

Quasi-linear functional differential equations with Property A

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Abstract

We study oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + F(u)(t) = 0, \quad (0.1)$$

where $n \geq 2$ and $F: C(R_+; R) \rightarrow L_{\text{loc}}(R_+; R)$ is a continuous mapping. Sufficient conditions are established for this equation to have the so-called Property A. The obtained results are also new for the generalized Emden–Fowler type ordinary differential equation. The method by which the oscillatory properties of Eq. (0.1) are established enables one to obtain optimal conditions for (0.1) to have Property A for sufficiently general equations (for some classes of functions the obtained sufficient conditions are necessary as well).

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1. Introduction

Let $\tau \in C(R_+; R_+)$, with $\lim_{t \rightarrow +\infty} \tau(t) = +\infty$. Let $\mathbf{V}(\tau)$ denote the set of continuous mappings $F: C(R_+; R) \rightarrow L_{\text{loc}}(R_+; R)$ satisfying the condition

$$F(x)(t) = F(y)(t) \quad \text{holds for any } t \in R_+ \text{ and } x, y \in C(R_+; R)$$

provided that $x(s) = y(s)$ for $s \geq \tau(t)$.

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This work concerns the study of oscillatory properties of the functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $n \geq 2$ and $F \in \mathbf{V}(\tau)$. For any $t_0 \in R_+$, let $\mathbf{H}_{t_0, \tau}$ denote the set of all functions $u \in C(R_+; R)$ satisfying the condition $u(t) \neq 0$ for $t \geq t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$ and $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$. It will always be assumed that

$$F(u)(t)u(t) \geq 0 \quad \text{for } t \geq t_0 \text{ and } u \in \mathbf{H}_{t_0, \tau}, \tag{1.2}$$

holds.

Let $t_0 \in R_+$. A function $u : [t_0, +\infty) \rightarrow R$ is said to be a proper solution of Eq. (1.1) if it is locally continuous along with its derivatives of order up to and including $n - 1$, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$, there exists a function $\bar{u} \in C(R_+; R)$ such that $\bar{u} \equiv u(t)$ on $[t_0, +\infty)$, and the equality $\bar{u}^{(n)}(t) + F(\bar{u})(t) = 0$ holds for $t \in [t_0, +\infty)$. A proper solution of Eq. (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution is said to be nonoscillatory.

Definition 1.1. [1] We say that Eq. (1.1) has Property **A** if any proper solution u is oscillatory if n is even, and is either oscillatory or satisfies

$$|u^{(i)}(t)| \downarrow 0 \quad \text{as } t \uparrow +\infty \quad (i = 1, \dots, n - 1) \tag{1.3}$$

if n is odd.

The higher order nonlinear ordinary differential equation

$$u^{(n)}(t) + \sum_{i=1}^m p_i(t) |u(t)|^{\eta_i(t)} \text{sign } u(t) = 0, \tag{1.4}$$

where $p_i \in L_{\text{loc}}(R_+; R)$, $\eta_i \in C(R_+; (0, +\infty))$ ($i = 1, \dots, m$) is a special case of Eq. (1.1). When $m = 1$ and $\eta_1(t) \equiv 1$, Eq. (1.4) is a linear ordinary differential equation whose oscillatory properties are studied in [1–5] well enough. When $m = 1$ and $\eta_1(t) \equiv \eta > 0$, $\eta \neq 1$, (1.4) becomes the essentially nonlinear ordinary differential equation of Emden–Fowler type. Oscillatory properties of its solutions in case $n = 2$ were first investigated by Atkinson [6], Kiguradze [7] who gave sufficient conditions for their behavior in case n is even and $\eta > 1$, and Ličko and Švec [8] who established the necessary and sufficient conditions for even and odd n and also for $0 < \eta < 1$ and $\eta > 1$. A number of survey papers and monographs have been written on various aspects of oscillation of nonlinear differential equations; we refer the reader to Kartsatos [9], Kiguradze and Chanturia [2], Ladde et al. [10], Györi and Ladas [11], Erbe et al. [12], Agarwal et al. [13], and Koplatadze and Chanturia [14]. Analogous problems for the equations of type (1.1) in case where the operator F has a nonlinear minorant and the exponent of the phase coordinate is constant, were studied in the monograph [15]. The problems in case of a linear minorant were considered in [16]. As for equations of type (1.4), the case $\eta_i(t) \not\equiv \text{const}$ ($i = 1, \dots, m$) was studied in [17–19].

In the present paper we consider only the case where $\lim_{t \rightarrow +\infty} \eta_i(t) = 1$ ($i = 1, \dots, m$). In that case the oscillatory properties of solutions of Eq. (1.4) depend essentially on the rate at which the functions $1 - \eta_i(t)$ ($i = 1, \dots, m$) tend to zero as $t \rightarrow +\infty$. In subsequent papers essentially nonlinear equations will be considered, and the necessary and sufficient conditions of new type will be obtained for the given equation to have Property **A**.

2. Some auxiliary lemmas

In the sequel, $\tilde{C}_{loc}^{n-1}([t_0, +\infty))$ will denote the set of all functions $u : [t_0, +\infty) \rightarrow R$ absolutely continuous on any finite subinterval of $[t_0, +\infty)$ along with their derivatives of order up to and including $n - 1$.

Lemma 2.1. (Kiguradze [2]) *Let $u \in \tilde{C}_{loc}^{n-1}([t_0, +\infty))$ satisfy $u(t) > 0$ and $u^{(n)}(t) \leq 0$ for $t \geq t_0$, and $u^{(n)}(t) \neq 0$ in any neighborhood of $+\infty$. Then there exist $t_1 \geq t_0$ and $l \in \{0, \dots, n - 1\}$ such that $l + n$ is odd and*

$$\begin{aligned} u^{(i)}(t) &> 0 \quad \text{for } t \geq t_1 \quad (i = 0, \dots, l - 1), \\ (-1)^{i+l} u^{(i)}(t) &> 0 \quad \text{for } t \geq t_1 \quad (i = l, \dots, n - 1). \end{aligned} \tag{2.1}$$

Note. In case $l = 0$, we mean that the second inequality in (2.1) holds.

Lemma 2.2. [15] *Let $u \in \tilde{C}_{loc}([t_0, +\infty))$ and (2.1) be satisfied for some $l \in \{1, \dots, n - 1\}$ with $l + n$ odd. Then*

$$\int_{t_0}^{+\infty} t^{n-l-1} |u^{(n)}(t)| dt < +\infty. \tag{2.2}$$

Moreover, if

$$\int_{t_0}^{+\infty} t^{n-l} |u^{(n)}(t)| dt = +\infty, \tag{2.3}$$

then there exists $t_* \geq t_0$ such that

$$\frac{u^{(i)}(t)}{t^{l-i}} \downarrow, \quad \frac{u^{(i)}(t)}{t^{l-i-1}} \uparrow +\infty \quad (i = 0, \dots, l - 1), \tag{2.4}$$

$$u(t) \geq \frac{t^{l-1}}{l!} u^{(l-1)}(t) \quad \text{for } t \geq t_* \tag{2.5}$$

and

$$u(t) \geq \frac{1}{(l-1)!(n-l-1)!} \int_{t_*}^t (t-s)^{l-1} \int_s^{+\infty} (\xi-s)^{n-l-1} |u^{(n)}(\xi)| d\xi ds \quad \text{for } t \geq t_*. \tag{2.6}$$

Lemma 2.3. *Let $t_0 \in R_+$, $\varphi; \psi \in C([t_0, +\infty), (0, +\infty))$, ψ be a nonincreasing function, and*

$$\lim_{t \rightarrow +\infty} \varphi(t) = +\infty, \tag{2.7}$$

$$\liminf_{t \rightarrow +\infty} \psi(t) \tilde{\varphi}(t) = 0, \tag{2.8}$$

where $\tilde{\varphi}(t) = \inf\{\varphi(s) : s \geq t \geq t_0\}$. Then there exists a sequence $\{t_k\}$ such that $t_k \uparrow +\infty$ as $k \uparrow +\infty$ and

$$\tilde{\varphi}(t_k) = \varphi(t_k), \quad \psi(t) \tilde{\varphi}(t) \geq \psi(t_k) \tilde{\varphi}(t_k), \quad t_0 \leq t \leq t_k \quad (k = 1, 2, \dots), \tag{2.9}$$

$$\lim_{k \rightarrow +\infty} \psi(t_k) \varphi(t_k) = 0. \tag{2.10}$$

Proof. The proof of the existence of the sequence $\{t_k\}$ satisfying (2.9) can be found in [15] (see Lemma 7.1). As for the condition (2.10), it immediately follows from (2.8) and (2.9). \square

Everywhere below we assume that the inequality

$$|F(u)(t)| \geq \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\mu_i(s)} d_s r_i(s, t) \quad \text{for } t \geq t_0, u \in \mathbf{H}_{t_0, \tau}, \tag{2.11}$$

holds, where

$$\begin{aligned} \mu_i &\in C(R_+; (0, +\infty)) \quad (i = 1, \dots, m), \\ \tau_i; \sigma_i &\in C(R_+; R_+), \quad \tau_i(t) \leq \sigma_i(t) \quad \text{for } t \in R_+, \\ \lim_{t \rightarrow +\infty} \tau_i(t) &= +\infty \quad (i = 1, \dots, m), \\ r_i : R_+ \times R_+ &\rightarrow R_+ \quad \text{are measurable in } t \text{ and nondecreasing} \\ &\quad \text{in } s \text{ functions } (i = 1, \dots, m). \end{aligned} \tag{2.12}$$

Besides, everywhere below we suppose that

$$\limsup_{t \rightarrow +\infty} t^{|1-\mu_i(t)|} < +\infty \quad (i = 1, \dots, m). \tag{2.13}$$

Let $t_0 \in R_+$. By \mathbf{U}_{l, t_0} we denote the set of all proper solutions of Eq. (1.1) satisfying the condition (2.1_l),

$$\mathbf{A}_{l, u} = \left\{ \lambda \mid \lambda \in [l - 1, l], \lim_{t \rightarrow +\infty} \frac{u(t)}{t^\lambda} = +\infty \right\}, \quad u \in \mathbf{U}_{l, t_0}.$$

Remark 2.1. In the definition of the set $\mathbf{A}_{l, u}$ we assume that if there is no $\lambda \in [l - 1, l]$ such that $\lim_{t \rightarrow +\infty} u(t)/t^\lambda = +\infty$, then $\mathbf{A}_{l, u} = \emptyset$.

3. The necessary conditions of the existence of solutions of type (2.1_l)

The results of this section play an important role in establishing the sufficient conditions for Eq. (1.1) to have Property **A**.

Theorem 3.1. Let $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13) be fulfilled, $l \in \{1, \dots, n - 1\}$, with $l + n$ odd,

$$\int_{t_0}^{+\infty} t^{n-i} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(i-1)\mu_i(s)} d_s r_i(s, t) dt = +\infty \quad (i = l, l + 1), \tag{3.1_i}$$

$\mathbf{U}_{l, t_0} \neq \emptyset$ for some $t_0 \in R_+$. Then there exists $\lambda_0 \in [l - 1, l]$ such that

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l, 1}(t, \varepsilon, \lambda_0) \right) \leq (l - 1)!(n - l - 1)!, \tag{3.2_l}$$

where

$$\begin{aligned} \rho_{l,1}(t, \varepsilon, \lambda_0) &= t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_0^t (t-s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ &\times \int_s^{+\infty} (\xi-s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds, \end{aligned} \tag{3.3}$$

$$\tau_*(t) = \inf\{\tau(s) : s \geq t\}, \quad \tau(t) = \min\{t, \tau_i(t) : i = 1, \dots, m\}, \tag{3.4}$$

$$h_{1\varepsilon}(\lambda_0) = \begin{cases} 0 & \text{for } \lambda_0 = l - 1, \\ \varepsilon & \text{for } \lambda_0 \in (l - 1, l], \end{cases} \quad h_{2\varepsilon}(\lambda_0) = \begin{cases} 0 & \text{for } \lambda_0 = l, \\ \varepsilon & \text{for } \lambda_0 \in [l - 1, l). \end{cases} \tag{3.5}$$

Proof. Let $t_0 \in R_+$ and $U_{l,t_0} \neq \emptyset$. Show that there exists $\lambda_0 \in [l - 1, l]$, such that inequality (3.2) is fulfilled. By the definition of the set U_{l,t_0} , Eq. (1.1) has a proper solution $u \in U_{l,t_0}$ satisfying the condition (2.1_l). In view of (1.2), (2.1_l), (2.5), (2.11), and (3.1_l) it is clear that the condition (2.3) holds. Thus by Lemma 2.2,

$$\frac{u(t)}{t^{l-1}} \uparrow +\infty, \quad \frac{u(t)}{t^l} \downarrow \quad \text{for } t \geq t_1, \tag{3.6}$$

$$\begin{aligned} u(t) &\geq \frac{1}{(l-1)!(n-l-1)!} \int_{t_1}^t (t-s)^{l-1} \int_s^{+\infty} (\xi-s)^{n-l-1} \\ &\times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} |u(\xi_1)|^{\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \quad \text{for } t \geq t_1, \end{aligned} \tag{3.7}$$

where $t_1 \geq t_0$ is sufficiently large. According to (3.6) it is clear that $l - 1 \in \mathbf{A}_{l,u}$ and $l \notin \mathbf{A}_{l,u}$. Therefore we have

$$\mathbf{A}_{l,u} \subset [l - 1, l) \quad \text{and} \quad \lambda_0 = \sup \mathbf{A}_{l,u} \in [l - 1, l].$$

Taking into account to the second condition of (3.6), using (3.1_{l+1}) we easily show, that $u(t)/t^l \downarrow 0$ as $t \uparrow +\infty$. Therefore, according to (3.5), for sufficiently small ε we get

$$\lim_{t \rightarrow +\infty} \frac{u(t)}{t^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} = +\infty, \quad \liminf_{t \rightarrow +\infty} \frac{u(t)}{t^{\lambda_0 + h_{2\varepsilon}(\lambda_0)}} = 0, \tag{3.8}$$

$$l - 1 \leq \lambda_0 - h_{1\varepsilon}(\lambda_0) \leq \lambda_0 + h_{2\varepsilon}(\lambda_0) \leq l. \tag{3.9}$$

Denote

$$\tilde{\varphi}(t) = \inf \left\{ \left(\frac{u(s)}{s^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(s)} : s \geq t \geq t_1 \right\}, \tag{3.10}$$

where

$$\mu(t) = \min\{\mu_i(t) : i = 1, \dots, m\}. \tag{3.11}$$

Show that

$$\liminf_{t \rightarrow +\infty} t^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \tilde{\varphi}(t) = 0. \tag{3.12}$$

Indeed, by (3.10) we have

$$\begin{aligned}
 t^{-(h_{1\varepsilon}(\lambda_0)+h_{2\varepsilon}(\lambda_0))} \tilde{\varphi}(t) &\leq t^{-(h_{1\varepsilon}(\lambda_0)+h_{2\varepsilon}(\lambda_0))} \left(\frac{u(t)}{t^{\lambda_0-h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(t)} \\
 &= t^{(\mu(t)-1)(h_{1\varepsilon}(\lambda_0)+h_{2\varepsilon}(\lambda_0))} \left(\frac{u(t)}{t^{\lambda_0+h_{2\varepsilon}(\lambda_0)}} \right)^{\mu(t)}. \tag{3.13}
 \end{aligned}$$

On the other hand, using (2.13), we obtain

$$\limsup_{t \rightarrow +\infty} t^{(\mu(t)-1)(h_{1\varepsilon}(\lambda_0)+h_{2\varepsilon}(\lambda_0))} < +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mu(t) = 1.$$

Thus according to the second condition of (3.8), from (3.13) follows the validity of (3.12) for any $\varepsilon > 0$ satisfying the condition (3.9). Using (3.4), (3.11) and the first condition of (3.8), from (3.7) we get

$$\begin{aligned}
 u(\tau_*(t)) &\geq \frac{1}{(l-1)!(n-l-1)!} \int_{t_2}^{\tau_*(t)} (\tau_*(t)-s)^{l-1} \int_s^{+\infty} (\xi-s)^{n-l-1} \\
 &\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \left(\frac{u(\xi_1)}{\xi_1^{\lambda_0-h_{1\varepsilon}(\lambda_0)}} \right)^{\mu_i(\xi_1)} \xi_1^{(\lambda_0-h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \\
 &\geq \frac{1}{(l-1)!(n-l-1)!} \int_{t_2}^{\tau_*(t)} (\tau_*(t)-s)^{l-1} \int_s^{+\infty} (\xi-s)^{n-l-1} \\
 &\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \left(\frac{u(\xi_1)}{\xi_1^{\lambda_0-h_{1\varepsilon}(\lambda_0)}} \right)^{\mu_i(\xi_1)} \xi_1^{(\lambda_0-h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \quad \text{for } t \geq t_2^*,
 \end{aligned}$$

where t_2 and t_2^* are sufficiently large numbers. Taking into account (3.4) and (3.10), from the latter inequality we obtain

$$\begin{aligned}
 u(\tau_*(t)) &\geq \frac{1}{(l-1)!(n-l-1)!} \int_{t_2}^{\tau_*(t)} (\tau_*(t)-s)^{l-1} \tilde{\varphi}(\tau_*(s)) \\
 &\quad \times \int_s^{+\infty} (\xi-s)^{n-l-1} \sum_{i=1}^n \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0-h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \quad \text{for } t \geq t_2^*. \tag{3.14}
 \end{aligned}$$

In view of (3.8), (3.10) and (3.12), it is obvious that the functions

$$\varphi(t) = \left(\frac{u(t)}{t^{\lambda_0-h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(t)} \quad \text{and} \quad \psi(t) = t^{-(h_{1\varepsilon}(\lambda_0)+h_{2\varepsilon}(\lambda_0))}$$

satisfy the conditions of Lemma 2.3. Thus there exists a sequence $\{t_k\}$ such that $t_k \uparrow +\infty$ as $k \uparrow +\infty$ and

$$\psi(\tau_*(t_k))\tilde{\varphi}(\tau_*(t_k)) \leq \psi(\tau_*(t))\tilde{\varphi}(\tau_*(t)) \quad \text{for } t_2 \leq t \leq t_k, \tag{3.15}$$

$$\tilde{\varphi}(\tau_*(t_k)) = \varphi(\tau_*(t_k)) = \left(\frac{u(\tau_*(t_k))}{(\tau_*(t_k))^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(\tau_*(t_k))}, \quad k = k_0, k_0 + 1, \dots, \tag{3.16}$$

$$\lim_{k \rightarrow +\infty} \psi(\tau_*(t_k))\varphi(\tau_*(t_k)) = 0, \tag{3.17}$$

where the functions $\tilde{\varphi}$ and μ are defined by (3.10) and (3.11), respectively, and $k_0 \in N$ is sufficiently large. By (3.15) and (3.16) from (3.14) we get

$$\begin{aligned} u(\tau_*(t_k)) &\geq \frac{(\tau_*(t_k))^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))}\tilde{\varphi}(\tau_*(t_k))}{(l-1)!(n-l-1)!} \\ &\times \int_{t_2}^{\tau_*(t_k)} (\tau_*(t_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ &\times \int_s^{+\infty} (\xi - s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \\ &= \frac{(\tau_*(t_k))^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} (u(\tau_*(t_k)))^{\mu(\tau_*(t_k))}}{(l-1)!(n-l-1)! (\tau_*(t_k))^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu(\tau_*(t_k))}} \\ &\times \int_{t_2}^{\tau_*(t_k)} (\tau_*(t_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \int_s^{+\infty} (\xi - s)^{n-l-1} \\ &\times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds, \quad k = k_0, k_0 + 1, \dots \end{aligned} \tag{3.18}$$

On the other hand, it is evident that there exists a subsequence $\{t'_k\}_{k=1}^{+\infty} \subset \{t_k\}_{k=1}^{+\infty}$ such that at least one of the following two conditions is fulfilled:

$$1 - \mu(\tau_*(t'_k)) \geq 0, \quad k = 1, 2, \dots, \tag{3.19}$$

or

$$1 - \mu(\tau_*(t'_k)) \leq 0, \quad k = 1, 2, \dots \tag{3.20}$$

Let the condition (3.19) hold. Then from (3.18) we obtain

$$\begin{aligned} &\left[\left(\frac{u(\tau_*(t'_k))}{(\tau_*(t'_k))^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(\tau_*(t'_k))} (\tau_*(t'_k))^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \right]^{\frac{1 - \mu(\tau_*(t'_k))}{\mu(\tau_*(t'_k))}} \\ &\times \left[(\tau_*(t'_k))^{\frac{1 - \mu(\tau_*(t'_k))}{\mu(\tau_*(t'_k))}} \right]^{(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \\ &\geq \frac{(\tau_*(t'_k))^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)}}{(l-1)!(n-l-1)!} \int_{t_2}^{\tau'_*(t_k)} (\tau_*(t'_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \end{aligned}$$

$$\begin{aligned} & \times \int_s^{+\infty} (\xi - s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds, \\ & k = k_0, k_0 + 1, \dots \end{aligned} \tag{3.21}$$

According to (3.17) and (3.19), for sufficiently large k we have

$$\left[\left(\frac{u(\tau_*(t'_k))}{(\tau_*(t'_k))^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{\mu(\tau_*(t'_k))} (\tau_*(t'_k))^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \right]^{\frac{1 - \mu(\tau_*(t'_k))}{\mu(\tau_*(t'_k))}} \leq 1. \tag{3.22}$$

On the other hand, due to (2.13)

$$\limsup_{k \rightarrow +\infty} (\tau_*(t'_k))^{\frac{1 - \mu(\tau_*(t'_k))}{\mu(\tau_*(t'_k))}} = c < +\infty.$$

Therefore taking into account (3.22), from (3.21) we get

$$\begin{aligned} & (l - 1)!(n - l - 1)!(1 + c)^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ & \geq (\tau_*(t'_k))^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_{t_2}^{\tau_*(t'_k)} (\tau_*(t'_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ & \times \int_s^{+\infty} (\xi - s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \end{aligned}$$

for sufficiently large k . In view of $\lim_{\varepsilon \rightarrow 0} (h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)) = 0$, the latter inequality yields

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0^+} \left(\limsup_{k \rightarrow +\infty} (\tau_*(t'_k))^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \right. \\ & \times \int_{t_2}^{\tau_*(t'_k)} (\tau_*(t'_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \int_s^{+\infty} (\xi - s)^{n-l-1} \\ & \left. \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \right) \leq (l - 1)!(n - l - 1)!. \end{aligned} \tag{3.23}$$

In view of (3.3_l) it is clear that (3.23) implies inequality (3.2_l). Thus, if (3.19) holds, the validity of the theorem is proved.

Suppose now that inequality (3.20) takes place. From (3.18) we obtain

$$\begin{aligned} & \left(\frac{u(\tau_*(t'_k))}{(\tau_*(t'_k))^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{1 - \mu(\tau_*(t'_k))} \geq \frac{(\tau_*(t'_k))^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)}}{(l - 1)!(n - l - 1)!} \\ & \times \int_{t_2}^{\tau_*(t'_k)} (\tau_*(t'_k) - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \int_s^{+\infty} (\xi - s)^{n-l-1} \\ & \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds, \quad k = k_0, k_0 + 1, \dots \end{aligned} \tag{3.24}$$

According to the first condition of (3.8), taking into account (3.2), for sufficiently large k we get

$$\left(\frac{u(\tau_*(t'_k))}{(\tau_*(t'_k))^{\lambda_0 - h_{1\varepsilon}(\lambda_0)}} \right)^{1 - \mu(\tau_*(t'_k))} \leq 1.$$

Therefore from (3.24) immediately follows (3.23). Thus inequality (3.2_l) is fulfilled, which proves the theorem. \square

Theorem 3.2. *Let $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_l) be fulfilled, $l \in \{1, \dots, n - 1\}$, with $l + n$ odd,*

$$\liminf_{t \rightarrow +\infty} \frac{\tau_i(t)}{t} > 0 \quad (i = 1, \dots, m) \tag{3.25}$$

and $\mathbf{U}_{l,t_0} \neq \emptyset$ for some $t_0 \in \mathbf{R}_+$. Then there exists $\lambda_0 \in [l - 1, l]$ such that

$$\limsup_{\varepsilon \rightarrow 0+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,2}(t, \varepsilon, \lambda_0) \right) \leq (l - 1)!(n - l - 1)!, \tag{3.26_l}$$

where

$$\begin{aligned} \rho_{l,2}(t, \varepsilon, \lambda_0) &= t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_0^t (t - s)^{l-1} s^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \int_s^{+\infty} (\xi - s)^{n-l-1} \\ &\times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds. \end{aligned} \tag{3.27_l}$$

Proof. By (2.13), (3.25) and (3.1_l) it is clear that condition (3.1_{l+1}) holds. Therefore, in view of Theorem 3.1, to prove Theorem 3.2 it is sufficient to show that the inequality (3.2_l) implies inequality (3.26_l), where the functions $\rho_{l,1}$ and $\rho_{l,2}$ are given by (3.3_l) and (3.27_l), respectively. Indeed, according to (3.25) there exist $c > 0$ and $T > 0$ such that $\tau_*(t) \geq ct$ for $t \geq T$, where $\tau_*(t)$ is defined by (3.4). Therefore taking into account (3.3_l), we obtain

$$\begin{aligned} \rho_{l,1}(t, \varepsilon, \lambda_0) &\geq t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_0^T (t - s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ &\times \int_s^{+\infty} (\xi - s)^{n-l-1} \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \\ &+ c^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_T^t (t - s)^{l-1} s^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ &\times \int_s^{+\infty} (\xi - s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{2\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds. \end{aligned} \tag{3.28}$$

Since $\lambda_0 + h_{2\varepsilon}(\lambda_0) > l - 1$, for any $\varepsilon > 0$ we have

$$\begin{aligned} & \lim_{t \rightarrow +\infty} t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_0^T (t-s)^{l-1} (\tau_*(s))^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \\ & \times \int_s^{+\infty} (\xi-s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds = 0. \end{aligned}$$

Therefore because of (3.3_l) and (3.27_l) from (3.28) we get

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \rho_{l,2}(t, \varepsilon, \lambda_0) \\ & \leq c^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \liminf_{t \rightarrow +\infty} \left(t^{-\lambda_0 - h_{2\varepsilon}(\lambda_0)} \int_T^t (t-s)^{l-1} s^{h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)} \right. \\ & \times \int_s^{+\infty} (\xi-s)^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds \\ & \left. = c^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))} \liminf_{t \rightarrow +\infty} \rho_{l,1}(t, \varepsilon, \lambda_0) \leq (l-1)!(n-l-1)! c^{-(h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0))}. \right. \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0+} (h_{1\varepsilon}(\lambda_0) + h_{2\varepsilon}(\lambda_0)) = 0$, taking if in the latter inequality the upper limit as $\varepsilon \rightarrow 0+$ we obtain inequality (3.26_l), which proves the validity of the theorem. \square

Theorem 3.3. *Suppose that $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_l), and (3.25) are fulfilled, $l \in \{1, \dots, n-1\}$ with $l+n$ odd and $\mathbf{U}_{l,t_0} \neq \emptyset$ for some $t_0 \in R_+$. Then there exists $\lambda_0 \in [l-1, l]$ such that*

$$\limsup_{\varepsilon \rightarrow 0+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon, \lambda_0) \right) \leq \prod_{i=0; i \neq l}^{n-1} |\lambda_0 - i|, \tag{3.29_l}$$

where

$$\rho_{l,3}(t, \varepsilon, \lambda_0) = t^{l-\lambda_0 + h_{1\varepsilon}(\lambda_0)} \int_t^{+\infty} \xi^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0 - h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi. \tag{3.30_l}$$

Proof. It is sufficient to see that under the conditions of Theorem 3.2 the inequality (3.26_l) implies (3.29_l). If that is not the case, we can find positive sequences $\{\varepsilon_k\}_{k=1}^{+\infty}$, $\varepsilon_0 > 0$, and $\{t_k\}_{k=1}^{+\infty}$, $t_k \in R_+$, such that $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$ and

$$\rho_{l,3}(t, \varepsilon_k, \lambda_0) \geq \prod_{i=0; i \neq l}^{n-1} |\lambda_0 - i| + \varepsilon_0 \quad \text{for } t \geq t_k. \tag{3.31}$$

Consider the case where $l = n - 1$. Then due to (3.31) taking into account (3.27_l), since $\lambda_0 + h_{2\varepsilon}(\lambda_0) > n - 2$ we get

$$\begin{aligned} \rho_{n-1,2}(t, \varepsilon_k, \lambda_0) &\geq \left(\prod_{i=0}^{n-2} |\lambda_0 - i| + \varepsilon_0 \right) t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} \int_{t_k}^t (t-s) s^{\lambda_0 + 1 - n + h_{2\varepsilon_k}(\lambda_0)} ds \\ &= \frac{(n-2)! (\prod_{i=0}^{n-2} |\lambda_0 - i| + \varepsilon_0)}{\prod_{i=0}^{n-2} (\lambda_0 + h_{2\varepsilon_k}(\lambda_0) - i)} \\ &\quad \times t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} (t^{\lambda_0 + h_{2\varepsilon_k}(\lambda_0)} + o(t^{\lambda_0 + h_{2\varepsilon_k}(\lambda_0)})) \quad \text{for } t \geq t_k. \end{aligned}$$

Passing to the limit in the latter inequality, we obtain

$$\liminf_{t \rightarrow +\infty} \rho_{n-1,2}(t, \varepsilon_k, \lambda_0) \geq \frac{(n-2)! (\prod_{i=0}^{n-2} |\lambda_0 - i| + \varepsilon_0)}{\prod_{i=0}^{n-2} (\lambda_0 + h_{2\varepsilon_k}(\lambda_0) - i)}.$$

Hence taking into account $\lim_{k \rightarrow +\infty} h_{2\varepsilon_k}(\lambda_0) = 0$, we have

$$\limsup_{k \rightarrow +\infty} \left(\liminf_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon_k, \lambda_0) \right) > (n-2)!$$

which contradicts the inequality (3.26_{n-1}). The obtained contradiction in the case where $l = n - 1$ proves the validity of the theorem.

Now assume that $l \in \{1, \dots, n - 3\}$. Using (3.27_l), we have

$$\begin{aligned} \rho_{l,2}(t, \varepsilon_k, \lambda_0) &\geq -t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} \int_1^t (t-s)^{l-1} s^{h_{1\varepsilon_k}(\lambda_0) + h_{2\varepsilon_k}(\lambda_0)} \\ &\quad \times \int_s^{+\infty} \left(1 - \frac{s}{\xi}\right)^{n-l-1} d \int_{\xi}^{+\infty} \xi_1^{n-l-1} \\ &\quad \times \sum_{i=1}^m \int_{\tau_i(\xi_1)}^{\sigma_i(\xi_1)} \xi_2^{(\lambda_0 - h_{1\varepsilon_k}(\lambda_0))\mu_i(\xi_2)} d_{\xi_2} r_i(\xi_2, \xi) d\xi_1 ds \\ &= t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} \int_1^t (t-s)^{l-1} s^{h_{1\varepsilon_k}(\lambda_0) + h_{2\varepsilon_k}(\lambda_0)} \int_s^{+\infty} \left(\left(1 - \frac{s}{\xi}\right)^{n-l-1} \right)' \\ &\quad \times \int_{\xi}^{+\infty} \xi_1^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi_1)}^{\sigma_i(\xi_1)} \xi_2^{(\lambda_0 - h_{1\varepsilon_k}(\lambda_0))\mu_i(\xi_2)} d_{\xi_2} r_i(\xi_2, \xi_1) d\xi_1 d\xi ds. \end{aligned}$$

Since $\frac{d}{d\xi} \left(1 - \frac{s}{\xi}\right)^{n-l-1} \geq 0$ for $\xi \geq s \geq 1$, according to (3.30_l) and (3.31) the latter inequality implies

$$\begin{aligned} \rho_{l,2}(t, \varepsilon_k, \lambda_0) &\geq \left(\prod_{i=0; i \neq l}^{n-1} |\lambda_0 - i| + \varepsilon_0 \right) t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} \int_{t_k}^t (t-s)^{l-1} s^{h_{1\varepsilon_k}(\lambda_0) + h_{2\varepsilon_k}(\lambda_0)} \\ &\quad \times \int_s^{+\infty} \xi^{\lambda_0 - l - h_{1\varepsilon_k}(\lambda_0)} \left(\left(1 - \frac{s}{\xi}\right)^{n-l-1} \right)' d\xi ds \end{aligned}$$

$$\begin{aligned}
 &= \left(\prod_{i=0; i \neq l}^{n-1} |\lambda_0 - i| + \varepsilon_0 \right) (l + h_{1\varepsilon_k}(\lambda_0) - \lambda_0) t^{-\lambda_0 - h_{2\varepsilon_k}(\lambda_0)} \\
 &\quad \times \int_{t_k}^t (t-s)^{l-1} s^{h_{1\varepsilon_k}(\lambda_0) + h_{2\varepsilon_k}(\lambda_0)} \int_s^{+\infty} (\xi-s)^{n-l-1} \xi^{\lambda_0 - n - h_{1\varepsilon_k}(\lambda_0)} d\xi \\
 &= \frac{(l-1)!(n-l-1)! \left(\prod_{i=0, i \neq l}^{n-1} |\lambda_0 - i| + \varepsilon_0 \right) (l + h_{1\varepsilon_k}(\lambda_0) - \lambda_0) (1 + o(1))}{\prod_{i=0}^{l-1} (\lambda_0 + h_{2\varepsilon_k}(\lambda_0) - i) \prod_{i=l}^{n-1} (\lambda_0 - i - h_{1\varepsilon_k}(\lambda_0))}
 \end{aligned}$$

for sufficiently large k . Therefore we have

$$\limsup_{k \rightarrow +\infty} \left(\liminf_{t \rightarrow +\infty} \rho_{l,2}(t, \varepsilon_k, \lambda_0) \right) > (l-1)!(n-l-1)!,$$

which contradicts (3.26_l). The obtained contradiction proves inequality (3.29_l). \square

Theorem 3.4. Suppose $F \in \mathbf{V}(t)$ and the conditions of Theorem 3.3 are fulfilled. Then there exists $\lambda_0 \in [l-1, l]$ such that

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,4}(t, \varepsilon, \lambda_0) \right) \leq \prod_{i=0}^{n-1} |\lambda_0 - i|, \tag{3.32_l}$$

where

$$\rho_{l,4}(t, \varepsilon, \lambda_0) = t \int_t^{+\infty} \xi^{n-2-\lambda_0+h_{1\varepsilon}(\lambda_0)} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0-h_{1\varepsilon}(\lambda_0))\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi \tag{3.33_l}$$

and $h_{1\varepsilon}(\lambda_0)$ is defined by (3.5).

Proof. Since the conditions of Theorem 3.3 are fulfilled, to prove the theorem it is sufficient to show that inequality (3.29_l) implies (3.32_l), where $\rho_{l,3}$ and $\rho_{l,4}$ are defined by (3.30_l) and (3.33_l), respectively.

Let $\lambda_0 \in [l-1, l]$. Show that (3.29_l) implies (3.32_l). We consider two cases: $\lambda_0 = l-1$ and $\lambda_0 \in (l-1, l]$. In the first case $h_{1\varepsilon}(l-1) = 0$ since $\rho_{l,3}(t, \varepsilon, l-1) = \rho_{l,4}(t, \varepsilon, l-1)$ and hence (3.32_l) is true. If $\lambda_0 \in (l-1, l]$, then due to the first condition of (3.5), we have $h_{1\varepsilon}(\lambda_0) = \varepsilon$. Assume now that (3.32_l) is invalid. Then there exist positive sequences $\{\varepsilon_k\}_{k=1}^{+\infty}$, $\varepsilon_0 > 0$, and $\{t_k\}_{k=1}^{+\infty}$, $t_k \in R_+$, such that $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$, $\lambda_0 - (l-1) - \varepsilon_k > 0$, $k = 1, 2, \dots$, and

$$\rho_{l,4}(t, \varepsilon_k, \lambda_0) \geq \left(\varepsilon_0 + \prod_{i=0}^{n-1} |\lambda_0 - i| \right) \text{ for } t \geq t_k, k = 1, 2, \dots \tag{3.34}$$

According to (3.30_l) we have

$$\begin{aligned}
 \rho_{l,3}(t, \varepsilon_k, \lambda_0) &= -t^{l-\lambda_0+\varepsilon_k} \int_t^{+\infty} s^{l-\lambda_0-\varepsilon_k} d \int_s^{+\infty} \xi^{n-2-\lambda_0+\varepsilon_k} \\
 &\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0-\varepsilon_k)\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi
 \end{aligned}$$

$$\begin{aligned}
 &= t \int_t^{+\infty} \xi^{n-2-\lambda_0+\varepsilon_k} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0-\varepsilon_k)\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi \\
 &\quad + (\lambda_0 - (l - 1) - \varepsilon_k) t^{l-\lambda_0+\varepsilon_k} \int_t^{+\infty} s^{-l+\lambda_0-\varepsilon_k} \int_s^{+\infty} \xi^{n-2-\lambda_0+\varepsilon_k} \\
 &\quad \times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda_0-\varepsilon_k)\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi ds.
 \end{aligned}$$

Since $\lambda_0 - (l - 1) - \varepsilon_k > 0$, by (3.33_l) and (3.34) we get

$$\rho_{l,3}(t, \varepsilon_k, \lambda_0) \geq \left(\varepsilon_0 + \prod_{i=0}^{n-1} |\lambda_0 - i| \right) \frac{1}{l - \lambda_0 + \varepsilon_k} \quad \text{for } t \geq t_k, \quad k = 1, 2, \dots$$

Therefore

$$\liminf_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon_k, \lambda_0) \geq \left(\varepsilon_0 + \prod_{i=0}^{n-1} |\lambda_0 - i| \right) \frac{1}{l - \lambda_0 + \varepsilon_k}, \quad k = 1, 2, \dots,$$

whence we obtain

$$\limsup_{k \rightarrow +\infty} \left(\liminf_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon_{k_j}, \lambda_0) \right) > \prod_{i=0; i \neq l}^{n-1} |\lambda_0 - i|.$$

This contradicts inequality (3.29_l). Thus the theorem is proved. \square

4. The sufficient conditions of nonexistence of solutions of type (2.1_l)

Theorem 4.1. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_i) ($i = l, l + 1$) are fulfilled, $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and for any $\lambda \in [l - 1, l]$*

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,1}(t, \varepsilon, \lambda) \right) > (l - 1)!(n - l - 1)!, \tag{4.1}$$

where the function $\rho_{l,1}(t, \varepsilon, \lambda)$ is defined by (3.3_l). Then Eq. (1.1) has no solution of type (2.1_l).

Proof. Assume the contrary. Let there exist $t_0 \in R_+$ such that $\mathbf{U}_{l,t_0} \neq \emptyset$. Thus Eq. (1.1) has a proper solution $u : [t_0, +\infty) \rightarrow (0, +\infty)$ satisfying the condition (2.1_l). Since the conditions of Theorem 3.1 are fulfilled, there exists $\lambda_0 \in [l - 1, l]$ such that inequality (3.2_l) holds, which contradicts (4.1). \square

Using Theorems 3.2–3.4, we can analogously prove Theorems 4.2–4.4 below.

Theorem 4.2. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_l) and (3.25) are fulfilled, $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and for any $\lambda \in [l - 1, l]$*

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,2}(t, \varepsilon, \lambda) \right) > (l - 1)!(n - l - 1)!, \tag{4.2}$$

where the function $\rho_{l,2}(t, \varepsilon, \lambda)$ is defined by (3.27_l). Then Eq. (1.1) has no solution of type (2.1_l).

Theorem 4.3. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_l) and (3.25) are fulfilled, $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and for any $\lambda \in [l - 1, l]$

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon, \lambda) \right) > \prod_{i=0; i \neq l}^{n-1} |\lambda - i|, \tag{4.3_l}$$

where the function $\rho_{l,3}(t, \varepsilon, \lambda)$ is defined by (3.30_l). Then Eq. (1.1) has no solution of type (2.1_l).

Theorem 4.4. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.1_l) and (3.25) are fulfilled, $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and for any $\lambda \in [l - 1, l]$

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,4}(t, \varepsilon, \lambda) \right) > \prod_{i=0}^{n-1} |\lambda - i|, \tag{4.4_l}$$

where the function $\rho_{l,4}(t, \varepsilon, \lambda)$ is defined by (3.33_l). Then Eq. (1.1) has no solution of type (2.1_l).

Remark 4.1. It is obvious that if the conditions of one of Theorems 4.1–4.4 are fulfilled, then the differential inequality

$$u^{(n)}(t) \operatorname{sign} u(t) + \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\mu_i(s)} d_s r_i(s, t) \leq 0$$

has no solution of type (2.1_l), where $l \in \{1, \dots, n - 1\}$ with $l + n$ odd.

5. Functional differential equations with Property A

Relying on the results obtained in Section 4, in this section we establish the sufficient conditions for Eq. (1.1) to have Property A.

Theorem 5.1. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13) are fulfilled, and for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$, the conditions (3.1_i) ($i = l, l + 1$) and (4.1_l) hold. If for odd n

$$\int_{t_1}^{+\infty} t^{n-1} \sum_{i=1}^m (r_i(\sigma_i(t), t) - r_i(\tau_i(t), t)) dt = +\infty, \tag{5.1}$$

then Eq. (1.1) has Property A.

Proof. Let Eq. (1.1) have a proper nonoscillatory solution $u : [t_0, +\infty) \rightarrow (0, +\infty)$ (the case $u(t) < 0$ is similar). Then by (1.1), (1.2) and Lemma 2.1, there exists $l \in \{0, \dots, n - 1\}$ such that $l + n$ is odd and the condition (2.1_l) holds. Since the conditions of Theorem 4.1 are fulfilled for any $l \in \{1, \dots, n - 1\}$, where $l + n$ is odd, then $l \notin \{1, \dots, n - 1\}$. Therefore n is odd and $l = 0$. Let us show that the conditions (1.3) hold. If that is not the case, since $\mu_i(t) \rightarrow 1$ as $t \rightarrow +\infty$ ($i = 1, \dots, m$), there exists $c > 0$ such that $|u(t)|^{\mu_i(t)} \geq c$ for sufficiently large t . According to (2.1_l) and (2.11) from (1.1), we have

$$\sum_{i=1}^m (n - i - 1)! t_1^i |u^{(i)}(t_1)| \geq c \int_{t_1}^t s^{n-1} \sum_{i=1}^m (r_i(\sigma_i(s), s) - r_i(\tau_i(s), s)) ds$$

for $t \geq t_1$ with t_1 sufficiently large. The latter inequality contradicts the condition (5.1). Thus (1.3) is fulfilled. This proves that Eq. (1.1) has Property A. \square

Using Theorem 4.2, we can analogously prove

Theorem 5.2. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.25) are fulfilled, and for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$ the conditions (3.1_l) and (4.2_l) hold. If for odd n the condition (5.1) is fulfilled, then Eq. (1.1) has Property A.*

Theorem 5.3. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.25) are fulfilled, and for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$ the conditions (4.3_l) holds. If for odd n the condition (5.1) is fulfilled, then Eq. (1.1) has Property A.*

Proof. If we take into account Theorem 4.3, it is sufficient to prove that (4.3_l) implies (3.1_l), where $l \in \{1, \dots, n - 1\}$ with $l + n$ odd. Suppose the contrary. Then there exists $l \in \{1, \dots, n - 1\}$ with $l + n$ odd such that

$$\int_{t_1}^{+\infty} t^{n-l} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(l-1)\mu_i(s)} d_s r_i(s, t) dt < +\infty. \tag{5.2}$$

On the other hand, taking into account (3.30_l), when $\lambda = l - 1$ (since in this case $h_1(\varepsilon) = 0$ (see (3.6)), we get

$$\begin{aligned} \rho_{l,3}(t, \varepsilon, l - 1) &= t \int_t^{+\infty} \xi^{n-l-1} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(l-1)\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi \\ &\leq \int_t^{+\infty} t^{n-l} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(l-1)\mu_i(\xi_1)} d_{\xi_1} r_i(\xi_1, \xi) d\xi. \end{aligned}$$

Therefore according to (5.2) we have $\lim_{t \rightarrow +\infty} \rho_{l,3}(t, \varepsilon, l - 1) = 0$ which contradicts (4.3_l). The obtained contradiction proves that the condition (3.1_l) is fulfilled with $l + n$ odd. Thus the validity of the theorem becomes obvious. \square

Corollary 5.1. *Suppose $F \in \mathbf{V}(\tau)$, (1.2) holds and for some $t_0 \in R_+$*

$$|F(u)(t)| \geq \sum_{i=1}^m p_i(t) \int_{\alpha_i t}^{\beta_i t} |u(s)|^{\mu_i(s)} ds \quad \text{for } t \geq t_0, u \in \mathbf{H}_{t_0, \tau}, \tag{5.3}$$

where

$$0 < \alpha_i < \beta_i, \quad p_i \in L_{\text{loc}}(R_+; R_+) \quad (i = 1, \dots, m), \tag{5.4}$$

$$\mu_i(t) = 1 + \frac{d_i}{\ln t}, \quad d_i \in R \quad (i = 1, \dots, m). \tag{5.5}$$

If, moreover, for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} t^{l-\lambda+h_{1\varepsilon}(\lambda)} \int_t^{+\infty} \xi^{n-l-h_{1\varepsilon}(\lambda)} \times \sum_{i=1}^m p_i(\xi) (\beta_i^{1+\lambda} - \alpha_i^{1+\lambda}) e^{\lambda d_i} d\xi \right) > \prod_{i=-1; i \neq l}^{n-1} |\lambda - i|, \tag{5.6_l}$$

where $h_{1\varepsilon}(\lambda)$ is defined by the first condition of (3.5), then Eq. (1.1) has Property A.

Proof. According to (5.3)–(5.6_l), we can easily show that the conditions of Theorem 5.3 are fulfilled, where $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s, t) = p_i(t)s$ ($i = 1, \dots, m$) which proves the validity of the corollary. \square

Using Theorem 4.4, analogously to Theorem 5.3 we can prove

Theorem 5.4. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (2.11)–(2.13), (3.25) are fulfilled, and for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$ the conditions (4.4_l) hold. If for odd n the condition (5.1) is fulfilled, then Eq. (1.1) has Property A.

Corollary 5.2. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (5.3)–(5.5) hold and for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} \sum_{i=1}^m p_i(s) (\beta_i^{1+\lambda} - \alpha_i^{1+\lambda}) e^{\lambda d_i} ds > \prod_{i=-1}^{n-1} |\lambda - i|. \tag{5.7_l}$$

Then Eq. (1.1) has Property A.

Proof. To prove the corollary, note that according to (5.3)–(5.5) and (5.7_l) the conditions of Theorem 5.4 are fulfilled, where $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s, t) = p_i(t)s$ ($i = 1, \dots, m$). \square

Corollary 5.3. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (5.3)–(5.5) hold,

$$p_i(t) = c_i p(t) + o(t^{-1-n}) \quad (i = 1, \dots, m), \tag{5.8}$$

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-1} p(s) ds > \max\{\varphi(\lambda): \lambda \in [l - 1, l], l \in \{1, \dots, n - 1\}, l + n \text{ is odd}\}, \tag{5.9}$$

where $p \in L_{\text{loc}}(R_+; R_+)$, $c_i \in (0, +\infty)$ ($i = 1, \dots, m$),

$$\varphi(\lambda) = \prod_{i=-1}^{n-1} |\lambda - i| \left(\sum_{i=1}^m c_i (\beta_i^{1+\lambda} - \alpha_i^{1+\lambda}) e^{\lambda d_i} \right)^{-1}. \tag{5.10}$$

Then Eq. (1.1) has Property A.

Proof. According to (5.8)–(5.10), inequality (5.7_l) is obviously fulfilled for any $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda \in [l - 1, l]$. Therefore the conditions of Corollary 5.2 are fulfilled, which proves the validity of the corollary. \square

Corollary 5.4. Suppose $0 < \alpha_i < \beta_i < +\infty$, $c_i \in (0, +\infty)$, $d_i \in R$ ($i = 1, \dots, m$). Then for the equation

$$u^{(n)}(t) + \sum_{i=1}^m \frac{c_i}{t^{n+1}} \int_{\alpha_i t}^{\beta_i t} |u(s)|^{1+\frac{d_i}{\ln s}} \operatorname{sign} u(s) ds = 0, \quad t \geq t_0, \tag{5.11}$$

with t_0 sufficiently large, to have Property **A** it is sufficient and necessary that

$$\max\{\varphi(\lambda): \lambda \in [l - 1, l], l \in \{1, \dots, n - 1\}, l + n \text{ is odd}\} < 1, \tag{5.12}$$

where $\varphi(\lambda)$ is given by (5.10).

Proof. According to (5.12) the sufficiency follows from Corollary 5.3. Show the necessity. Let (5.12) be violated. Since $\varphi(l - 1) = 0$, there obviously exists $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda_0 \in [l - 1, l)$ such that

$$\prod_{i=-1}^{n-1} |\lambda_0 - i| = \sum_{i=1}^m c_i (\beta_i^{1+\lambda_0} - \alpha_i^{1+\lambda_0}) e^{\lambda_0 d_i}.$$

If we take into account the latter inequality and the fact that $l + n$ is odd, we will see that $u(t) = t^{\lambda_0}$ is a solution of type (2.1_l) of Eq. (5.11). Therefore Eq. (5.11) has not Property **A**, which proves the necessity. \square

6. Differential equations with deviating arguments with Property **A**

Throughout this section, it is assumed that instead of (2.11) the inequality

$$|F(u)(t)| \geq \sum_{i=1}^m p_i(t) |u(\delta_i(t))|^{\eta_i(t)} \quad \text{for } t \geq t_0, u \in \mathbf{H}_{t_0, \tau}, \tag{6.1}$$

holds with $t_0 \in R_+$ sufficiently large. Here we assume that

$$p_i \in L_{\text{loc}}(R_+; R_+), \quad \eta_i \in C(R_+; (0, +\infty)), \tag{6.2}$$

$$\delta_i \in C(R_+; (0, +\infty)), \quad \lim_{t \rightarrow +\infty} \delta_i(t) = +\infty \quad (i = 1, \dots, m),$$

$$\limsup_{t \rightarrow +\infty} (\delta_i(t))^{|1-\eta_i(t)|} < +\infty \quad (i = 1, \dots, m). \tag{6.3}$$

Theorem 6.1. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.1)–(6.3) are fulfilled, and for odd n

$$\int_{t_0}^{+\infty} t^{n-1} \sum_{i=1}^m p_i(t) dt = +\infty. \tag{6.4}$$

If, moreover, for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd

$$\int_{t_0}^{+\infty} t^{n-i} \sum_{j=1}^m p_j(t) \delta_j^{i-1}(t) dt = +\infty \quad (i = l, l + 1) \tag{6.5}$$

and

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,1}^\delta(t, \varepsilon, \lambda) \right) > (l-1)!(n-l-1)!, \tag{6.6_l}$$

where

$$\begin{aligned} \rho_{l,1}^\delta(t, \varepsilon, \lambda) &= t^{-\lambda-h_{2\varepsilon}(\lambda)} \int_0^t (t-s)^{l-1} (\delta_*(s))^{h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda)} \int_s^{+\infty} (\xi-s)^{n-l-1} \\ &\quad \times \sum_{i=1}^m p_i(\xi) (\delta_i(\xi))^{(\lambda-h_{1\varepsilon}(\lambda))\eta_i(\xi)} d\xi ds, \end{aligned} \tag{6.7_l}$$

$$\delta_*(t) = \inf\{\delta(s) : s \geq t\}, \quad \delta(t) = \min\{t, \delta_i(t) : i = 1, \dots, m\}, \tag{6.8}$$

and the functions $h_{i\varepsilon}(\lambda)$ ($i = 1, 2$) are defined by (3.5), then Eq. (1.1) has Property **A**.

Proof. In view of (6.1), the inequality (2.11) clearly holds with

$$\begin{aligned} \tau_i(t) &= \delta_i(t) - 1, \quad \sigma_i(t) = \delta_i(t), \quad r_i(s, t) = p_i(t)e(s - \delta_i(t)), \\ \mu_i(\delta_i(t)) &= \eta_i(t) \quad (i = 1, \dots, m), \end{aligned} \tag{6.9}$$

where

$$e(t) = \begin{cases} 0 & \text{for } t \in (-\infty, 0), \\ 1 & \text{for } t \in [0, +\infty). \end{cases} \tag{6.10}$$

Therefore, taking into account (6.2)–(6.10), we can easily check that the conditions of Theorem 5.1 are satisfied, which proves the validity of the theorem. \square

Theorem 6.2. Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.1)–(6.3) are fulfilled, and for odd n the condition (6.4) holds along with

$$\liminf_{t \rightarrow +\infty} \frac{\delta_i(t)}{t} > 0 \quad (i = 1, \dots, m). \tag{6.11}$$

If, moreover, for any $l \in \{1, \dots, n-1\}$ and $\lambda \in [l-1, l]$ with $l+n$ odd there takes place (6.5_l) and

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,2}^\delta(t, \varepsilon, \lambda) \right) > (l-1)!(n-l-1)!, \tag{6.12_l}$$

where

$$\begin{aligned} \rho_{l,2}^\delta(t, \varepsilon, \lambda) &= t^{-\lambda-h_{2\varepsilon}(\lambda)} \int_0^t (t-s)^{l-1} s^{h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda)} \int_s^{+\infty} (\xi-s)^{n-l-1} \\ &\quad \times \sum_{i=1}^m p_i(\xi) (\delta_i(\xi))^{(\lambda-h_{1\varepsilon}(\lambda))\eta_i(\xi)} d\xi ds, \end{aligned} \tag{6.13_l}$$

then Eq. (1.1) has Property **A**.

Proof. According to (6.1)–(6.5_l), (6.11)–(6.13_l), it is easy to see that the conditions of Theorem 5.2 are fulfilled, where the functions $\tau_i(t)$, $\sigma_i(t)$, $r_i(s, t)$ and $\mu_i(t)$ ($i = 1, \dots, m$) are defined by (6.9) and (6.10). \square

Using Theorem 5.3, we can analogously prove

Theorem 6.3. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.1)–(6.3), (6.11) are fulfilled, and for odd n the condition (6.4) holds. If, moreover, for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ there takes place (6.5_l) and*

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,3}^\delta(t, \varepsilon, \lambda) \right) > \prod_{i=0; i \neq l}^{n-1} |\lambda - i|, \tag{6.14_l}$$

where

$$\rho_{l,3}^\delta(t, \varepsilon, \lambda) = t^{l-\lambda-h_{1\varepsilon}(\lambda)} \int_t^{+\infty} \xi^{n-l-1} \sum_{i=1}^m p_i(\xi) (\delta_i(\xi))^{(\lambda-h_{1\varepsilon}(\lambda))\eta_i(\xi)} d\xi \tag{6.15_l}$$

with $h_{1\varepsilon}(\lambda)$ given by the first equality of (3.5), then Eq. (1.1) has Property A.

Corollary 6.1. *Suppose $F \in \mathbf{V}(\tau)$, (1.2) holds and*

$$|F(u)(t)| \geq \sum_{i=1}^m p_i(t) |u(\alpha_i t)|^{1 + \frac{d_i}{\ln \alpha_i t}}, \quad t \geq t_0, \quad u \in \mathbf{H}_{t_0, \tau}, \tag{6.16}$$

with $t_0 \in R_+$ sufficiently large, where

$$p_i \in L_{\text{loc}}(R_+; R_+), \quad \alpha_i \in (0, +\infty), \quad d_i \in R \quad (i = 1, \dots, m). \tag{6.17}$$

If, moreover, for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} t^{l-\lambda+h_{1\varepsilon}(\lambda)} \int_t^{+\infty} s^{n-l-1+\lambda-h_{1\varepsilon}(\lambda)} \sum_{i=1}^m \alpha_i^\lambda e^{\lambda d_i} p_i(s) ds \right) \\ > \prod_{i=0; i \neq l}^{n-1} |\lambda - i|, \end{aligned} \tag{6.18_l}$$

then Eq. (1.1) has Property A.

Proof. It is sufficient to note that according to (6.17) and (6.18_l), in case of inequality (6.16) the conditions of Theorem 6.3 are fulfilled. \square

Theorem 6.4. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.1)–(6.3), (6.11) are fulfilled, and for odd n the condition (6.4) holds. If, moreover, for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd there takes place (6.5_l) and*

$$\limsup_{\varepsilon \rightarrow 0^+} \left(\liminf_{t \rightarrow +\infty} \rho_{l,4}^\delta(t, \varepsilon, \lambda) \right) > \prod_{i=0}^{n-1} |\lambda - i|, \tag{6.19_l}$$

where

$$\rho_{l,4}^\delta(t, \varepsilon, \lambda) = t \int_t^{+\infty} s^{n-2-\lambda+h_{1\varepsilon}(\lambda)} \sum_{i=1}^m p_i(s) (\delta_i(s))^{(\lambda-h_{1\varepsilon}(\lambda))\eta_i(s)} ds, \tag{6.20_l}$$

then Eq. (1.1) has Property A.

The proof of this theorem is analogous to that of Theorem 6.2, with Theorem 5.4 used instead of Theorem 5.2.

Corollary 6.2. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.16)–(6.17) hold and for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd*

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} \sum_{i=1}^m \alpha_i^\lambda e^{\lambda d_i} p_i(s) ds > \prod_{i=0}^{n-1} |\lambda - i|. \tag{6.21_l}$$

Then Eq. (1.1) has Property **A**.

Proof. Using (1.2), (6.16), (6.17), and (6.21_l) we can conclude that the conditions of Theorem 6.4 are fulfilled, where $\delta_i(t) = \alpha_i t$, $\eta_i(t) = 1 + \frac{d_i}{\ln \alpha_i t}$ ($i = 1, \dots, m$). \square

Corollary 6.3. *Suppose $F \in \mathbf{V}(\tau)$, the conditions (1.2), (6.16)–(6.17) hold and*

$$p_i(t) = c_i p(t) + o(t^{-n}) \quad (i = 1, \dots, m), \tag{6.22}$$

where $p \in L_{loc}(R_+; R_+)$, $c_i \in (0, +\infty)$ ($i = 1, \dots, m$). For Eq. (1.1) to have Property **A**, it is sufficient that

$$\begin{aligned} &\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds \\ &> \max\{\varphi(\lambda): \lambda \in [l - 1, l], l \in \{1, \dots, n - 1\}, l + n \text{ is odd}\}, \end{aligned} \tag{6.23}$$

where

$$\varphi(\lambda) = \prod_{i=0}^{n-1} |\lambda - i| \left(\sum_{i=1}^m c_i \alpha_i^\lambda e^{\lambda d_i} \right)^{-1}. \tag{6.24}$$

Proof. It is sufficient to note that according to (6.22)–(6.24), for any $l \in \{1, \dots, n - 1\}$ and $\lambda \in [l - 1, l]$ with $l + n$ odd inequality (6.21_l) takes place. \square

Corollary 6.4. *Let $c_i, \alpha_i \in (0, +\infty)$, $d_i \in R$ ($i = 1, \dots, m$). The equation*

$$u^{(n)}(t) + \sum_{i=1}^m \frac{c_i}{t^n} |u(\alpha_i t)|^{1 + \frac{d_i}{\ln \alpha_i t}} \text{sign } u(\alpha_i t) = 0, \quad t \geq t_0, \tag{6.25}$$

has Property **A** if and only if

$$\max\{\varphi(\lambda): \lambda \in [l - 1, l], l \in \{1, \dots, n - 1\}, l + n \text{ is odd}\} < 1, \tag{6.26}$$

where $\varphi(\lambda)$ is defined by (6.24).

Proof. The sufficiency follows from Corollary 6.3. Let us prove the necessity.

Suppose

$$\max\{\varphi(\lambda): \lambda \in [l - 1, l], l \in \{1, \dots, n - 1\}, l + n \text{ is odd}\} \geq 1.$$

Then there exist $l \in \{1, \dots, n - 1\}$ with $l + n$ odd and $\lambda_0 \in [l - 1, l]$ such that

$$\prod_{i=0}^{n-1} |\lambda_0 - i| = \sum_{i=1}^m c_i \alpha_i^{\lambda_0} e^{\lambda_0 d_i}.$$

It is easy to see that t^{λ_0} is a solution of (6.25). Thus Eq. (6.25) has not Property A, which proves the necessity. \square

7. Some auxiliary lemmas for Volterra type differential inequalities

Consider the following differential equation:

$$u^{(n)}(t) \operatorname{sign} u(t) + \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} |u(s)|^{\mu_i(s)} d_s r_i(s, t) \leq 0, \quad t \geq t_1, \tag{7.1}$$

where $t_1 \in R_+$, the functions τ_i, σ_i, r_i and μ_i ($i = 1, \dots, m$) satisfy the conditions (2.12) and (2.13). Furthermore, everywhere below in this section we assume that one of the following conditions is fulfilled:

$$\sigma_i(t) \leq t, \quad \mu_i(t) \leq 1 \quad \text{for } t \in R_+ \quad (i = 1, \dots, m) \tag{7.2}$$

or

$$\tau_i(t) \geq t, \quad \mu_i(t) \geq 1 \quad \text{for } t \in R_+ \quad (i = 1, \dots, m). \tag{7.3}$$

Now consider the differential equation with deviating arguments

$$u^{(n)}(t) \operatorname{sign} u(t) + p(t) |u(\delta(t))| \leq 0, \tag{7.4}$$

where $n \geq 2$, $p \in L_{\text{loc}}(R_+; R_+)$, $\lim_{t \rightarrow +\infty} \delta(t) = +\infty$. Throughout this section it will be assumed that the following condition is fulfilled:

$$\int_{t_0}^{+\infty} \delta_0^{n-1}(t) p(t) dt = +\infty, \tag{7.5}$$

where

$$\delta_0(t) = \min\{t, \delta(t)\}. \tag{7.6}$$

Lemma 7.1. *Let $\delta(t) \leq t$ for $t \in R_+$. For the differential inequality (7.4) to have Property A it is necessary and sufficient that it have no solution satisfying (2.1_{n-1}).*

Lemma 7.2. *Let $\delta(t) \geq t$ for $t \in R_+$. Then when n is even (when n is odd) for (7.4) to have Property A it is necessary and sufficient that it have no solution satisfying (2.1₁) ((2.1₂) and (2.1_{n-1})).*

The proof of Lemmas 7.1 and 7.2 see in [15, Lemmas 5.2 and 5.4].

Lemma 7.3. *Let the conditions (7.2) be fulfilled. Then for the differential inequality (7.1) to have Property A it is necessary and sufficient that it have no solution of type (2.1_{n-1}).*

Proof. The necessity is obvious. Show the sufficiency. Let the inequality have no solution of type (2.1_{n-1}) and prove that it has Property **A**. First of all note that since (7.1) has no solution of type (2.1_{n-1}), according to Lemma 4.1 from [15],

$$\int_{+\infty}^{\infty} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(n-1)\mu_i(s)} d_s r_i(s, t) dt = +\infty. \tag{7.7}$$

On the other hand, if we take into account (7.2) and (7.7), we obtain

$$\int_{+\infty}^{\infty} t^{n-k-1} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} \xi^{k\mu_i(\xi)} d_\xi r_i(\xi, t) dt = +\infty \quad (k = 0, \dots, n - 1). \tag{7.8k}$$

Now suppose that the differential inequality (7.1) has not Property **A** and $u : [t_0, +\infty) \rightarrow R$ is a proper nonoscillatory solution of the differential inequality (7.1). Then by Lemma (2.1), without loss of generality, we can assume that the function u satisfies the condition (2.1_l), where $l \in \{0, \dots, n - 3\}$ with $l + n$ odd. As it was shown in the proof of Theorem 5.1, if n is odd and $l = 0$, then according to (7.8₀), the conditions (1.3) hold. Therefore, since by the assumption the differential inequality has not Property **A**, (7.1) has a solution of type (2.1_l), where $l \in \{1, \dots, n - 3\}$ with $l + n$ odd. By (2.1_l), there exist $c > 0$ and $t_1 > t_0$ such that $u(t) \geq ct^{l-1}$ for $t \geq t_1$. Therefore from (7.1) we have

$$|u^{(n)}(t)| \geq c_0 \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} \xi^{(l-1)\mu_i(\xi)} d_s r_i(s, t)$$

for t sufficiently large, where $c_0 \in (0, c)$. Thus by (7.8_{l-1}) it is obvious that the condition (2.3) is fulfilled. Hence the conditions of Lemma 2.2 hold. Taking into account the first condition of (2.4₀) and (7.8_l), from (7.1) it is easily obtained that $u(t)/t^l \rightarrow 0$ as $t \rightarrow \infty$. Thus $u(t) \leq t^l$ for t sufficiently large. Therefore, by (2.4₀), (7.1) and (7.2) the function u for sufficiently large t satisfies the differential inequality

$$u^{(n)}(t) + \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{l\mu_i(s)} d_s r_i(s, t) \frac{u(\bar{\sigma}(t))}{(\bar{\sigma}(t))^l} \leq 0, \tag{7.9}$$

where $\bar{\sigma}(t) = \max\{\sigma_i(t) : i = 1, \dots, m\}$. On the other hand, according to (7.2) and (7.8_{n-1}), the condition (7.5) is fulfilled, where

$$\delta_0(t) = \bar{\sigma}(t) \quad \text{and} \quad p(t) = \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{l\mu_i(s)} d_s r_i(s, t) (\bar{\sigma}(t))^{-l}.$$

Therefore by Lemma 7.1, inequality (7.9) has a solution u_1 of type (2.1_{n-1}). Since $l \leq n - 3$, it is obvious that $u_1(t)/t^l \uparrow +\infty$ as $t \uparrow +\infty$. Therefore by (7.2), (7.9) for sufficiently large t , u_1 is a solution of type (2.1_{n-1}) of the differential inequality (7.1). This contradicts the conditions of the lemma. The obtained contradiction proves the validity of the lemma. \square

Lemma 7.4. *Let the condition (7.3) be fulfilled and for odd n (7.8₀) hold. Then for the differential inequality (7.1) to have Property **A** it is necessary and sufficient that in the case of even n (odd n) the differential inequality (7.1) have no solution of type (2.1₁) ((2.1₂) and (2.1_{n-1})).*

Proof. The necessity is obvious. Show the sufficiency. Consider the case where n is even and prove that the differential inequality (7.1) has Property **A**. First of all note that since inequality (7.1) has no solution of type (2.1₁), due to Lemma 4.1 from [15], (7.8₁) is fulfilled. Therefore, by (7.3) the condition (7.8_k) is fulfilled for $k = 1, \dots, n - 1$. Now suppose that the differential inequality (7.1) has not Property **A** and $u : [t_0, +\infty) \rightarrow R$ is a proper nonoscillatory solution of the differential inequality (7.1). Then using (7.1) and Lemma 2.1, without loss of generality, we can assume that the condition (2.1_l) is fulfilled, where $l \in \{3, \dots, n - 1\}$ and $l + n$ is odd. According to (7.8_{l-1}), (2.1_l) and (7.1), it is obvious that the condition (2.3) is fulfilled. Therefore according to the second condition of (2.4₀) and (7.3), from (7.1) it follows that for sufficiently large t u is a proper solution of type (2.1_l) of the differential inequality

$$u^{(n)}(t) + \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(l-1)\mu_i(s)} d_s r_i(s, t) \frac{u(t)}{t^{l-1}} \leq 0. \tag{7.10}$$

On the other hand, by (7.8_{l-1}) the condition (7.5) is fulfilled with

$$\delta_0(t) = t, \quad p(t) = t^{1-l} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(l-1)\mu_i(s)} d_s r_i(s, t).$$

Therefore by Lemma 7.2, the differential inequality (7.10) has a proper solution u_1 of type (2.1₁). Since for the function u_1 the conditions of Lemma 2.2 are fulfilled, we have $u_1(t)/t^{l-1} \downarrow$ for $t \uparrow +\infty$. Therefore, by (7.3), from (7.10) we obtain that for sufficiently large t the function u_1 is a proper solution of type (2.1₁) of the differential inequality (7.1), which contradicts the conditions of lemma in case of even n . The obtained contradiction proves that the differential inequality (7.1) has Property **A**. As for the case of odd n , by the reasoning analogous to the above, we will show that the differential inequality (7.1) has no proper solution of type (2.1_l), where $l = \{2, \dots, n - 1\}$ with $l + n$ odd. On the other hand, if the differential inequality (7.1) has a proper solution of type (2.1₀), then using (7.8₀), we can easily show that the function u satisfies the condition (1.3). Hence in the case of odd n inequality (7.1) has Property **A**, which proves the validity of the lemma. \square

8. Functional differential equations with a Volterra type minorant with Property **A**

Theorem 8.1. *Let $F \in \mathbf{V}(\tau)$ and (1.2), (2.11)–(2.13), (7.2) and (7.8_{n-1}) be fulfilled. Then the condition (4.1_{n-1}) is sufficient for Eq. (1.1) to have Property **A**.*

Proof. First of all note that (7.8_{n-1}) and (7.2) imply the validity of (7.8_k) for any $k \in \{0, \dots, n - 1\}$. Suppose now that Eq. (1.1) has not Property **A**. Then by Lemma 2.1, (1.1) has a proper nonoscillatory solution $u : [t_0, +\infty) \rightarrow R$ satisfying the condition (2.1_l), where $l \in \{1, \dots, n - 1\}$ with $l + n$ odd (if n is odd and $l = 0$, then according to (7.8₀), the conditions (1.3) hold). By (2.11), u is a proper solution of (7.1) with t_1 sufficiently large. Since $l \in \{1, \dots, n - 1\}$ with $l + n$ odd, due to Lemma 7.3 the differential inequality (7.1) has a solution of type (2.1_{n-1}). On the other hand, if the conditions of Theorem 4.1 with $l = n - 1$ are fulfilled, according to Remark 4.1, (7.1) has no solution of type (2.1_{n-1}). The obtained contradiction proves the validity of the theorem. \square

Theorem 8.2. Let $F \in \mathbf{V}(\tau)$ and (1.2), (2.11)–(2.13), (3.25), (7.2) and (7.8 $_{n-1}$) be fulfilled. Then for Eq. (1.1) to have Property **A** it is sufficient that one of the three conditions (4.2 $_{n-1}$) or (4.3 $_{n-1}$) or (4.4 $_{n-1}$) hold.

Proof. The proof is analogous to that of Theorem 8.1, with the use of Theorems 4.2, 4.3 or 4.4, respectively. \square

In the case of a Volterra type minorant we can formulate various corollaries just as in the general case, but we restrict ourselves with ones whose conditions has simpler form.

Corollary 8.1. Let $F \in \mathbf{V}(\tau)$ and the conditions (1.2), (5.3)–(5.5), (5.8) be fulfilled, where $p \in L_{\text{loc}}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\beta_i \leq 1$ ($i = 1, \dots, m$). Then the condition

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-1} p(s) ds > \max\{\varphi(\lambda): \lambda \in [n-2, n-1]\} \quad (8.1)$$

is sufficient for Eq. (1.1) to have Property **A**, where $\varphi(\lambda)$ is given by equality (5.10).

Corollary 8.2. Let $0 < \alpha_i < \beta_i \leq 1$, $c_i \in (0, +\infty)$, $d_i \in R$ ($i = 1, \dots, m$). Then the condition

$$\max\{\varphi(\lambda): \lambda \in [n-2, n-1]\} < 1, \quad (8.2)$$

is necessary and sufficient for Eq. (5.11) to have Property **A**, where $\varphi(\lambda)$ is given by equality (5.10).

If we take into account Remark 4.1 and Lemma 7.3, the validity of Corollaries 8.1 and 8.2 follows from Corollaries 5.3 and 5.4.

Corollary 8.3. Let $F \in \mathbf{V}(\tau)$ and the conditions (1.2), (5.3)–(5.5), (6.22) be fulfilled, where $p \in L_{\text{loc}}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\alpha_i \leq 1$ ($i = 1, \dots, m$). Then condition (8.1) is sufficient for Eq. (1.1) to have Property **A**, where $\varphi(\lambda)$ is given by equality (6.24).

Corollary 8.4. Let $c_i \in (0, +\infty)$, $0 < \alpha_i \leq 1$, $d_i \in R$ ($i = 1, \dots, m$). Then for Eq. (6.25) to have Property **A** it is necessary and sufficient that condition (8.2) be fulfilled, where $\varphi(\lambda)$ is given by (6.24).

If we take into account Remark 4.1 and Lemma 7.3, the validity of Corollaries 8.3 and 8.4 follows from Corollaries 6.3 and 6.4.

Theorem 8.3. Let $F \in \mathbf{V}(\tau)$ and the conditions (1.2), (2.11)–(2.13), (7.3), (7.8 $_0$) be fulfilled. Then for Eq. (1.1) to have Property **A** it is sufficient that for even n (odd n) the condition (4.2 $_1$) ((4.1 $_2$) and (4.1 $_{n-1}$)) hold.

Proof. The proof of Theorem 8.3 is analogous to that of Theorem 8.1, Lemma 7.3 being used instead of Lemma 7.4. \square

Analogously we can prove

Theorem 8.4. Let $F \in \mathbf{V}(\tau)$ and the conditions (1.2), (2.11)–(2.13), (3.25), (7.3) and (7.8₀) be fulfilled. Then for Eq. (1.1) to have Property **A** it is sufficient that one of the following conditions (4.2₁) or (4.3₁) or (4.4₁) ((4.2₂) and (4.2_{n-1}) or (4.3₂) and (4.3_{n-1}) or (4.4₂) and (4.4_{n-1})) hold for even n (odd n).

Corollary 8.5. Let $F \in \mathbf{V}(\tau)$ and (1.2), (5.3)–(5.5), (5.8) be fulfilled, where $p \in L_{\text{loc}}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\alpha_i \geq 1$ ($i = 1, \dots, m$). Then for Eq. (1.1) to have Property **A** it is sufficient that the condition

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-1} p(s) ds > \max\{\varphi(\lambda): \lambda \in [0, 1]\}, \tag{8.3}$$

be fulfilled for even n , and the condition

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-1} p(s) ds > \max\{\varphi(\lambda): \lambda \in [1, 2] \cup [n - 2, n - 1]\} \tag{8.4}$$

for odd n , where $\varphi(\lambda)$ is defined by (5.10).

Corollary 8.6. Let $c_i \in (0, +\infty)$, $1 \leq \alpha_i < \beta_i < +\infty$, $d_i \in R$ ($i = 1, \dots, m$). Then for Eq. (5.11) to have Property **A** it is necessary and sufficient that the condition

$$\max\{\varphi(\lambda): \lambda \in [0, 1]\} < 1, \tag{8.5}$$

be fulfilled for even n , and the condition

$$\max\{\varphi(\lambda): \lambda \in [1, 2] \cup [n - 2, n - 1]\} < 1 \tag{8.6}$$

for odd n , where $\varphi(\lambda)$ is defined by (5.10).

If we take into account Remark 4.1 and Lemma 7.4, the validity of Corollaries 8.5 and 8.6 follows from Corollaries 5.3 and 5.4.

Corollary 8.7. Let $F \in \mathbf{V}(\tau)$ and the conditions (1.2), (6.16), (6.17), (6.22) be fulfilled, where $p \in L_{\text{loc}}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\alpha_i \geq 1$ ($i = 1, \dots, m$). Then for Eq. (1.1) to have Property **A** it is sufficient that the condition (8.3) be fulfilled for even n , and the condition (8.4) for odd n , where $\varphi(\lambda)$ is defined by (6.24).

Corollary 8.8. Let $c_i \in (0, +\infty)$, $\alpha_i \in [1, +\infty)$, $d_i \in R$ ($i = 1, \dots, m$). Then for Eq. (6.25) to have Property **A** it is necessary and sufficient that the condition (8.5) be fulfilled for even n , and the condition (8.6) for odd n , where $\varphi(\lambda)$ is defined by (6.24).

If we take into account Remark 4.1 and Lemma 7.4, the validity of Corollaries 8.7 and 8.8 follows from Corollaries 6.3 and 6.4.

9. Generalized ordinary differential equations of Emden–Fowler type

Here we give sufficient conditions for Eq. (1.4) to have Property **A**. The results of this section are the consequences of those of the previous ones, but we present them because in this case the conditions have quite a simple form.

Theorem 9.1. *Let*

$$\limsup_{t \rightarrow +\infty} t^{|1-\eta_i(t)|} < +\infty \quad (i = 1, \dots, m), \tag{9.1}$$

*the condition (6.22) be fulfilled, where $p \in L_{loc}(R_+; R_+)$, $c_i \in (0, +\infty)$ ($i = 1, \dots, m$). Then for Eq. (1.4) to have Property **A**, it is sufficient that*

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds > \max\{\varphi(\lambda): \lambda \in [0, n-1]\},$$

where

$$\varphi(\lambda) = -\left(\sum_{i=1}^m c_i \gamma_i^\lambda\right)^{-1} \lambda(\lambda-1)\dots(\lambda-n+1), \tag{9.2}$$

$$\gamma_i = \liminf_{t \rightarrow +\infty} t^{\eta_i(t)-1} \quad (i = 1, \dots, m). \tag{9.3}$$

Theorem 9.2. *Let $c_i \in (0, +\infty)$, $d_i \in R$ ($i = 1, \dots, m$). Then for the equation*

$$u^{(n)}(t) + \frac{1}{t^n} \sum_{i=1}^m c_i |u(t)|^{1+\frac{d_i}{m}} \text{sign } u(t) = 0, \quad t \geq 2, \tag{9.4}$$

*to have Property **A**, it is necessary and sufficient that*

$$\max\left\{-\left(\sum_{i=1}^m c_i e^{\lambda d_i}\right)^{-1} \lambda(\lambda-1)\dots(\lambda-n+1): \lambda \in [0, n-1]\right\} < 1.$$

Theorem 9.3. *Let the conditions (9.1), (6.22) be fulfilled, where $p \in L_{loc}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\eta_i(t) \leq 1$ for $t \in R_+$ ($i = 1, \dots, m$). Then for Eq. (1.4) to have Property **A**, it is sufficient that*

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds > \max\{\varphi(\lambda): \lambda \in [0, n-1]\},$$

where $\varphi(\lambda)$ is defined by the equalities (9.2) and (9.3).

Theorem 9.4. *Let $c_i \in (0, +\infty)$, $d_i \in (-\infty, 0]$. Then for Eq. (9.4) to have Property **A**, it is necessary and sufficient that*

$$\max\left\{-\left(\sum_{i=1}^m c_i e^{\lambda d_i}\right)^{-1} \lambda(\lambda-1)\dots(\lambda-n+1): \lambda \in [n-2, n-1]\right\} < 1.$$

Theorem 9.5. *Let the conditions (9.1), (6.22) be fulfilled, where $p \in L_{loc}(R_+; R_+)$, $c_i \in (0, +\infty)$, $\eta_i(t) \geq 1$ ($i = 1, \dots, m$). Then for Eq. (1.4) to have Property **A**, it is sufficient that*

$$\liminf_{t \rightarrow +\infty} t \int_t^{+\infty} s^{n-2} p(s) ds > \max\{\varphi(\lambda): \lambda \in [0, 1]\}$$

for even n , and

$$\liminf_{t \rightarrow +\infty} \int_t^{+\infty} s^{n-2} p(s) ds > \max \{ \varphi(\lambda) : \lambda \in [1, 2] \cup [n-2, n-1] \}$$

for odd n , where $\varphi(\lambda)$ is defined by (9.2) and (9.3).

Theorem 9.6. Let $c_i \in (0, +\infty)$, $d_i \in [0, +\infty)$. Then for Eq. (9.4) to have Property **A**, it is necessary and sufficient that

$$\max \left\{ - \left(\sum_{i=1}^m c_i e^{\lambda d_i} \right)^{-1} \lambda(\lambda-1) \cdots (\lambda-n+1) : \lambda \in [0, 1] \right\} < 1$$

for even n , and

$$\max \left\{ - \left(\sum_{i=1}^m c_i e^{\lambda d_i} \right)^{-1} \lambda(\lambda-1) \cdots (\lambda-n+1) : \lambda \in [1, 2] \cup [n-2, n-1] \right\} < 1$$

for odd n .

References

- [1] V.A. Kondrat'ev, On oscillation of solutions of the equation $y^{(n)} + p(x)y = 0$, Tr. Mosk. Mat. Obs. 10 (1961) 419–436 (in Russian).
- [2] I.T. Kiguradze, T.A. Chanturia, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations, Kluwer Academic, Dordrecht, 1993.
- [3] R. Koplatadze, Differential equations with deviating argument that have the properties **A** and **B**, Differ. Uravn. 25 (11) (1989) 1897–1909; English transl.: Differ. Equ. 25 (11) (1989) 1332–1342 (1990).
- [4] M.K. Grammatikopoulos, R. Koplatadze, G. Kvinikadze, Linear functional differential equations with Property **A**, J. Math. Anal. Appl. 284 (2003) 294–314.
- [5] R. Koplatadze, G. Kvinikadze, I.P. Stavroulakis, Properties **A** and **B** of n th order linear differential equations with deviating argument, Georgian Math. J. 6 (1999) 553–566.
- [6] F.V. Atkinson, On second-order nonlinear oscillations, Pacific J. Math. 5 (1) (1955) 643–647.
- [7] I.T. Kiguradze, On oscillatory solutions of some ordinary differential equations, Soviet Math. Dokl. 144 (1962) 33–36.
- [8] I. Ličko, M. Švec, Le caractere oscillatoire des solutions de l'équation $y^{(n)} + f(x)y^\alpha = 0$, $n > 1$, Czechoslovak Math. J. 13 (1963) 481–489.
- [9] A.G. Kartsatos, Recent results on oscillation of solutions of forced and perturbed nonlinear differential equations of even order, in: J.R. Graef (Ed.), Stability of Dynamical Systems, Dekker, New York, 1977.
- [10] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Dekker, New York, 1987.
- [11] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.
- [12] L.H. Erbe, Q. Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations, Dekker, New York, 1995.
- [13] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic, Dordrecht, 2002.
- [14] R. Koplatadze, T. Canturia, On Oscillatory Properties of Differential Equation with a Deviating Argument, Tbilisi State Univ. Press, Tbilisi, 1977 (in Russian).
- [15] R. Koplatadze, On oscillatory properties of solutions of functional differential equations, Mem. Differential Equations Math. Phys. 3 (1994) 1–179.
- [16] R. Koplatadze, On higher order functional differential equations with Property **A**, Georgian Math. J. 11 (2) (2004) 307–336.

- [17] R. Koplatadze, Generalized ordinary differential equations of Emden–Fowler type with Properties **A** and **B**, Proc. A. Razmadze Math. Inst. 136 (2004) 145–148.
- [18] R. Koplatadze, G. Kvinikadze, On oscillatory properties of generalized ordinary differential equations of Emden–Fowler type, Mem. Differential Equations Math. Phys. 34 (2005) 153–156.
- [19] J. Graef, R. Koplatadze, G. Kvinikadze, Nonlinear functional differential equations with Properties **A** and **B**, J. Math. Anal. Appl. 306 (2005) 136–160.