Georgian Mathematical Journal
Volume 10 (2003), Number 1, 63–76

ON THE OSCILLATION OF SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

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Abstract. For the differential equation

$$u'(t) + \sum_{i=1}^{m} p_i(t)u(\tau_i(t)) = 0,$$

where $p_i \in L_{loc}(R_+; R_+)$, $\tau_i \in C(R_+; R_+)$, $\tau_i(t) \le t$ for $t \in R^+$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, \dots, m)$, optimal integral conditions for the oscillation of all solutions are established.

2000 Mathematics Subject Classification: 34C10, 34K11.
Key words and phrases: Retarded arguments, proper solutions, oscillation.

1. Introduction

Consider the differential equation

$$u'(t) + \sum_{i=1}^{m} p_i(t)u(\tau_i(t)) = 0, \tag{1.1}$$

where $p_i \in L_{loc}(R_+; R_+)$, $\tau_i \in C(R_+; R_+)$, $\tau_i(t) \le t$ for $t \in R_+$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, \dots, m)$.

The first systematic study for the oscillation of all solutions of equation (1.1) for the case of constant coefficients and constant delays was made by Myshkis [18]. Since then a number of papers have been devoted to this subject. For the case m=1 the reader is referred to the papers [2–7, 10, 12–14, 16, 18], while for the case $m > 1$ to [1, 6, 9, 11, 15, 17]. The difficulties connected with the study of specific properties of solutions of delay differential equations are emphasized in the monograph by Hale [8]. In [12] the following statement is proved.

Theorem 1.1. *Let* $m = 1$,

$$\liminf_{t\to+\infty} \int_{\tau_1(t)}^{t} p_1(s)ds > \frac{1}{e}.$$

Then equation (1.1) *is oscillatory.*

In the case $m > 1$ there are some difficulties in finding optimal conditions for the oscillation of solutions of (1.1). In the present paper we make an attempt at carrying out in this direction. Several sufficient oscillation conditions for the

ISSN 1072-947X / \$8.00 /

case of several delays are contained in [1, 9, 15, 17]. It is to be pointed out that the technique used in [17] cannot be applied for equation (1.1).

2. Formulation of the Main Results

Throughout the paper we will assume that $p_i : R_+ \to R_+$ $(i = 1, \dots, m)$ are locally integrable functions, $\tau_i : R_+ \to R_+$ $(i = 1, \dots, m)$ are continuous functions, and

$$p_i(t) \geq 0, \quad \tau_i(t) \leq t \ \text{ for } \ t \in R_+, \quad \lim_{t\to+\infty} \tau_i(t) = +\infty \ \ (i = 1, \dots, m). \tag{2.1}$$

Let $a \in R_+$. Denote $a_0 = \inf\{\tau_*(t) : t \geq a\}$, $\tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$.

Definition 2.1. A continuous function $u : [a_0, +\infty) \to R$ is called a proper solution of equation (1.1) in $[a, +\infty)$ if it is absolutely continuous in each finite segment contained in $[a, +\infty)$ and satisfies (1.1) almost everywhere on $[a, +\infty)$ and $\sup\{|u(s)| : s \geq t\} > 0$ for $t \geq a_0$.

Definition 2.2. A proper solution of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to infinity; otherwise it is said to be non-oscillatory.

Definition 2.3. Equation (1.1) is said to be oscillatory if its every proper solution is oscillatory.

Theorem 2.1. *Let condition* (2.1) *hold, for some* $i \in \{1, \dots, m\}$,

$$\liminf_{t\to+\infty} \int_{\tau_i(t)}^{t} p_i(s)ds > 0, \tag{2.2}$$

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^{t} \overline{p}(s)ds < +\infty \tag{2.3}$$

and

$$\inf\left\{\liminf_{t\to+\infty} \exp\left(\lambda \int_0^t \overline{p}(s)ds\right)\right. \\ \left. \times \sum_{i=1}^{m} \int_t^{+\infty} p_i(s) \exp\left(-\lambda \int_0^{\tau_i(s)} \overline{p}(\xi)d\xi)ds\right) : \lambda \in (0, \infty)\right\} > 1, \tag{2.4}$$

where

$$\overline{p}(t) = \sum_{i=1}^{m} p_i(t), \quad \sigma(t) = \inf\{\tau_*(s) : s \geq t \geq 0\}, \\ \tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}. \tag{2.5}$$

Then equation (1.1) *is oscillatory.*

Remark 2.1. Condition (2.3) is not an essential restriction because if for some $i \in \{1, \dots, m\}$,

$$\limsup_{t\to+\infty} \int\limits_{\tau_i(t)}^{t} p_i(s)ds > 1,$$

then equation (1.1) is oscillatory (see, e.g., [13]).

Theorem 2.2. *Let conditions* (2.2), (2.3) *be fulfilled,* $\overline{p}(t) > 0$ *for sufficiently large* t, *and*

$$\liminf_{t\to+\infty} \int\limits_{\tau_i(t)}^{t} \overline{p}(s)ds = \alpha_i > 0 \quad (i = 1, \dots, m). \tag{2.6}$$

If, moreover, for some $t_0 \in R_+$,

$$\inf\left\{ \frac{1}{\lambda} \operatorname*{vrai\,inf}_{t\geq t_0} \left(\frac{1}{\overline{p}(t)} \sum_{i=1}^{m} p_i(t) e^{\alpha_i \lambda} \right) : \lambda \in (0, +\infty) \right\} > 1, \tag{2.7}$$

then equation (1.1) *is oscillatory.*

Theorem 2.3. *Let conditions* (2.2), (2.3), (2.6) *be fulfilled, and* $\overline{p}(t) > 0$ *for sufficiently large* t. *Let, moreover, for some* $t_0 \in R_+$,

$$\operatorname*{vrai\,inf}_{t\geq t_0} \left(\frac{1}{\overline{p}(t)} \sum_{i=1}^{m} \alpha_i p_i(t) \right) > \frac{1}{e}. \tag{2.8}$$

Then equation (1.1) *is oscillatory.*

Theorem 2.4. *If conditions* (2.2), (2.3), (2.6) *are fulfilled, and*

$$\min\{\alpha_i : i = 1, \dots, m\} > \frac{1}{e}, \tag{2.9}$$

then equation (1.1) *is oscillatory.*

Theorem 2.5. *Let* $\tau_i(t)$ $(i = 1, \dots, m)$ *be nondecreasing,*

$$\int\limits_{0}^{\infty} |\, p_i(t) - p_j(t) \,|\, dt < +\infty \quad (i, j = 1, \dots, m), \tag{2.10}$$

$$\liminf_{t\to+\infty} \int\limits_{\tau_i(t)}^{t} p_i(s)ds = \beta_i > 0 \quad (i = 1, \dots, m) \tag{2.11}$$

and

$$\min\left\{ \sum_{i=1}^{m} \frac{e^{\beta_i \lambda}}{\lambda} : \lambda \in (0, +\infty) \right\} > 1. \tag{2.12}$$

Then equation (1.1) *is oscillatory.*

Theorem 2.6. *Let conditions* (2.10), (2.11) *hold, and*

$$\sum_{i=1}^{m} \beta_i > \frac{1}{e}. \tag{2.13}$$

Then equation (1.1) *is oscillatory.*

Remark 2.2. It is obvious that Theorem 2.6 coincides with Theorem 1.1 for the case $m = 1$.

3. Auxiliary Statements

Lemma 3.1. *Let $p : R_+ \to R_+$ be a summable function in every finite segment, $\tau : R_+ \to R_+$ be a continuous and nondecreasing function, and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. If, moreover,*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > 0 \tag{3.1}$$

and $u : [a_0, +\infty) \to (0, +\infty)$ is a solution of the equation

$$u'(t) + p(t)u(\tau(t)) = 0, \tag{3.2}$$

then there exists $\lambda > 0$ such that

$$\lim_{t\to+\infty} u(t) \left\{ \exp\left(\lambda \int_0^t p(s)ds \right) \right\} = +\infty. \tag{3.3}$$

Proof. First we will show that

$$\limsup_{t\to+\infty} \frac{u(\tau(t))}{u(t)} < +\infty. \tag{3.4}$$

By virtue of (3.1) there are $c > 0$ and $t_0 \in R_+$ such that

$$\int_{\tau(t)}^{t} p(s)ds \geq c \text{ for } t \geq t_0.$$

Thus for any $t > t_0$ there exists $t^* > t$ such that

$$\int_{t}^{t^*} p(s)ds = \frac{c}{2}, \quad \int_{\tau(t^*)}^{t} p(s)ds \geq \frac{c}{2}. \tag{3.5}$$

Without loss of generality we can assume that $u(\tau(t)) > 0$ for $t \geq t_0$. In view of (3.5) from (3.2) we have

$$u(t) \geq \int_{t}^{t^*} p(s)u(\tau(s))ds \geq u(\tau(t^*)) \int_{t}^{t^*} p(s)ds = \frac{c}{2}u(\tau(t^*))$$

and

$$u(\tau(t^*)) \geq \int\limits_{\tau(t^*)}^{t} p(s)u(\tau(s))ds \geq \frac{c}{2}u(\tau(t)).$$

The last two inequalities result in $u(t) \geq (c^2/4)u(\tau(t))$. This, in view of the arbitrariness of t, means that (3.4) is valid. Thus from (3.2) we get

$$u(t) = u(t_0)\exp\left(-\int\limits_{t_0}^{t} p(s)\frac{u(\tau(s))}{u(s)}ds\right) \geq u(t_0)\exp\left(-\frac{4}{c^2}\int\limits_{t_0}^{t} p(s)ds\right). \tag{3.6}$$

On the other hand, from (3.1) it obviously follows that

$$\int\limits_{t_0}^{+\infty} p(s)ds = +\infty.$$

Therefore, according to (3.6), there exists $\lambda > 0$ such that (3.3) is satisfied. $\square$

Lemma 3.2. *Let* (3.1) *be fulfilled, $p, q : R_+ \to R_+$ be summable functions in every finite segment, $\tau, \tau_0 : R_+ \to R_+$ be continuous functions,*

$$\begin{gathered}\lim_{t\to+\infty}\tau(t) = \lim t \to +\infty\tau_0(t) = +\infty,\\ q(t) \geq p(t), \quad \tau_0(t) \leq \tau(t) \leq t \quad \textit{for} \quad t \geq t_0.\end{gathered} \tag{3.7}$$

If, moreover, $v : [t_0, +\infty) \to (0, +\infty)$ is a solution of the inequality

$$v'(t) + q(t)v(\tau_0(t)) \leq 0, \tag{3.8}$$

then equation (3.2) *has a solution $u : [t_1, +\infty) \to (0, +\infty)$ satisfying the condition*

$$0 < u(t) \leq v(t) \quad \textit{for} \quad t \geq t_1, \tag{3.9}$$

where $t_1 \geq t_0$ is a sufficiently large number.

Proof. Let $v : [t_0, +\infty) \to (0, +\infty)$ be a solution of inequality (3.8). By (3.1) and (3.7) there is $t_1 > t_0$ such that $v(\tau_0(t)) > 0$ for $t > t_1$ and

$$\int\limits_{\tau(t)}^{t} p(s)ds > 0 \quad \text{for} \quad t \geq t_1. \tag{3.10}$$

From (3.8) we have

$$v(t) \geq \int\limits_{t}^{+\infty} q(s)v(\tau(s))ds \quad \text{for} \quad t \geq t_1. \tag{3.11}$$

Denote $t_1^* = \inf\{\tau(t) : t \geq t_1\}$ and consider the sequence of functions $u_i : [t_1^*, +\infty) \to [0, +\infty)$ $(i = 1, 2, 3, \dots)$ defined by the following equalities:

$$u_1(t) = v(t) \quad \text{for} \quad t \geq t_1^*,$$

$$u_i(t) = \begin{cases} \int\limits_t^{+\infty} p(s) u_{i-1}(\tau(s)) ds & \text{for } t \geq t_1 \\ v(t) - v(t_1) + u_i(t_1) & \text{for } t_1^* \leq t < t_1 \end{cases} \qquad (i = 2, 3, \dots). \tag{3.12}$$

On account of the last inequality of (3.7) and conditions (3.10), (3.11) it is clear that $0 < u_i(t) \leq u_{i-1}(t) \leq v(t)$ $(i = 2, 3, \dots)$ for $t \geq t_1$. Thus $0 \leq u(t) \leq v(t)$ for $t \geq t_1$, where $u(t) = \lim_{i \to +\infty} u_i(t)$. Let us show that $u(t) > 0$ for $t \geq t_1$. Otherwise there is $t_2 \geq t_1$such that $u(t) \equiv 0$ for $t \geq t_2$ and $u(t) > 0$ for $t \in [t_1^*, t_2)$. Denote by U the set of points t satisfying $\tau(t) = t_2$, and put $t_* = \min U$. Evidently $t_* \geq t_2$. Therefore, by (3.10) and (3.12), we get

$$u(t_2) = \int\limits_{t_2}^{+\infty} p(s) u(\tau(s)) ds \geq \int\limits_{\tau(t^*)}^{t^*} p(s) u(\tau(s)) ds > 0.$$

The obtained contradiction proves that $u(t) > 0$ for $t \geq t_1$. Consequently we have $0 < u(t) \leq v(t)$ for $t \geq t_1$. □

Lemma 3.3. *Let condition* (2.1) *hold, for some* $i \in \{1, \dots, m\}$,

$$\liminf_{t \to +\infty} \int\limits_{\tau_i(t)}^{t} p_i(s) ds > 0, \tag{3.13}$$

and $u : [t_0, +\infty) \to (0, +\infty)$ *be a positive solution of equation* (1.1). *Then there exists* $\lambda > 0$ *such that*

$$\lim_{t \to +\infty} u(t) \exp\left(\lambda \int\limits_0^t p_i(s) ds\right) = +\infty. \tag{3.14}$$

Proof. It is obvious that u is a solution of the differential inequality

$$u'(t) + p_i(t) u(\tau_i(t)) \leq 0 \quad \text{for} \quad t \geq t_1,$$

where $t_1 > t_0$ is a sufficiently large number. Thus, taking into account (3.13) and Lemmas 3.1, 3.2, there exists $\lambda > 0$ such that (3.14) is fulfilled. □

Lemma 3.4. *Let* $t_0 \in R_+$, $\varphi, \psi \in C([t_0, +\infty); (0, +\infty))$, $\psi(t)$ *be non-increasing and*

$$\lim_{t \to +\infty} \varphi(t) = +\infty, \quad \liminf_{t \to +\infty} \psi(t) \widetilde{\varphi}(t) = 0,$$

where $\widetilde{\varphi}(t) = \inf\{\varphi(s) : s \geq t \geq t_0.\}$ *Then there exists an increasing sequence of points* $\{t_k\}_{k=1}^{+\infty}$ *such that* $t_k \uparrow +\infty$ *as* $k \uparrow +\infty$ *and*

$$\widetilde{\varphi(t_k)} = \varphi(t_k), \quad \psi(t) \widetilde{\varphi}(t) \geq \psi(t_k) \widetilde{\varphi}(t_k) \quad \textit{for} \quad t_0 \leq t \leq t_k.$$

For the proof of Lemma 3.4 see [11, Lemma 7.1].

4. Proof of the Main results

Proof of Theorem 2.1. Assume the contrary. Let equation (1.1) have a non-oscillatory proper solution $u : [t_0, +\infty) \to (0, +\infty)$. According to condition (2.2) and Lemma 3.3, there exists $\lambda > 0$ such that

$$\lim_{t\to+\infty} u(t)\exp\left(\lambda\int_0^t \overline{p}(s)ds\right) = +\infty, \tag{4.1}$$

where the function $\overline{p}(t)$ is defined by the first equality of (2.5).

Denote by Λ the set of all λ satisfying condition (4.1), and put $\lambda_0 = \inf\Lambda$. Since $u(t)$ is non-increasing, in view of (4.1) it is obvious that $\lambda_0 \geq 0$. By the definition of λ_0 and condition (2.4), there exist $\varepsilon > 0$ and $\lambda^* > \lambda_0$ such that

$$\liminf_{t\to+\infty}\left\{\exp\left(\lambda^*\int_0^t \overline{p}(s)ds\right)\sum_{i=1}^m\int_t^{+\infty} p_i(\xi)\exp\left(-\lambda\int_0^{\tau_i(\xi)}\overline{p}(s)ds\right)d\xi\right\}$$
$$> (1+\varepsilon)e^{(1+M)\varepsilon}, \tag{4.2}$$

$$\lim_{t\to+\infty} u(t)\exp\left(\lambda^*\int_0^t \overline{p}(\xi)d\xi\right) = +\infty, \tag{4.3}$$

$$\liminf_{t\to+\infty}\exp\left((\lambda^*-\varepsilon)\int_0^t \overline{p}(\xi)d\xi\right) = 0, \tag{4.4}$$

where

$$M = \limsup_{t\to+\infty}\int_{\sigma(t)}^t \overline{p}(s)ds. \tag{4.5}$$

Due to (4.3) and (4.4) it is clear that the functions φ and ψ satisfy the conditions of Lemma 3.4 where

$$\varphi(t) = u(\sigma(t))\exp\left(\lambda^*\int_0^{\sigma(t)}\overline{p}(s)ds\right), \quad \psi(t) = \exp\left(-\varepsilon\int_0^t \overline{p}(s)ds\right)$$

and the function $\sigma(t)$ is defined by the last two equalities of (2.5). Therefore, by Lemma 3.4, there exists an increasing sequence of points $\{t_k\}_{k=1}^{+\infty}$ such that

$$\widetilde{\varphi}(t_k)\exp\left(-\varepsilon\int_0^{t_k}\overline{p}(s)ds)\right) \leq \widetilde{\varphi}(t)\exp\left(-\varepsilon\int_0^t \overline{p}(s)ds\right) \quad \text{for } t_0 \leq t \leq t_k, \tag{4.6}$$

$$\widetilde{\varphi}(t_k) = u(\sigma(t_k))\exp\left(\lambda^*\int_0^{\sigma(t_k)}\overline{p}(s)ds\right). \tag{4.7}$$

If we take into account the definition of the function $\sigma(t)$ (see condition (2.5)), it becomes clear that

$$\begin{aligned}\widetilde{\rho}_i(t) &= \inf\left\{u(\tau_i(s))\exp\left(\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right) : s\geq t\right\}\\ &\geq \inf\left\{u(\sigma(s))\exp\left(\lambda^*\int\limits_0^{\sigma(s)}\overline{p}(\xi)d\xi\right) : s\geq t\right\} = \widetilde{\varphi}(t)\quad (i=1,\dots,m).\end{aligned}$$

Thus from (1.1) we get

$$\begin{aligned}u(\sigma(t_k)) &\geq \sum_{i=1}^m\left(\int\limits_{\sigma(t_k)}^{t_k} p_i(s)u(\tau_i(s))ds + \int\limits_{t_k}^{+\infty} p_i(s)u(\tau_i(s))ds\right)\\ &\geq \sum_{i=1}^m\int\limits_{\sigma(t_k)}^{t_k} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)\widetilde{\rho}_i(s)ds\\ &\quad+\sum_{i=1}^m\int\limits_{t_k}^{+\infty} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)\widetilde{\rho}_i(s)ds\\ &\geq \sum_{i=1}^m\int\limits_{\sigma(t_k)}^{t_k} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)\widetilde{\varphi}(s)ds\\ &\quad+\sum_{i=1}^m\int\limits_{t_k}^{+\infty} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)\widetilde{\varphi}(s)ds\end{aligned}$$

whence, in view of (4.6), we find

$$\begin{aligned}u(\sigma(t_k)) &\geq \sum_{i=1}^m \widetilde{\varphi}(t_k)\exp\left(-\varepsilon\int\limits_0^{t_k}\overline{p}(\xi)d\xi\right)\\ &\times\int\limits_{\sigma(t_k)}^{t_k}\exp\left(\varepsilon\int\limits_0^{s}\overline{p}(\xi)d\xi\right)p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)ds\\ &\quad+\sum_{i=1}^m\widetilde{\varphi}(t_k)\int\limits_{t_k}^{+\infty} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)ds\\ &= \widetilde{\varphi}(t_k)\sum_{i=1}^m\int\limits_{t_k}^{+\infty} p_i(s)\exp\left(-\lambda^*\int\limits_0^{\tau_i(s)}\overline{p}(\xi)d\xi\right)ds - \widetilde{\varphi}(t_k)\exp\left(-\varepsilon\int\limits_0^{t_k}\overline{p}(s)ds\right)\end{aligned}$$

$$\times \sum_{i=1}^{m} \int\limits_{\sigma(t_k)}^{t_k} \exp\left(\varepsilon \int\limits_0^s \overline{p}(\xi)d\xi\right) d \int\limits_s^{+\infty} p_i(\xi) \exp\left(-\lambda^* \int\limits_0^{\tau_i(\xi)} \overline{p}(\xi_1\right) d\xi_1)d\xi$$

$$= \widetilde{\varphi}(t_k) \sum_{i=1}^{m} \exp\left(-\varepsilon \int\limits_{\sigma(t_k)}^{t_k} \overline{p}(s)ds\right) \int\limits_{\sigma(t_k)}^{+\infty} p_i(s) \exp\left(-\lambda^* \int\limits_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds.$$

By (4.7), for sufficiently large k we obtain

$$e^{-(1+M)\varepsilon} \sum_{i=1}^{m} \exp\left(\lambda^* \int\limits_0^{\sigma(t_k)} \overline{p}(\xi)d\xi\right) \int\limits_{\sigma(t_k)}^{+\infty} p_i(s) \exp\left(-\lambda^* \int\limits_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds \ \le 1$$

Consequently,

$$\liminf_{t\to+\infty} \exp\left(\lambda^* \int\limits_0^t \overline{p}(\xi)d\xi\right) \sum_{i=1}^{m} \int\limits_t^{+\infty} p_i(s) \exp\left(-\lambda^* \int\limits_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds \le e^{(1+M)\varepsilon}.$$

This contradicts inequality (4.2) and the proof of the theorem is complete. □

Proof of Theorem 2.2. It suffices to show that conditions (2.6) and (2.7) imply inequality (2.4). Indeed, by (2.6) and (2.7) there exist $\varepsilon > 0$ and $t_1 > t_0$ such that

$$\int\limits_{\tau_i(t)}^{t} \overline{p}(s)ds > \alpha_i - \varepsilon \text{ for } t \ge t_1 \quad (i = 1, \dots, m) \tag{4.8}$$

and for any $\lambda \in (0, +\infty)$,

$$\frac{1}{\overline{p}(t)} \sum_{i=1}^{m} p_i(t) e^{\lambda(\alpha_i - \varepsilon)} \ge (1+\varepsilon)\lambda \quad \text{for} \quad t \ge t_1. \tag{4.9}$$

According to (4.8) and (4.9), for any $\lambda \in (0, +\infty)$ we find

$$\begin{aligned}
&\exp\left(\lambda \int\limits_0^t \overline{p}(s)ds\right) \sum_{i=1}^{m} \int\limits_t^{+\infty} p_i(s) \exp\left(-\lambda \int\limits_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds \\
&\quad \ge \exp\left(\lambda \int\limits_0^t \overline{p}(s)ds\right) \int\limits_t^{+\infty} \sum_{i=1}^{m} p_i(s) e^{\lambda(\alpha_i-\varepsilon)} \exp\left(-\lambda \int\limits_0^s \overline{p}(\xi)d\xi\right) ds \\
&\quad \ge \lambda(1+\varepsilon) \exp\left(\lambda \int\limits_0^t \overline{p}(s)ds\right) \int\limits_t^{+\infty} \exp\left(-\lambda \int\limits_0^s \overline{p}(\xi)d\xi\right) \overline{p}(s)ds \\
&\quad = 1+\varepsilon \quad \text{for} \quad t \ge t_1.
\end{aligned}$$

Therefore condition (2.4) holds and the proof of the theorem is complete. □

Proof of Theorem 2.3. From (2.8), using the inequality $e^x \geq ex$, clearly follows (2.7). This completes the proof. □

Proof of Theorem 2.4. It is enough to show that (2.9) yields (2.4). Indeed, according to (2.9) there exist $t_1 \in R_+$ and $\varepsilon > 0$ such that

$$\int_{\tau_i(t)}^{t} \overline{p}(s)ds \geq \frac{1+\varepsilon}{e} \text{ for } t \geq t_1 \quad (i = 1, \dots, m).$$

Thus for any $\lambda \in (0, +\infty)$ we have

$$\begin{aligned}
&\exp\left(\lambda \int_0^t \overline{p}(s)ds\right) \sum_{i=1}^{m} \int_t^{+\infty} p_i(s) \exp\left(-\lambda \int_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds \\
&\geq e^{\frac{(1+\varepsilon)\lambda}{e}} \exp\left(\lambda \int_0^t \overline{p}(s)ds\right) \int_t^{+\infty} \overline{p}(s) \exp\left(-\lambda \int_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds \\
&\geq \frac{e(1+\varepsilon)\lambda}{\lambda e} = 1 + \varepsilon.
\end{aligned}$$

Consequently, (2.4) is satisfied. □

Proof of Theorem 2.5. Below we will assume that

$$\limsup_{t \to +\infty} \int_{\tau_i(t)}^{t} p_i(s)ds \leq 1 \quad (i = 1, \dots, m).$$

Otherwise it is easy to show that (1.1) is oscillatory. Thus, by virtue of (2.10), condition (2.3) is satisfied. Therefore it is enough to show that inequality (2.4) holds. Due to (2.11) and (2.12) there exist $t_1 \in R_+$ and $\varepsilon \in (0, \beta_i)$ such that

$$\int_{\tau_i(t)}^{t} p_i(s)ds > \beta_i - \varepsilon \quad \text{fort} \geq t_1 \quad (i = 1, \dots, m) \tag{4.10}$$

and for any $\lambda \in (0, +\infty)$,

$$\sum_{i=1}^{m} \frac{e^{(\beta_i - \varepsilon)\lambda}}{\lambda} > 1 + \varepsilon. \tag{4.11}$$

Put

$$p_i(t) - p_1(t) = q_i(t), \eta(t,s) = \exp\left(-\lambda \sum_{i=1}^{m} \left| \int_t^{\tau_i(s)} q_i(\xi)d\xi \right| \right) \quad \text{for} \;\; s \geq t \geq t_1$$

$(i = 1, \dots, m)$ and

$$\psi(t,\lambda) = \exp\left(\lambda \int\limits_0^t \overline{p}(s)ds\right) \sum_{i=1}^m \int\limits_t^{+\infty} p_i(s) \exp\left(-\lambda \int\limits_0^{\tau_i(s)} \overline{p}(\xi)d\xi\right) ds.$$

According to (4.10), (4.11), for any $\lambda \in (0, +\infty)$ we have

$$\psi(t,\lambda) = \exp\left(\lambda \sum_{i=1}^m \int\limits_0^t q_i(s))ds\right) \exp\left(\lambda m \int\limits_0^t p_1(s)ds\right)$$

$$\times \sum_{i=1}^m \int\limits_t^{+\infty} (q_i(s) + p_1(s)) \exp\left(-\lambda \sum_{i=1}^m \int\limits_0^{\tau_i(s)} q_i(\xi)d\xi\right) \exp\left(-\lambda m \int\limits_0^{\tau_i(s)} p_1(\xi)d\xi\right) ds$$

$$\geq \exp\left(\lambda m \int\limits_0^t p_1(s)ds\right) \sum_{i=1}^m \int\limits_t^{+\infty} m p_1(s) \exp\left(-\lambda m \int\limits_0^{\tau_i(s)} p_1(\xi)d\xi\right) \eta(t,s)ds$$

$$- \exp\left(\lambda m \int\limits_0^t p_1(s)ds\right) \int\limits_t^{+\infty} \sum_{i=1}^m |q_i(s)|\eta(t,s)$$

$$\times \exp\left(-\lambda m \int\limits_0^s p_1(\xi)d\xi\right) \exp\left(\lambda m \int\limits_{\tau_i(s)}^s p_1(\xi)d\xi\right) ds.$$

Therefore, if we take into account the condition $\lim\limits_{t\to+\infty} \eta(t,s) = 1$, then by (2.3) and (2.10) we obtain for any $\lambda \in (0, +\infty)$,

$$\liminf_{t\to+\infty} \psi(t,\lambda) \geq \liminf_{t\to+\infty} \exp\left(\lambda m \int\limits_0^t p_1(s)ds\right)$$

$$\times \sum_{i=1}^m \int\limits_t^{+\infty} m p_1(s) \exp\left(-\lambda m \int\limits_0^s p_1(\xi)d\xi\right) e^{(\beta_i - \varepsilon)\lambda} ds$$

$$-\limsup_{t\to+\infty} e^{\lambda m(1+M)} \int\limits_t^{+\infty} \sum_{i=1}^m |q_i(\xi)|\eta(t,\xi)d\xi$$

$$= \liminf_{t\to+\infty} \exp\left(\lambda m \int\limits_0^t p_1(s)ds\right) \int\limits_t^{+\infty} m p_1(s) \exp\left(-\lambda m \int\limits_0^s p_1(\xi)d\xi\right)$$

$$\times \sum_{i=1}^m e^{(\beta_i - \varepsilon)\lambda} = \sum_{i=1}^m \frac{e^{(\beta_i - \varepsilon)\lambda}}{\lambda}.$$

Consequently, according to (4.11) inequality (4.2) evidently holds. The proof is complete. □

The validity of Theorem 2.6 easily follows from Theorem 2.5 if we take into consideration the inequality $e^x \geq ex$.

Remark 4.1. As it is noted in the Introduction, several sufficient conditions for the oscillation of equation (1.1) for $\tau_i(t) = t - \tau_i$ $(i = 1, \dots, m)$, where τ_i $(i = 1, \dots, m)$ are positive constants, are established in [1,15,17], while a non-integral condition is given in [9] for $\tau_i(t) = t - T_i(t)$, where T_i are continuous and positive-valued functions on $[0,\infty)$. However, as the following example indicates, even in the case of constant coefficients and constant delays none of the conditions in the said papers [1,9,15,17] is satisfied, while the conditions of Theorem 2.5 are satisfied.

Example 4.1. Consider the equation

$$u'(t) + u(t-\tau) + u(t - (1/e - \tau)) = 0, \tag{4.12}$$

where $\tau \in (0, \frac{1}{e}), \tau \neq 1/2e$. It is easy to see that none of the conditions in [1,9,15,17] is satisfied. However we will show that the conditions of Theorem 2.5 are satisfied. To this end it suffices to show the validity of the inequality

$$\min\left\{\left(\frac{e^{\tau\lambda}}{\lambda} + \frac{e^{(\frac{1}{e}-\tau)\lambda}}{\lambda}\right) : \lambda \in (0,\infty)\right\} > 1. \tag{4.13}$$

Since

$$\min\left\{\frac{e^{\tau\lambda}}{\lambda} : \lambda \in (0,\infty)\right\} = \tau e, \quad \min\left\{\frac{e^{(\frac{1}{e}-\tau)\lambda}}{\lambda} : \lambda \in (0,\infty)\right\} = 1 - \tau e$$

and the functions $\frac{e^{\tau\lambda}}{\lambda}$, $\frac{e^{(\frac{1}{e}-\tau)\lambda}}{\lambda}$ attain their minima at different points, it is clear that (4.13) is valid. According to Theorem 2.5 all solutions of equation (4.12) oscillate.

Acknowledgement

The research was supported by the Greek Ministry of Development in the framework of Bilateral S&T Cooperation between the Hellenic Republic and the Republic of Georgia.

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(Received 26.06.2002)

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