_1^{(2)}(s,f) = -\frac{\beta_0}{\beta_1}-\frac{1}{\pi\beta_1}\,\mathrm{Im}\left(\frac{1}{\alpha^{(2)}(-s)}\right), \tag{24}$$

$$\mathfrak{A}_2^{(2)}(s,f) = \frac{1}{\pi\beta_1}\,\mathrm{Im}\log\left(1+\frac{\beta_1}{\beta_0}\alpha^{(2)}(-s)\right), \tag{25}$$

$$\mathfrak{A}_3(s,f) = -\frac{\beta_0}{\beta_1}\frac{1}{\pi\beta_1}\,\mathrm{Im}\left\{\log\left(1+\frac{\beta_1}{\beta_0}\alpha^{(2)}(-s)\right)-\frac{\beta_1}{\beta_0}\alpha^{(2)}(-s)\right\}, \tag{26}$$

$$\mathfrak{A}_4(s,f) = \left(-\frac{\beta_0}{\beta_1}\right)^2\frac{1}{\pi\beta_1}\,\mathrm{Im}\left\{\log\left(1+\frac{\beta_1}{\beta_0}\alpha^{(2)}(-s)\right)-\frac{\beta_1}{\beta_0}\alpha^{(2)}(-s)+\frac{\beta_1^2}{2\beta_0^2}\alpha^{(2)2}(-s)\right\}, \tag{27}$$

[3]For $f>6$, formula (23) has a different form [13], [14].

[4]One-loop formulas for the timelike functions are given in [6].

and so on. Here, we introduce the notation

$$\alpha^{(2)}(-s) = \alpha_s^{(2)}(-s - i0, f) = -\frac{\beta_0}{\beta_1}\frac{1}{1 + W_1(z_s)}, \tag{28}$$

where $z_s = (s/\Lambda^2)^{-1/b_1} e^{i\pi(1/b_1-1)-1}/b_1$. This form is convenient because we can use different approximate two-loop functions expressed, for example, in terms of "double logarithms" instead of the Lambert function. These relations thus allow obtaining the analogue of the coupling function on the spacelike domain in the timelike domain without difficult calculations of the spectral integrals.

We note that the function $\mathfrak{A}_n(s, f)$ is proportional to the remainder after the $(n-2)$th term of the Taylor expansion of the function $\log\bigl(1 + (\beta_1/\beta_0)\alpha^{(2)}(-s)\bigr)$ in powers of $(\beta_1/\beta_0)\alpha^{(2)}(-s)$.

By the asymptotic properties of the W-function [19], we immediately obtain the result in [18]: we have

$$\mathfrak{A}_1^{(2)}(s, f) \to \frac{1}{\beta_0} \quad \text{and} \quad \mathfrak{A}_n^{(2)}(s, f) \to 0 \quad \text{for } n > 1 \tag{29}$$

as $s \to 0$.

Function (24) has the formal expansion

$$\mathfrak{A}_1^{(2)}(s, f) = \alpha_s^{(2)}(s, f) - \frac{1}{3}\pi^2\beta_0^2\alpha_s^{(2)3}(s, f) - \frac{5}{6}\pi^2\beta_1\beta_0\alpha_s^{(2)4}(s, f) - \dots, \tag{30}$$

where $\alpha_s^{(2)}(s, f)$ is the exact solution of two-loop RG equation (22). The higher-order functions $\mathfrak{A}_n$, $n = 1, 2, \dots$, have analogous expansions. Substituting them in representation (3), we obtain the expansion for $R(s)$ in powers of the traditional coupling function α_s. But the coefficients of this series contain extra π^2-factors. Similar expansions (in the powers of the iterative coupling function) for timelike observables were previously introduced in [7], [8] (see also [20], [21]). In [21], the contribution of π^2-factors was calculated up to the terms α_s^5 in $R_{e^+e^-}$ and R_τ, and it was established that they give the leading contribution to the expansion coefficients $R_{e^+e^-}(s) = \sigma_{\text{tot}}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. Different quantities in timelike domain were recently considered for $f = 3, 4, 5$ [22]. The higher-order π^2-effects were taken into account, and it was shown that the experimental value of $\alpha_s(M_z^2)$ depends considerably on these effects.

4. Three-loop approximation

After the Padé transformation, the third-order β-function becomes

$$\beta_{\text{Padé}} = -\beta_0\alpha_s^2\left(1 + \frac{\beta_1\alpha_s}{\beta_0 - \beta_0\beta_2\beta_1^{-1}\alpha_s}\right).$$

The solution of RG equation (6) for this case was found in [13]. It is given by

$$\alpha_{\text{Padé}}^{(3)}(Q^2, f) = -\frac{\beta_0}{\beta_1}\frac{1}{1 - \beta_0\beta_2\beta_1^{-2} + W_{-1}(\xi)}, \tag{31}$$

where $\xi = -\exp\bigl[(\beta_0\beta_2/\beta_1^2)(Q^2/\Lambda^2)^{-1/b_1}\bigr]/eb_1$. For the spectral functions $\rho_{\text{Padé},n}(\sigma, f) \equiv \tilde{\rho}_{\text{Padé},n}(t, f)$ (for $0 \le f \le 6$), we obtain

$$\tilde{\rho}_{\text{Padé},n}^{(3)}(t, f) = -\left(\frac{\beta_0}{\beta_1}\right)^n \operatorname{Im}\left(-\frac{1}{1 - \beta_2\beta_0\beta_1^{-2} + W_{-1}(Z(t))}\right)^n, \tag{32}$$

where $Z(t) = \exp\bigl(\beta_0\beta_2/\beta_1^2 - t/b_1 - i(1/b_1 - 1)\pi\bigr)/eb_1$. In the case of weight function (32), spectral integral (10) yields

$$\mathfrak{A}^{(3)}_{\text{Padé},1}(s,f) = -\frac{1}{\pi\beta_0}\left(\frac{1}{\eta}\operatorname{Im}\log\bigl(W_1(Z_s)\bigr) + \left(1-\frac{1}{\eta}\right)\operatorname{Im}\log\bigl(\eta + W_1(Z_s)\bigr) - \pi\right), \tag{33}$$

$$\mathfrak{A}^{(3)}_{\text{Padé},2}(s,f) = \frac{1}{\pi\beta_1}\left(\frac{1}{\eta^2}\operatorname{Im}\log\left(\frac{W_1(Z_s)}{\eta + W_1(Z_s)}\right) - \left(1-\frac{1}{\eta}\right)\operatorname{Im}\left(\frac{1}{\eta + W_1(Z_s)}\right)\right), \tag{34}$$

where $\eta = 1 - \beta_0\beta_2/\beta_1^2$ and $Z_s = (s/\Lambda^2)^{-1/b_1}\exp\bigl(-\eta + i(1/b_1 - 1)\pi\bigr)/b_1$. For $n > 2$, we find that

$$\mathfrak{A}^{(3)}_{\text{Padé},n}(s,f) = \frac{p_n}{\eta^{n-2}}\operatorname{Im}\left\{\frac{1}{\eta^2}\left[\log\left(1 - \frac{\eta}{\eta + W_1(Z_s)}\right) + \right.\right.$$
$$\left.\left. + \sum_{N=1}^{n-2}\left(\frac{\eta}{\eta + W_1(Z_s)}\right)^N \frac{1}{N}\right] + \frac{\eta^{n-3}(1-\eta)}{(n-1)\bigl(\eta + W_1(Z_s)\bigr)^{n-1}}\right\}, \tag{35}$$

where $p_n = (-\beta_0/\beta_1)^{n-2}/(\pi\beta_1)$.

5. Multi-loop case

As shown in [15], we can expand the higher-order coupling function in a series in powers of the two-loop function,

$$\alpha_s^{(k)} = \sum_{n=1}^{\infty} c_n^{(k)} \alpha_s^{(2)n}, \tag{36}$$

where $c_1^{(k)} = 1$. The analyticized (spacelike and timelike) versions of (36) are written in the form

$$\mathcal{A}_1^{(k)} = \sum_{n=1}^{\infty} c_n^{(k)} \mathcal{A}_n^{(2)}, \qquad \mathfrak{A}_1^{(k)} = \sum_{n=1}^{\infty} c_n^{(k)} \mathfrak{A}_n^{(2)}. \tag{37}$$

We can therefore regard the two-loop coupling function as the minimal basis for expanding the higher-order solutions. Every observable (except for the quantities with an anomalous dimensionality) is represented by the series $O^{(k)} = \sum_{n=1}^{\infty} O_n^{(k)} \mathcal{A}_n^{(2)}$. We note that the one-loop function cannot be used for this purpose because the multiloop functions have more complicated singularities.[5] But we can use the exact two-loop coupling function (expressed through the Lambert function) to describe the higher-order contributions (see analogous results in [16]).

By (29) and the second expansion in (37), we obtain the universal limiting behavior of the timelike coupling function in any order of the perturbation theory: $\mathfrak{A}_1^{(k)}(s,f) \to 1/\beta_0$ as $s \to 0$. This result was obtained in [18] by other methods. We can thus express the observables in any order in terms of the Lambert function. The obtained expressions have the correct analytic properties and a finite infrared limit.

6. Quark mass thresholds

In "massless" schemes, it is important to take the heavy quark mass thresholds into account [23]. In the APT context, this question was studied in [6], [22], where the special model spectral function

$$\rho_n(\sigma) = \rho_n^{f=3}(\sigma,\Lambda_3) + \sum_{f\geq 4}\Theta(\sigma - M_f^2)\bigl(\rho_n^f(\sigma,\Lambda_f) - \rho_n^{f-1}(\sigma,\Lambda_{f-1})\bigr) \tag{38}$$

[5] For example, the two-loop function contains the double logarithm $\log\log x$, which cannot be expanded in a power series in $1/\log x$.

was proposed. Here M_f corresponds to the quark with flavor f, and the quantities Λ_f are determined by the requirement of the continuity of the coupling function,

$$\alpha_s(M_f^2, f) = \alpha_s(M_f^2, f-1), \quad f = 4, 5, 6. \tag{39}$$

Strictly speaking, in MS-like schemes (except for the leading order), condition (39) should be modified [23], but it does not lead to a noticeable error in APT, and we therefore assume it without restrictions. In the case of exact solutions (22) and (31), relation (39) is solved explicitly for Λ_f. Substituting the above expressions for spectral functions (23) and (32) in (38) and using formulas (9) and (10), we construct the global functions $\mathcal{A}_n(Q^2)$ and $\mathfrak{A}_n(s)$.

7. Numerical results

The calculations were performed using the system Maple V (release 5), where all branches of the Lambert function are realized with an arbitrary accuracy. Here and hereafter, we fix the value $\Lambda_{f=3} = 0.4\,\mathrm{GeV}$ for the scale parameter.

Table 1

Q (GeV)	$\alpha^{(3)}_{\mathrm{num}}$	$\delta\alpha^{(3)}_{\mathrm{ts}}$	$\delta\alpha^{(3)}_{\mathrm{Padé}}$	$\delta\alpha^{(3)}_{\mathrm{it}}$
0.8	0.76491	1.9	−18.9	−15.5
0.9	0.63323	0.8	−7.4	−9.3
1.0	0.55414	0.46	−4.4	−6.1
1.1	0.50028	0.33	−3.0	−4.3
1.2	0.46075	0.21	−2.3	−3.3
1.4	0.40587	0.15	−1.5	−2.2
1.6	0.36901	0.11	−1.1	−1.6
1.8	0.34220	0.08	−0.9	−1.3
2.0	0.32165	0.06	−0.7	−1.1
2.2	0.30527	0.05	−0.6	−1.0
2.6	0.28059	0.04	−0.5	−0.9

The comparison of different approximating functions with the exact three-loop solution $\alpha^{(3)}_{\mathrm{num}}$ of the RG equation: Percentage deviations of the functions from $\alpha^{(3)}_{\mathrm{num}}$ are given.

In Table 1, different approximations of the three-loop coupling function are compared with the exact numerical solution $\alpha^{(3)}_{\mathrm{num}}$ of the RG equation. Here, $\alpha^{(3)}_{\mathrm{it}}$ is the third-order iterative solution [7]

$$\alpha^{(3)}_{\mathrm{it}}(Q^2) = \frac{1}{\beta_0 L} - \frac{\beta_1}{\beta_0^3}\frac{\log L}{L^2} + \frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}(\log^2 L - \log L - 1) + \frac{\beta_2}{\beta_0}\right), \tag{40}$$

where $L = \log Q^2/\Lambda^2_{\overline{MS}}$. Usually, formula (40) is also used in the timelike domain. Here, α_{ts} denotes the truncated series (36): $\alpha^{(3)}_{\mathrm{ts}} = \sum_{i=1}^{5} c_i^{(3)} \alpha_s^{(2)i}$, where $c_1^{(3)} = 1$, $c_2^{(3)} = 0$ (this choice corresponds to the ordinary choice of the parameter Λ [24]), $c_3^{(3)} = \beta_2/\beta_0$, $c_4^{(3)} = 0$, and $c_5^{(3)} = (5/3)\beta_2^2/\beta_0^2$. The best accuracy is achieved by $\alpha^{(3)}_{\mathrm{ts}}$: in the domain $Q \geq 0.9\,\mathrm{GeV}$, it practically coincides with the exact solution. The accuracy of the Padé approximation decreases for $Q \leq 1.6\,\mathrm{GeV}$, while the iterative solution allows obtaining the accuracy of one per cent only for $Q \geq 2.3\,\mathrm{GeV}$. Comparison with the numerical solution shows that the accuracy of the expression $\alpha^{(3)}_{\mathrm{ts}}$ obtained by the expansion in powers of the two-loop functions (with the first four expansion terms) is (approximately by one order) greater than that of the other approximations.

Table 2

$\sqrt{s}$ (GeV)	$\mathfrak{A}^{(3)}_{\mathrm{ts},1}$	$\delta\mathfrak{A}^{(2)}_1$	$\delta\mathfrak{A}^{(3)}_{\mathrm{Padé},1}$	$\mathfrak{A}^{(3)}_{\mathrm{ts},2}$	$\delta\mathfrak{A}^{(2)}_2$	$\delta\mathfrak{A}^{(3)}_{\mathrm{Padé},2}$
0.4	0.50161	1.27	0.11	0.13089	−1.99	0.28
0.6	0.42540	1.90	0.03	0.11420	−0.18	0.29
0.8	0.37711	2.14	−0.04	0.09985	1.04	0.21
1.0	0.34373	2.19	−0.08	0.08859	1.75	0.12
1.4	0.30034	2.11	−0.11	0.07291	2.39	−0.01
2.0	0.26272	1.91	−0.12	0.05897	2.60	−0.09
2.6	0.23975	1.74	−0.11	0.05056	2.57	−0.12

Comparison of the timelike approximating functions $\mathfrak{A}^{(2)}_n$, $\mathfrak{A}^{(3)}_{\mathrm{Padé},n}$, and $\mathfrak{A}^{(3)}_{\mathrm{ts},n}$ for $f=3$, where $\delta\mathfrak{A}^{(2)}_n = \big(\mathfrak{A}^{(3)}_{\mathrm{ts},n} - \mathfrak{A}^{(2)}_n\big)/\mathfrak{A}^{(3)}_{\mathrm{ts},n} \times 100$ and so on.

In Tables 2 and 3, the second- and third-order results are given for the timelike and spacelike functions in the domain with three flavors ($0.4\,\mathrm{GeV} < \sqrt{s} < 2.6\,\mathrm{GeV}$). The functions with the index ts give the best approximation (see Table 1) and can therefore serve as a comparison standard. In the third order, different approximating functions are compared. The relative deviation of $\mathfrak{A}^{(2)}_1$ from $\mathfrak{A}^{(3)}_{\mathrm{ts},1}$ is less than 2.2 %, the deviation of $\mathfrak{A}^{(3)}_{\mathrm{Padé},1}$ from $\mathfrak{A}^{(3)}_{\mathrm{ts},1}$ is less than 0.12 %. In the Euclidean case, the agreement between the considered functions is even better (see Table 3). For the second functions ($n=2$), the corresponding deviations are of the same order: $|\delta\mathfrak{A}^{(2)}_2| < 2.6\,\%$ and $|\delta\mathfrak{A}^{(3)}_{\mathrm{Padé},2}| < 0.3\,\%$.

Table 3

Q (GeV)	$\mathcal{A}^{(2)}_1$	$\mathcal{A}^{(3)}_{\mathrm{Padé},1}$	$\mathcal{A}^{(3)}_{\mathrm{ts},1}$	$\mathcal{A}^{(2)}_2$	$\mathcal{A}^{(3)}_{\mathrm{Padé},2}$	$\mathcal{A}^{(3)}_{\mathrm{ts},2}$
0.4	0.50785	0.51159	0.51193	0.11891	0.11714	0.11727
0.6	0.43844	0.44359	0.44376	0.10718	0.10629	0.10644
0.8	0.39341	0.39907	0.39912	0.09703	0.09678	0.09691
1.0	0.36138	0.36716	0.36713	0.08871	0.08888	0.08899
1.4	0.31822	0.32381	0.32369	0.07621	0.07686	0.07692
2.0	0.27916	0.28420	0.28403	0.06389	0.06479	0.06482
2.6	0.25454	0.25906	0.25888	0.05578	0.05675	0.05675

The Euclidean approximating functions $\mathcal{A}^{(k)}_n(Q^2, f=3)$ in the second and third orders.

We observe the noticeable asymmetry [25] $\delta_{\mathrm{as}} = \big(\mathcal{A}^{(3)}_{\mathrm{tr},1}(Q^2,f) - \mathfrak{A}^{(3)}_{\mathrm{tr},1}(Q^2,f)\big)/\mathcal{A}^{(3)}_{\mathrm{tr},1}(Q^2,f) \times 100$ (cf. Tables 2 and 3); it increases from 2% for $Q=\sqrt{s}=0.4\,\mathrm{GeV}$ to 7.5% for $Q=\sqrt{s}=2.0\,\mathrm{GeV}$. In Table 4, we give the results for the global functions $\mathfrak{A}^{(3)}_{\mathrm{ts},1}(s)$ and $\mathcal{A}^{(3)}_{\mathrm{ts},1}(Q^2)$ in the interval $0.4\,\mathrm{GeV} < Q, \sqrt{s} < 90\,\mathrm{GeV}$. Here, $\Lambda_{f=3} = 0.4\,\mathrm{GeV}$, and the values of Λ_f, $f=4,5,6$, are determined by (39)). The numerical values of quark masses are fixed as follows: $M_1=M_2=M_3=0$, $M_4=1.3\,\mathrm{GeV}$, $M_5=4.3\,\mathrm{GeV}$, and $M_6=170\,\mathrm{GeV}$.

Table 4

$\sqrt{s}$, Q (GeV)	$\mathfrak{A}^{(3)}_{\mathrm{ts},1}(s)$	$\mathcal{A}^{(3)}_{\mathrm{ts},1}(Q^2)$	$\sqrt{s}$, Q (GeV)	$\mathfrak{A}^{(3)}_{\mathrm{ts},1}(s)$	$\mathcal{A}^{(3)}_{\mathrm{ts},1}(Q^2)$
0.4	0.52016	0.53038	5	0.21155	0.22536
0.6	0.44395	0.46209	10	0.17907	0.18759
0.8	0.39566	0.41730	20	0.15510	0.16057
1.0	0.36229	0.38516	50	0.13174	0.13505
1.4	0.31864	0.34139	70	0.12484	0.12767
2	0.28004	0.30132	90	0.12014	0.12270

Comparison of the global timelike and spacelike functions.

8. Conclusion

In our opinion, the main result in the present paper is obtaining the exact two-loop expressions for the QCD invariant coupling function in the timelike domain (see (24)–(27)). In turn, this allowed using the method (proposed in [15]) of expanding the multiloop functions in powers of the two-loop ones to obtain the three-loop expressions. The results of this method were compared with the exact numerical three-loop functions as well as the expressions obtained by the Padé approximation. These numerical estimates support the method of expanding in powers of the two-loop function (see Table 1).

Using the obtained expressions, we constructed the global functions with quark mass thresholds. The functions $\{\mathcal{A}_n\}$ and $\{\mathfrak{A}_n\}$ were previously calculated based on the iterative solution of RG equation (40) (see [6], [9], [10]). We showed that in the two-loop case, the iterative solution in the infrared region leads to errors of the order 4–5% for the functions $\{\mathcal{A}_n\}$ and $\{\mathfrak{A}_n\}$. On the other hand, the accuracy of the experimental data (for example, in the measurements of the quantities $R_{e^+e^-}(s)$ and R_τ) is steadily increasing, and more precise theoretical formulas are therefore needed. We note that an alternative approach was proposed in [11], where the RG equation was solved numerically in the complex plane.

We emphasize that obtaining the explicit two-loop expressions is something more than just increasing the accuracy of analysis or facilitating numerical calculations because it is possible [12] to express the three- and four-loop expressions in terms of the known two-loop ones. This leads to an attractive result: expressions for the invariant coupling function can be obtained with an arbitrary accuracy with respect to the loop number. Moreover, they have the correct analyticity properties and reveal a regular behavior in the infrared region.

Acknowledgments. The authors are sincerely grateful to D. V. Shirkov for the numerous valuable discussions and recommendations and to A. L. Kataev, A. A. Pivovarov, A. V. Sidorov, I. L. Solovtsov, and O. P. Solovtsova for the critical remarks and helpful discussions.

This work was supported in part by the Russian Foundation for Basic Research (Grant Nos. 99-01-00091 and 00-15-96691).

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