Available at www.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 284 (2003) 294–314

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Linear functional differential equations with Property A

M.K. Grammatikopulos,[a] R. Koplatadze,[b,*] and G. Kvinikadze[b]

[a] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*
[b] *A. Razmadze Mathematical Institute, Georgian Academy of Sciences, M. Aleksidze St. 1, Tbilisi 380093, Georgia*

Received 28 January 2003

Submitted by R.P. Agarwal

Abstract

In the paper the general linear functional differential equation with several distributed deviations is considered. Sufficient conditions for the equation to have Property A (see Definition 1.2 below) are established. The obtained results are new even for Eq. (1.4).

Keywords: Linear functional differential equations; Distributed deviations; Oscillatory solution; Property A

1. Introduction

Consider the differential equation

$$u^{(n)}(t) + \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} u(s)\, d_s r_i(s,t) = 0, \tag{1.1}$$

where $n \geqslant 2$, $\sigma_i, \tau_i \in C(R_+; (0,\infty))$, $\tau_i(t) \leqslant \sigma_i(t)$ for $t \in R_+$, and $\lim_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, \dots, m)$, while the functions $r_i : R_+ \times R_+ \to R$ are nondecreasing in the first argument and Lebesgue integrable in the second argument on any finite subsegment of $[0, +\infty)$.

* Corresponding author.
E-mail addresses: mgrammat@cc.uoi.gr (M.K. Grammatikopulos), roman@rmi.acnet.ge (R. Koplatadze), giokvi@rmi.acnet.ge (G. Kvinikadze).

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00356-1

Definition 1.1. Let $t_0 \in R_+$. A function $u \in C(R_+; R)$ is called a proper solution of Eq. (1.1) on $[t_0, +\infty)$, if it is locally absolutely continuous on $[t_0, +\infty)$ along with its derivatives up to the order $n-1$ inclusively, almost everywhere on $[t_0, +\infty)$ satisfies (1.1) and $\sup\{|u(s)|\colon\ s \geqslant t\} > 0$ for $t \geqslant t_0$. A proper solution is called oscillatory, if it has a sequence of zeros tending to $+\infty$. Otherwise, it is called nonoscillatory.

Definition 1.2 [1]. We say that Eq. (1.1) has Property A, if any of its solutions is oscillatory when n is even, and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{for } t \uparrow +\infty \ (i = 0, \ldots, n-1) \tag{1.2}$$

when n is odd.

Oscillatory properties of ordinary differential equations have been studied since long. As early as in 1893, Kneser [2] obtained sufficient conditions for the equation

$$u^{(n)}(t) + p(t)u(t) = 0 \tag{1.3}$$

with $p \in L_{\mathrm{loc}}(R_+; R_+)$ to have Property A. Essential results in this direction were obtained by Kondrat'ev, Kiguradze, and Chanturia [1,3]. For the differential equation with deviating arguments

$$u^{(n)}(t) + \sum_{i=1}^{m} p_i(t)u\big(\delta_i(t)\big) = 0 \tag{1.4}$$

with

$$\begin{gathered} p_i \in L_{\mathrm{loc}}(R_+; R_+), \qquad \delta_i \in C(R_+; R_+), \\ \lim_{t\to+\infty} \delta_i(t) = +\infty \quad (i = 1, \ldots, m), \end{gathered} \tag{1.5}$$

which is a special case of (1.1), similar problems were considered in [4] (see also references therein). As to functional differential equations, both in linear and nonlinear cases they are studied well enough in [5]. In the present paper we give sufficient conditions for Eq. (1.1) to have Property A. The obtained results make more precise those obtained in [6] for Eq. (1.4) with $m = 1$.

Remark 1.1. Equation (1.1) can obviously be written as

$$u^{(n)}(t) + \int_{\tau(t)}^{\sigma(t)} u(s)\, d_s\, \tilde{r}(s, t) = 0, \tag{1.6}$$

where $\tau(t) = \min\{\tau_i(t)\colon\ 1 \leqslant i \leqslant m\}$, $\sigma(t) = \max\{\sigma_i(t)\colon\ 1 \leqslant i \leqslant m\}$, and $\tilde{r}(\cdot, t)$ is the sum of $r_i(\cdot, t)$ appropriately extended to $[\tau(t), \sigma(t)]$. For this reason, it may seem at the first sight that in (1.1) it suffices to consider only the case $m = 1$. However, doing so we would misconsider the specific character of (1.1). Writing (1.1) as (1.6) and applying the below obtained results with $m = 1$, we would obtain the results worse than those obtained by considering (1.1) directly.

2. Some auxiliary lemmas

In this section we give some auxiliary statements concerning some properties of monotone functions. In the sequel we denote by $\tilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$ the set of those functions $u:[t_0,+\infty)\to R$ which are absolutely continuous on any finite subsegment of $[t_0,+\infty)$ along with their derivatives up to the order $n-1$ inclusively.

First we formulate the following well-known lemma due to Kiguradze.

Lemma 2.1 [3]. *Let $u\in\tilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$, $u(t)>0$, $u^{(n)}(t)\leqslant 0$ for $t\geqslant t_0$, and $u^{(n)}(t)\not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1\geqslant t_0$ and $l\in\{0,\dots,n-1\}$ such that $l+n$ is odd and*

$$\begin{aligned}
&u^{(i)}(t)>0 \quad \text{for } t\geqslant t_1\ (i=0,\dots,l-1),\\
&(-1)^{i+l}u^{(i)}(t)>0 \quad \text{for } t\geqslant t_1\ (i=l,\dots,n-1),\\
&u^{(n)}(t)\leqslant 0 \quad \text{for } t\geqslant t_1.
\end{aligned}\tag{2.1$_l$}$$

Remark 2.1. If n is odd and $l=0$, then in (2.1_0) it is meant that only the second and the third inequalities are fulfilled.

Now we prove the following lemma describing the behavior of nonoscillatory functions.

Lemma 2.2. *Let $u\in\tilde{C}^{n-1}_{\rm loc}([t_0,+\infty))$ and (2.1_l) be fulfilled for some $l\in\{1,\dots,n-1\}$ with $l+n$ odd. Then*

$$\int^{+\infty} t^{n-l-1}\big|u^{(n)}(t)\big|\,dt<+\infty. \tag{2.2}$$

If, moreover,

$$\int^{+\infty} t^{n-l}\big|u^{(n)}(t)\big|\,dt=+\infty, \tag{2.3}$$

then there exists $t_*\geqslant t_0$ such that

$$\frac{u^{(i)}(t)}{t^{l-i}}\downarrow,\quad \frac{u^{(i)}(t)}{t^{l-i-1}}\uparrow\quad (i=0,\dots,l-1), \tag{2.4$_i$}$$

$$u(t)\geqslant\frac{t^{l-1}}{l!}u^{(l-1)}(t)\quad \text{for } t\geqslant t_*, \tag{2.5}$$

and

$$u^{(l-1)}(t)\geqslant\frac{t}{(n-l)!}\int_t^{+\infty}s^{n-l-1}\big|u^{(n)}(s)\big|\,ds+\frac{1}{(n-l)!}\int_{t_*}^{t}s^{n-l}\big|u^{(n)}(s)\big|\,ds \quad \text{for } t\geqslant t_*. \tag{2.6}$$

Proof. Taking into account (2.1_l), we deduce (2.2) from the identity

$$\sum_{j=i}^{k-1}\frac{(-1)^j t^{j-i}u^{(j)}(t)}{(j-i)!}=\sum_{j=i}^{k-1}\frac{(-1)^j t_0^{j-i}u^{(j)}(t_0)}{(j-i)!}+\frac{(-1)^{k-1}}{(k-i-1)!}\int_{t_0}^{t}s^{k-i-1}u^{(k)}(s)\,ds \tag{$2.7_{i,k}$}$$

with $i=l$ and $k=n$. The same equality implies

$$\sum_{j=l}^{n-1}\frac{t^{j-l}|u^j(t)|}{(j-l)!}\geqslant\frac{1}{(n-l-1)!}\int_t^{+\infty}s^{n-l-1}\big|u^{(n)}(s)\big|\,ds\quad\text{for } t\geqslant t_0. \tag{2.8}$$

Now assume that (2.3) holds. Then, using (2.1_l), from $(2.7_{l-1,n})$ we obtain

$$u^{(l-1)}(t)-tu^{(l)}(t)=\sum_{j=l+1}^{n-1}\frac{t^{j-l+1}|u^{(j)}(t)|}{(j-l+1)!}+\sum_{j=l-1}^{n-1}\frac{(-1)^{j+l-1}t_0^{j-l+1}u^{(j)}(t_0)}{(j-l+1)!}+\frac{1}{(n-l)!}\int_{t_0}^{t}s^{n-l}\big|u^{(n)}(s)\big|\,ds\quad\text{for } t\geqslant t_0. \tag{2.9}$$

Hence by (2.3) we have

$$\lim_{t\to+\infty}\big(u^{(l-1)}(t)-tu^{(l)}(t)\big)=+\infty. \tag{2.10}$$

For any $t\geqslant t_0$ and $i\in\{1,\dots,l\}$ denote

$$\gamma_i(t)=iu^{(l-i)}(t)-tu^{(l-i+1)}(t)=-t^{i+1}\big(t^{-i}u^{(l-i)}(t)\big)', \tag{2.11}$$

$$r_i(t)=tu^{(l-i+1)}(t)-(i-1)u^{(l-i)}(t)=t^i\big(t^{1-i}u^{(l-i)}(t)\big)'. \tag{2.12}$$

According to (2.10) and by L'Hôpital's rule we obtain

$$\lim_{t\to+iy}\frac{u^{(l-i)}(t)}{t^{i-1}}=+\infty\quad(i=1,\dots,l).$$

Therefore, in view of (2.12), there exist $\alpha_l\geqslant\cdots\geqslant\alpha_1$ such that $r_i(\alpha_i)>0$ $(i=1,\dots,l)$. Since $r_1(t)=tu^{(l)}(t)>0$ and $r'_{i+1}(t)=r_i(t)$ for $t\geqslant t_0$ $(i=1,\dots,l-1)$, we see that $r_i(t)>0$ for $t\geqslant\alpha_i$. Analogously, by (2.10) we have $\gamma_1(t)\to+\infty$ for $t\to+\infty$ and $\gamma'_{i+1}(t)=\gamma_i(t)$. Therefore $\gamma_i(t)\to+\infty$ as $t\to+\infty$ $(i=1,\dots,l)$. So, by (2.11) and (2.12), we obtain (2.4_i). On the other hand, by (2.11) we have

$$iu^{(l-i)}(t)\geqslant tu^{(l-i+1)}(t)\quad\text{for sufficiently large } t\ (i=1,\dots,l),$$

whence (2.5) follows.

Now we show that (2.6) is true. By (2.3) there is $t_* > t_0$ such that

$$\frac{1}{(n-l)!}\int_{t_0}^{t_*} s^{n-l}\left|u^{(n)}(s)\right| ds \geqslant \sum_{j=l-1}^{n-1} \frac{(-1)^{j+l} t_0^{j-l+1} u^{(j)}(t_0)}{(j-l+1)!}.$$

Therefore (2.8) and (2.9) imply (2.6). The proof of the lemma is complete. □

3. Main results

Let $j \in \{0, 1, \dots, n-1\}$ and $\varphi, \sigma \in C([t_0, +\infty); (0, +\infty))$. Denote

$$\eta_{\varphi,\sigma}(t) = \begin{cases} 1 & \text{for } \varphi(t) \leqslant \sigma(t), \\ \frac{\sigma(t)}{\varphi(t)} & \text{for } \varphi(t) > \sigma(t), \end{cases} \tag{3.1}$$

$$\rho_{ji}(t) = \int_{\tau_i(t)}^{\sigma_i(t)} s^j \, d_s r_i(s,t) \quad (i = 1, \dots, m). \tag{3.2_j}$$

The following four propositions are crucial in obtaining efficient sufficient conditions for Property A.

Proposition 3.1. *Let $l \in \{1, \dots, n-1\}$ with $l + n$ odd and there exist a nondecreasing function $\varphi \in C(R_+; R_+)$ such that*

$$\lim_{t\to+\infty} \varphi(t) = +\infty, \quad \varphi(t) \leqslant t \quad \textit{for } t \geqslant 0, \tag{3.3}$$

$$\begin{aligned} \limsup_{t\to+\infty} \Bigg\{ \sum_{i=1}^{m} \Bigg(& \varphi(t) \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds + \int_{\varphi(t)}^{t} \frac{s^{n-l-1}}{\sigma_i(s)} \varphi(s) \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds \\ & + \frac{1}{\varphi(t)} \int_0^{\varphi(t)} \frac{s^{n-l}}{\sigma_i(s)} \varphi(s) \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds \Bigg) \Bigg\} > l!(n-l)!, \end{aligned} \tag{3.4_l}$$

where the functions η_{φ,σ_i} and ρ_{li} $(i = 1, \dots, m)$ are defined by (3.1) *and* (3.2_l)*, respectively. Then Eq.* (1.1) *has no solution of the type* (2.1_l)*.*

Proof. First of all we will show that (3.3) and (3.4_l) imply

$$\int^{+\infty} t^{n-l} \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} s^{l-1} \, d_s r_i(s,t)\, dt = +\infty. \tag{3.5_l}$$

Indeed, otherwise by (3.3) and

$$\eta_{\varphi,\sigma_i}(t)\rho_{li}(t) \leqslant \sigma_i(t) \int_{\tau_i(t)}^{\sigma_i(t)} s^{l-1} \, d_s r_i(s,t) \quad (i = 1, \dots, m)$$

we obtain

$$\varphi(t)\int_t^{+\infty} s^{n-l-1}\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$\leqslant \int_t^{+\infty} s^{n-l}\sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds \to 0 \quad \text{as } t\to+\infty, \tag{3.6}$$

$$\int_{\varphi(t)}^{t} s^{n-l-1}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$\leqslant \int_{\varphi(t)}^{t} s^{n-l}\sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds \to 0 \quad \text{as } t\to+\infty,$$

$$\frac{1}{t}\int_0^t s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$= \frac{1}{t}\int_0^{t_*} s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$+ \frac{1}{t}\int_{t_*}^{t} s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$\leqslant \frac{1}{t}\int_0^{t_*} s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds$$
$$+ \int_{t_*}^{t} s^{n-l}\sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds\,ds$$
$$< \frac{1}{t}\int_0^{t_*} s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds + \varepsilon \quad \text{for } t\geqslant t_*, \tag{3.7}$$

where, given $\varepsilon>0$, t_* is chosen so that

$$\int_{t_*}^{+\infty} s^{n-l}\sum_{i=1}^m \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds < \varepsilon.$$

In view of arbitrariness of ε, the latter implies

$$\frac{1}{t}\int_0^t s^{n-l}\varphi(s)\sum_{i=1}^m \sigma_i^{-1}(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds \to 0 \quad \text{as } t\to+\infty. \tag{3.8}$$

Obviously, (3.6)–(3.8) contradict (3.4_l). This contradiction proves that (3.5_l) is fulfilled.

Now suppose that Eq. (1.1) has a proper nonoscillatory solution $u:[t_0,+\infty)\to(0,+\infty)$ satisfying (2.1_l) with $l\in\{1,\dots,n-1\}$ and $l+n$ odd. In view of (2.1_l) and (3.5_l), the function u satisfies the conditions of Lemma 2.2. Therefore conditions (2.4_{l-1}) are fulfilled and there exists $t_*\geqslant t_0$ such that

$$u^{(l-1)}\big(\varphi(t)\big)\geqslant \frac{\varphi(t)}{(n-l)!}\int_{\varphi(t)}^{+\infty} s^{n-l-1}\big|u^{(n)}(s)\big|\,ds + \frac{1}{(n-l)!}\int_{t_*}^{\varphi(t)} s^{n-l}\big|u^{(n)}(s)\big|\,ds \quad \text{for } t\geqslant t_*. \tag{3.9}$$

Hence from (1.1), in view of (2.5) and the first condition of (2.4_{l-1}), we have

$$\begin{aligned} u^{(l-1)}\big(\varphi(t)\big)\geqslant{}& \frac{\varphi(t)}{l!(n-l)!}\int_{\varphi(t)}^{+\infty} s^{n-l-1}\sum_{i=1}^m u^{(l-1)}\big(\sigma_i(s)\big)\sigma_i^{-1}(s)\int_{\tau_i(s)}^{\sigma_i(s)}\xi^l\,d_\xi\, r_i(\xi,s)\,ds \\ &+\frac{1}{l!(n-l)!}\int_{t_*}^{\varphi(t)} s^{n-l}\sum_{i=1}^m u^{(l-1)}\big(\sigma_i(s)\big)\sigma_i^{-1}(s)\int_{\tau_i(s)}^{\sigma_i(s)}\xi^l\,d_\xi\, r_i(\xi,s)\,ds \end{aligned}$$

for large t. (3.10)

On the other hand, by (2.1_l) and the first condition of (2.4_{l-1}) it is obvious that the inequalities

$$u^{(l-1)}\big(\sigma_i(t)\big)\geqslant \eta_{\varphi,\sigma_i}(t)u^{(l-1)}\big(\varphi(t)\big) \quad (i=1,\dots,m)$$

hold for large t, where the functions η_{φ,σ_i} are defined by (3.1). So (3.10) and (3.3) imply

$$\begin{aligned} u^{(l-1)}\big(\varphi(t)\big)\geqslant{}& \frac{1}{l!(n-l)!}\sum_{i=1}^m\Bigg(\varphi(t)\int_t^{+\infty}\frac{s^{n-l-1}}{\sigma_i(s)}\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds \\ &+\varphi(t)\int_{\varphi(t)}^{t}\frac{s^{n-l-1}}{\sigma_i(s)}\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds \\ &+\int_{t_1}^{\varphi(t)}\frac{s^{n-l}}{\sigma_i(s)}\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds\Bigg) \quad \text{for } t\geqslant t_1, \end{aligned} \tag{3.11}$$

where $t_1 > t_*$ is sufficiently large and the functions η_{φ,σ_i} and ρ_{li} $(i = 1, \dots, m)$ are defined by (3.1) and (3.2_l), respectively. According to the second condition of (2.4_{l-1}) we have

$$\varphi(t) \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds \geqslant \varphi(t)u^{(l-1)}\big(\varphi(t)\big) \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds \quad \text{for } t \geqslant t_1\ (i = 1, \dots, m). \tag{3.12}$$

On the other hand, the first condition of (2.4_{l-1}) and (3.3) imply

$$\begin{aligned}
&\varphi(t) \int_{\varphi(t)}^{t} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds + \int_{t_1}^{\varphi(t)} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s)\rho_{li}(s)u^{(l-1)}\big(\varphi(s)\big)\,ds \\
&\quad \geqslant u^{(l-1)}\big(\varphi(t)\big)\left(\int_{\varphi(t)}^{t} \frac{s^{n-l-1}}{\sigma_i(s)}\varphi(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds + \frac{1}{\varphi(t)}\int_{t_1}^{\varphi(t)} \frac{s^{n-l-1}}{\sigma_i(s)}\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\varphi(s)\,ds\right) \\
&\quad \text{for } t \geqslant t_1\ (i = 1, \dots, m).
\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), from (3.11) we obtain

$$\begin{aligned}
l!(n-l)! \geqslant \sum_{i=1}^{m}\Bigg(&\varphi(t)\int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds + \int_{\varphi(t)}^{t} \frac{s^{n-l-1}}{\sigma_i(s)}\varphi(s)\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\,ds \\
&+ \frac{1}{\varphi(t)}\int_{t_1}^{\varphi(t)} \frac{s^{n-l-1}}{\sigma_i(s)}\eta_{\varphi,\sigma_i}(s)\rho_{li}(s)\varphi(s)\,ds\Bigg) \quad \text{for } t \geqslant t_1.
\end{aligned}$$

But this contradicts (3.4_l). The obtained contradiction proves the proposition. □

Proposition 3.2. *Let $l \in \{1, \dots, n-1\}$ with $l + n$ is odd and there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that* (3.3) *is fulfilled and*

$$\limsup_{t\to+\infty}\Biggl\{\sum_{i=1}^{m}\Biggl(\varphi(t)\int_{t}^{+\infty} s^{n-l-1}\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds + \int_{\varphi(t)}^{t} s^{n-l-1}\varphi(s)\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds + \frac{1}{\varphi(t)}\int_{0}^{\varphi(t)} s^{n-l}\varphi(s)\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds\Biggr)\Biggr\} > l!(n-l)!, \tag{3.14_l}$$

where the functions η_{φ,τ_i} *and* ρ_{l-1i} $(i = 1,\dots,m)$ *are defined by* (3.1) *and* (3.2_{l-1}), *respectively. Then Eq.* (1.1) *has no solution of the type* (2.1_l).

Proof. As in the proof of Proposition 3.1, by (3.14_l) we can show that (3.5_l) is fulfilled. Therefore, if we suppose that Eq. (1.1) has a proper nonoscillatory solution satisfying (2.1_l) with $l \in \{1,\dots,n-1\}$ and $l+n$ odd, as in Proposition 3.1 we will conclude that (3.9) is fulfilled with t_* sufficiently large. On the other hand, (3.9) along with (1.1), (2.5), and the second condition of (2.4_{l-1}) implies

$$u^{(l-1)}\bigl(\varphi(t)\bigr) \geqslant \frac{\varphi(t)}{l!(n-l)!}\int_{\varphi(t)}^{+\infty} s^{n-l-1}\sum_{i=1}^{m} u^{(l-1)}\bigl(\tau_i(s)\bigr)\int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds + \frac{1}{l!(n-l)!}\int_{t_*}^{\varphi(t)} s^{n-l}\sum_{i=1}^{m} u^{(l-1)}\bigl(\tau_i(s)\bigr)\int_{\tau_i(s)}^{\sigma_i(s)} \xi^{l-1}\,d_\xi\, r_i(\xi,s)\,ds$$

for large t.

If we follow the arguments similar to those used in the proof of Proposition 3.1 with σ_i replaced by τ_i, we will see that the above inequality implies

$$l!(n-l)! \geqslant \sum_{i=1}^{m}\Biggl(\varphi(t)\int_{t}^{+\infty} s^{n-l-1}\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds + \int_{\varphi(t)}^{t} s^{n-l-1}\varphi(s)\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds + \frac{1}{\varphi(t)}\int_{t_*}^{\varphi(t)} s^{n-l}\varphi(s)\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\varphi(s)\,ds\Biggr) \quad \text{for large } t.$$

But this contradicts (3.14_l). The obtained contradiction proves the proposition. □

Proposition 3.3. *Let $l \in \{1, \dots, n-1\}$ with $l+n$ odd and there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that*

$$\varphi(t) \geqslant t \quad \text{for } t \in R_+ \tag{3.15}$$

and

$$\limsup_{t \to +\infty} \left\{ \sum_{i=1}^{m} \left(\varphi(t) \int_{\varphi(t)}^{+\infty} \frac{s^{n-l-1}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds + \int_{t}^{\varphi(t)} \frac{s^{n-l}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds \right.\right.$$
$$\left.\left. + \frac{1}{\varphi(t)} \int_0^t \frac{s^{n-l}}{\sigma_i(s)} \varphi(s) \eta_{\varphi,\sigma_i}(s) \rho_{li}(s)\, ds \right) \right\} > l!(n-l)!, \tag{3.16$_l$}$$

where the functions η_{φ,σ_i} and ρ_{li} $(i = 1, \dots, m)$ are defined by (3.1) *and* (3.2$_l$)*, respectively. Then Eq.* (1.1) *has no solution of the type* (2.1$_l$).

Proof. As in the proof of Proposition 3.1, from (3.16$_l$) it follows (3.5$_l$). Therefore, if we suppose that Eq. (1.1) has a proper nonoscillatory solution satisfying (2.1$_l$) with $l \in \{1, \dots, n-1\}$ and $l+n$ odd, as in the proof of Proposition 3.1 we see that (3.9) is fulfilled with t_* sufficiently large. On the other hand, (3.9) along with (1.1), (2.5), (3.15), and (2.4$_{l-1}$) implies

$$u^{(l-1)}\big(\varphi(t)\big) \geqslant \frac{\varphi(t)}{l!(n-l)!} \int_{\varphi(t)}^{+\infty} s^{n-l-1} \sum_{i=1}^{m} \eta_{\varphi,\sigma_i}(s) \rho_{li}(s) \frac{u^{(l-1)}(\varphi(s))}{\sigma_i(s)}\, ds$$
$$+ \sum_{i=1}^{m} \left(\int_t^{\varphi(t)} \frac{s^{n-l}}{\sigma_i(s)} \eta_{\varphi,\sigma_i}(s) \rho_{li}(s) u^{(l-1)}\big(\varphi(s)\big)\, ds \right.$$
$$\left. + \int_{t_*}^{t} \frac{s^{n-l}}{\sigma_i(s)} \varphi(s) \eta_{\varphi,\sigma_i}(s) \rho_{li}(s) \frac{u^{(l-1)}(\varphi(s))}{\varphi(s)}\, ds \right) \quad \text{for large } t. \tag{3.17}$$

By using the second condition of (2.4$_{l-1}$) in the first two addends of (3.17) and the first condition in the third one, we easily get the inequality opposite to (3.16$_l$). The obtained contradiction proves the proposition. □

In the same manner as in the cases of Propositions 3.1–3.3, we prove the following

Proposition 3.4. *Let $l \in \{1, \dots, n-1\}$ with $l+n$ odd and there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that* (3.15) *is fulfilled and*

$$\limsup_{t \to +\infty} \left\{ \sum_{i=1}^{m} \left(\varphi(t) \int_{\varphi(t)}^{+\infty} s^{n-l-1} \eta_{\varphi,\tau_i}(s) \rho_{l-1i}(s)\, ds + \int_t^{\varphi(t)} s^{n-l} \eta_{\varphi,\tau_i}(s) \rho_{l-1i}(s)\, ds \right.\right.$$

$$+\frac{1}{\varphi(t)}\int_0^t s^{n-l}\varphi(s)\eta_{\varphi,\tau_i}(s)\rho_{l-1i}(s)\,ds\Bigg)\Bigg\} > l!(n-l)!, \tag{3.18$_l$}$$

where the functions η_{φ,τ_i} *and* ρ_{l-1i} $(i=1,\dots,m)$ *are defined by* (3.1) *and* (3.2_{l-1}), *respectively. Then Eq.* (1.1) *has no solution of the type* (2.1_l).

4. Functional differential equations with Property A

In this section, using the previous results, we derive sufficient conditions under which the functional differential equation (1.1) has Property A.

Theorem 4.1. *Let there exist a nondecreasing function* $\varphi \in C(R_+;(0,+\infty))$ *satisfying* (3.3) *and such that for any* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd the condition* (3.4_l) *holds. Let, moreover, in the case of the odd* n *the condition*

$$\int^{+\infty} t^{n-1}\sum_{i=1}^m \big(r_i\big(\sigma_i(t),t\big) - r_i\big(\tau_i(t),t\big)\big)\,dt = +\infty \tag{4.1}$$

be fulfilled. Then Eq. (1.1) *has Property* A.

Proof. Suppose that Eq. (1.1) has a proper nonoscillatory solution $u:[t_0,+\infty)\to(0,+\infty)$. Then by (1.1) and Lemma 2.1 there exists $l\in\{0,\dots,n-1\}$ such that $l+n$ is odd and conditions (2.1_l) are fulfilled. According to (3.3), (3.4_l), and Proposition 3.1, we have $l\notin\{1,\dots,n-1\}$. Hence n is odd and $l=0$. We will show that in this case (1.2) holds. If this is not the case, then by (2.1_0) we have $\lim_{t\to+\infty}u(t)=c>0$. Therefore there is $t_*\in R$ such that $u(t)\geqslant c/2$ for $t\geqslant t_*$ and $\tau_i(t)\geqslant t_*$ for $t\geqslant t_1$ $(i=1,\dots,m)$, where $t_1\geqslant t_*$ is sufficiently large. Thus, in view of (2.1_0) from (1.1) we obtain

$$\sum_{i=0}^{n-1}(n-i-1)!t_1^i\big|u^{(i)}(t_1)\big|$$
$$\geqslant \frac{c}{2}\int_{t_1}^t s^{n-1}\sum_{i=1}^m\big(r_i\big(\sigma_i(s),s\big)-r_i\big(\tau_i(s),s\big)\big)\,ds \quad \text{for } t\geqslant t_1.$$

But this contradicts (4.1). The obtained contradiction proves (1.2). Consequently, Eq. (1.1) has Property A. □

Remark 4.1. Note that condition (4.1) is necessary for Eq. (1.1) to have Property A (see [5, Lemma 4.1]).

Corollary 4.1. *Let* $\alpha_i,\beta_i\in(0,+\infty)$, $\alpha_i<\beta_i$, $\beta_i\geqslant 1$, $p_i\in L_{\mathrm{loc}}(R_+;R_+)$ $(i=1,\dots,m)$, *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd,*

$$\limsup_{t\to+\infty}\sum_{i=1}^{m}\frac{(\beta_i^{l+1}-\alpha_i^{l+1})}{\beta_i}\left(t\int_t^{+\infty}s^{n-1}p_i(s)\,ds+\frac{1}{t}\int_0^t s^{n+1}p_i(s)\,ds\right)$$
$$>(l+1)!(n-l)!. \tag{4.2}$$

Then the equation

$$u^{(n)}(t)+\sum_{i=1}^{m}p_i(t)\int_{\alpha_i t}^{\beta_i t}u(s)\,ds=0 \tag{4.3}$$

has Property A.

Proof. It suffices to note that for Eq. (4.3), in view of (4.2), the conditions of Theorem 4.1 are fulfilled with $\varphi(t)\equiv t$. □

Corollary 4.1′ below shows that if a function $\tilde p\in L_{\mathrm{loc}}(R_+;R_+)$ is a common minorant of p_i, or all p_i are close to $\tilde p$ in the sense that "lim inf" in (4.4) is zero, than the $[n/2]$ conditions in (4.2) can be written in a compact form in terms of $\tilde p$. Analogous remarks are true for Corollaries 4.2′–4.4′.

Corollary 4.1′. *Let* $\alpha_i,\beta_i\in(0,+\infty)$, $\alpha_i<\beta_i$, $\beta_i\geqslant 1$, $p_i,\tilde p\in L_{\mathrm{loc}}(R_+;R_+)$ $(i=1,\dots,m)$, *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd*,

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\frac{\beta_i^{l+1}-\alpha_i^{l+1}}{\beta_i}\left(t\int_t^{+\infty}s^{n-1}\big(p_i(s)-\tilde p(s)\big)\,ds\right.$$
$$\left.+\frac{1}{t}\int_0^t s^{n+1}\big(p_i(s)-\tilde p(s)\big)\,ds\right)\geqslant 0. \tag{4.4}$$

Then for Eq. (4.3) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty}s^{n-1}\tilde p(s)\,ds+\frac{1}{t}\int_0^t s^{n+1}\tilde p(s)\,ds\right)$$
$$>\max\left\{(l+1)!(n-l)!\left(\sum_{i=1}^{m}\frac{\beta_i^{l+1}-\alpha_i^{l+1}}{\beta_i}\right)^{-1}:\ l\in\{1,\dots,n-1\},\right.$$
$$\left.l+n\ \textit{is odd}\right\}. \tag{4.5}$$

To prove Corollary 4.1′, it suffices to note that (4.4) and (4.5) imply (4.2).

By means of Propositions 3.2–3.4, Theorems 4.2–4.4 below can be proved analogously to Theorem 4.1.

Theorem 4.2. *Let there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ satisfying* (3.3) *and such that for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd condition* (3.14_l) *holds. Let, moreover, in the case of the odd n the condition* (4.1) *be fulfilled. Then Eq.* (1.1) *has Property* A.

Corollary 4.2. *Let $\alpha_i, \beta_i \in (0, +\infty)$, $\alpha_i \leqslant 1 \leqslant \beta_i$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \dots, m)$, and for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd,*

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \alpha_i \big(\beta_i^l - \alpha_i^l\big) \left(t \int_t^{+\infty} s^{n-1} p_i(s)\, ds + \frac{1}{t} \int_0^t s^{n+1} p_i(s)\, ds \right) > l l!(n-l)!. \tag{4.6}$$

Then Eq. (4.3) *has Property* A.

To prove the corollary, it suffices to note that for Eq. (4.3), in view of (4.6), the conditions of Theorem 4.2 are fulfilled with $\varphi(t) \equiv t$.

Corollary 4.2′. *Let $\alpha_i, \beta_i \in (0, +\infty)$, $\alpha_i \leqslant 1 \leqslant \beta_i$, $p_i, \tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \dots, m)$, and for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd,*

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \alpha_i \big(\beta_i^l - \alpha_i^l\big) \left(t \int_t^{+\infty} s^{n-1} \big(p_i(s) - \tilde{p}(s)\big)\, ds + \frac{1}{t} \int_0^t s^{n+1} \big(p_i(s) - \tilde{p}(s)\big)\, ds \right) \geqslant 0. \tag{4.7}$$

Then for Eq. (4.3) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty} \left(t \int_t^{+\infty} s^{n-1} \tilde{p}(s)\, ds + \frac{1}{t} \int_0^t s^{n+1} \tilde{p}(s)\, ds \right) > \max\left\{ \frac{l l!(n-l)!}{\sum_{i=1}^{m} \alpha_i (\beta_i^l - \alpha_i^l)} :\ l \in \{1, \dots, n-1\},\ l+n \text{ is odd} \right\}. \tag{4.8}$$

Since (4.7) and (4.8) imply (4.6), Corollary 4.2′ follows from Corollary 4.2.

Theorem 4.3. *Let there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ satisfying* (3.15) *and such that for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd condition* (3.16_l) *holds. Let, moreover, in the case of the odd n the condition* (4.1) *be fulfilled. Then Eq.* (1.1) *has Property* A.

Corollary 4.3. *Let $\alpha_i, \beta_i \in (0, +\infty)$, $\alpha_i < \beta_i$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \dots, m)$, and for any $l \in \{1, \dots, n-1\}$ with $l+n$ odd,*

$$\limsup_{t\to+\infty}\sum_{i=1}^{m}\bigl(\beta_i^{l+1}-\alpha_i^{l+1}\bigr)\left(\beta^* t\int_{\beta^* t}^{+\infty} s^{n-1}p_i(s)\,ds + \int_t^{\beta^* t} s^n p_i(s)\,ds + \frac{1}{t}\int_0^t s^{n+1}p_i(s)\,ds\right) > \beta^*(l+1)!(n-l)!, \tag{4.9$_l$}$$

where $\beta^* = \max\{1, \beta_i\colon i = 1,\dots,m\}$. *Then Eq.* (4.3) *has Property* A.

To prove the corollary, it suffices to note that in view of (4.9$_l$) the conditions of Theorem 4.3 are fulfilled for Eq. (4.3) with $\varphi(t) \equiv \beta^* t$.

Corollary 4.3′. *Let* $\alpha_i, \beta_i \in (0,+\infty)$, $\alpha_i < \beta_i$, $p_i, \tilde p \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1,\dots,m)$, *and for any* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd,*

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\bigl(\beta_i^{l+1}-\alpha_i^{l+1}\bigr)\left(\beta^* t\int_{\beta^* t}^{+\infty} s^{n-1}\bigl(p_i(s)-\tilde p(s)\bigr)\,ds + \int_t^{\beta^* t} s^n\bigl(p_i(s)-\tilde p(s)\bigr)\,ds + \frac{1}{t}\int_0^t s^{n+1}\bigl(p_i(s)-\tilde p(s)\bigr)\,ds\right) \geqslant 0. \tag{4.10$_l$}$$

Then for Eq. (4.3) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty}\left(\beta^* t\int_{\beta^* t}^{+\infty} s^{n-1}\tilde p(s)\,ds + \int_t^{\beta^* t} s^n\tilde p(s)\,ds + \frac{1}{t}\int_0^t s^{n+1}\tilde p(s)\,ds\right) > \max\left\{\frac{\beta^*(l+1)!(n-l)!}{\sum_{i=1}^m(\beta_i^{l+1}-\alpha_i^{l+1})}\colon l\in\{1,\dots,n-1\},\ l+n \text{ is odd}\right\}, \tag{4.11}$$

where $\beta^* = \max\{1, \beta_i\colon i = 1,\dots,m\}$.

Since (4.10$_l$) and (4.11) imply (4.9$_l$), Corollary 4.3′ follows from Corollary 4.3.

Theorem 4.4. *Let there exist a nondecreasing function* $\varphi \in C(R_+; (0,+\infty))$ *satisfying* (3.15) *and such that for any* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd condition* (3.18$_l$) *holds. Let, moreover, in the case of the odd* n *the condition* (4.1) *be fulfilled. Then Eq.* (1.1) *has Property* A.

Corollary 4.4. *Let* $\alpha_i, \beta_i \in (0,+\infty)$, $\alpha_i < \beta_i$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1,\dots,m)$, *and for any* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd,*

$$\limsup_{t\to+\infty}\sum_{i=1}^{m}\alpha_i\left(\beta_i^l-\alpha_i^l\right)\left(\beta^*t\int_{\beta^*t}^{+\infty}s^{n-1}p_i(s)\,ds+\int_{t}^{\beta^*t}s^n p_i(s)\,ds+\frac{1}{t}\int_{0}^{t}s^{n+1}p_i(s)\,ds\right)$$
$$>ll!(n-l)!\beta^*,$$

where $\beta^*=\max\{1,\beta_i\colon i=1,\dots,m\}$. *Then Eq.* (4.3) *has Property* A.

Corollary 4.4′. *Let* $\alpha_i,\beta_i\in(0,+\infty)$, $\alpha_i<\beta_i$, $p_i,\tilde p\in L_{\mathrm{loc}}(R_+;R_+)$ $(i=1,\dots,m)$, *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd,*

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\alpha_i\left(\beta_i^l-\alpha_i^l\right)\left(\beta^*t\int_{\beta^*t}^{+\infty}s^{n-1}\big(p_i(s)-\tilde p(s)\big)\,ds\right.$$
$$\left.+\int_{t}^{\beta^*t}s^n\big(p_i(s)-\tilde p(s)\big)\,ds+\frac{1}{t}\int_{0}^{t}s^{n+1}\big(p_i(s)-\tilde p(s)\big)\,ds\right)\geqslant 0.$$

Then for Eq. (4.3) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty}\left(\beta^*t\int_{\beta^*t}^{+\infty}s^{n-1}\tilde p(s)\,ds+\int_{t}^{\beta^*t}s^n\tilde p(s)\,ds+\frac{1}{t}\int_{0}^{t}s^{n+1}\tilde p(s)\,ds\right)$$
$$>\max\left\{\frac{\beta^*ll!(n-l)!}{\sum_{i=1}^{m}\alpha_i(\beta_i^l-\alpha_i^l)}\colon l\in\{1,\dots,n-1\},\ l+n\text{ is odd}\right\},$$

where $\beta^*=\max\{1,\beta_i\colon i=1,\dots,m\}$.

Remark 4.2. In all the theorems of this section the expression "there exists φ such that for any $l\dots$" can be replaced by "for any l there exists φ_l such that..." without affecting their validity.

Remark 4.3. Corollaries 4.1 and 4.2 show that the conditions of Theorem 4.1 can be fulfilled without those of Theorem 4.2 being satisfied, and vice versa. The situation is analogous for Theorems 4.3 and 4.4.

5. Volterra type equations with Property A

In the case where either

$$\sigma_i(t)\leqslant t\quad\text{for } t\in R_+\ (i=1,\dots,m)\tag{5.1}$$

or

$$\tau_i(t)\geqslant t\quad\text{for } t\in R_+\ (i=1,\dots,m)\tag{5.2}$$

hold, the $[n/2]$ conditions in Theorems 4.1–4.4 can be replaced by one or two.

Theorem 5.1. *Let inequalities* (5.1) *be fulfilled and there exist a nondecreasing function* $\varphi \in C(R_+; (0, +\infty))$ *satisfying conditions* (3.3) *such that either* (3.4_{n-1}) *or* (3.14_{n-1}) *is fulfilled. Then Eq.* (1.1) *has Property* A.

Proof. To prove the theorem, it suffices to note that, in view of (5.1), (3.3), and (3.4_{n-1}) ((5.1), (3.3), and (3.14_{n-1})), conditions (4.1) and (3.4_l) ((3.14_l)) are fulfilled for any l. Therefore all the conditions of Theorem 4.1 (Theorem 4.2) are fulfilled. This proves the theorem. □

Corollary 5.1. *Let* $0 < \alpha_i < \beta_i \leqslant 1$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \ldots, m)$, *and inequality* (4.9_{n-1}) *be fulfilled with* $\beta^* = 1$. *Then Eq.* (4.3) *has Property* A.

The validity of the corollary follows from Theorem 5.1 since in the case of Eq. (4.3) condition (4.9_{n-1}) implies (3.4_{n-1}) with $\varphi(t) \equiv t$.

Corollary 5.2. *Let* $0 < \alpha_i < \beta_i \leqslant 1$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \ldots, m)$, *and*

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \left(\beta_i^{n-1} - \alpha_i^{n-1}\right) \left(t \int_t^{+\infty} s^{n-1} p_i(s)\, ds + \alpha_* \int_{\alpha_* t}^{t} s^n p_i(s)\, ds + \frac{1}{t} \int_0^{\alpha_* t} s^{n+1} p_i(s)\, ds \right) > (n-1)!, \tag{5.3}$$

where $\alpha_* = \min\{\alpha_i\colon i = 1, \ldots, m\}$. *Then Eq.* (4.3) *has Property* A.

The validity of the corollary follows from Theorem 5.1 since in the case of Eq. (4.3) condition (5.3) implies (3.14_{n-1}) with $\varphi(t) \equiv \alpha_* t$.

Corollary 5.1′. *Let* $0 < \alpha_i < \beta_i \leqslant 1$, $p_i, \tilde p \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \ldots, m)$, *and inequalities* (4.10_{n-1}) *and* (4.11_{n-1}) *be fulfilled with* $\beta^* = 1$. *Then Eq.* (4.3) *has Property* A.

To prove the corollary, it suffices to note that (4.10_{n-1}) and (4.11_{n-1}) imply the conditions of Corollary 5.1.

Similarly, from Corollary 5.2 we can get

Corollary 5.2′. *Let* $0 < \alpha_i < \beta_i \leqslant 1$, $p_i, \tilde p \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \ldots, m)$, *and*

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \left(\beta_i^{n-1} - \alpha_i^{n-1}\right) \left(t \int_t^{+\infty} s^{n-1} \big(p_i(s) - \tilde p(s)\big)\, ds + \alpha_* \int_{\alpha_* t}^{t} s^n \big(p_i(s) - \tilde p(s)\big)\, ds + \frac{1}{t} \int_0^{\alpha_* t} s^{n+1} \big(p_i(s) - \tilde p(s)\big)\, ds \right) \geqslant 0.$$

Then for Eq. (4.3) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty}s^{n-1}\tilde{p}(s)\,ds+\alpha_*\int_{\alpha_* t}^{t}s^n\tilde{p}(s)\,ds+\frac{1}{t}\int_0^{\alpha_* t}s^{n+1}\tilde{p}(s)\,ds\right)$$
$$>\frac{(n-1)!}{\sum_{i=1}^m(\beta_i^{n-1}-\alpha_i^{n-1})},$$

where $\alpha_*=\min\{\alpha_i\colon i=1,\dots,m\}$.

Using Theorems 4.3 and 4.4, we can prove similarly to Theorem 5.1 the following

Theorem 5.2. *Let inequalities* (5.1) *be fulfilled and there exist a nondecreasing function* $\varphi\in C(R_+;(0,+\infty))$ *satisfying* (3.15) *such that either* (3.16_{n-1}) *or* (3.18_{n-1}) *is fulfilled. Then Eq.* (1.1) *has Property* A.

Theorem 5.3. *Let inequalities* (5.2) *be fulfilled and there exist a nondecreasing function* $\varphi\in C(R_+;(0,+\infty))$ *satisfying* (3.3) *such that either* (3.4_1) *or* (3.14_1) *holds if* n *is even and either* (3.4_2), (3.4_{n-1}) *or* (3.14_2), (3.14_{n-1}) *hold if* n *is odd. Let, moreover,* (4.1) *hold in the case where* n *is odd. Then Eq.* (1.1) *has Property* A.

Proof. It can be easily checked that, in view of (1.2) and (3.3), if (3.4_1) holds in the case where n is even and (3.4_2), (3.4_{n-1}) hold in the case where n is odd, then all the conditions of Theorem 4.1 are fulfilled. On the other hand, if (3.14_1) holds in the case where n is even and (3.14_2), (3.14_{n-1}) hold in the case where n is odd, all the conditions of Theorem 4.2 are fulfilled. This proves the theorem. □

Analogously can be proved the following

Theorem 5.4. *Let inequalities* (5.2) *be fulfilled and there exist a nondecreasing function* $\varphi\in C(R_+;(0,+\infty))$ *satisfying* (3.15) *such that either* (3.16_1) *or* (3.18_1) *holds if* n *is even and either* (3.16_2), (3.16_{n-1}) *or* (3.18_2), (3.18_{n-1}) *hold if* n *is odd. Let, moreover,* (4.1) *hold in the case where* n *is odd. Then Eq.* (1.1) *has Property* A.

Remark 5.1. In Theorems 5.3 and 5.4 one cannot ignore any of the conditions required in the case where n is odd. Otherwise the theorems will not be true.

Remark 5.2. It is clear from Corollaries 5.1 and 5.2 that, using the above theorems and choosing a function φ, one can obtain quite simple effective criteria for various types of Eq. (1.1) to have Property A.

6. Differential equations with deviating arguments

In this section we consider Eq. (1.4) with the functions p_i and δ_i $(i=1,\dots,m)$ satisfying (1.5). It is obvious that this equation is a particular case of (1.1). Below we will give some sufficient conditions for Eq. (1.4) to have Property A. The validity of the theorems of this section follows from the theorems of the previous section. Specifically, Theorem 6.1 is implied by Theorem 5.1, while Theorem 6.2 follows from Theorem 5.3.

Theorem 6.1. *Let*

$$\delta_i(t) \leqslant t \quad \text{for } t \in R_+ \ (i = 1, \ldots, m) \tag{6.1}$$

and the inequality

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \left(\delta_*(t) \int_t^{+\infty} p_i(s)\delta_i^{n-2}(s)\,ds + \int_{\delta_*(t)}^{t} \delta_*(s)\delta_i^{n-2}(s)p_i(s)\,ds + \frac{1}{\delta_*(t)} \int_0^{\delta_*(t)} s\delta_*(s)\delta_i^{n-2}(s)p_i(s)\,ds \right) > (n-1)!$$

holds, where

$$\delta_*(t) = \inf_{s\geqslant t}\bigl(\min\{\delta_i(s)\colon\ i = 1, \ldots, m\}\bigr). \tag{6.2}$$

Then Eq. (1.4) *has Property* A.

Corollary 6.1. *Let $\delta_i(t) \geqslant \alpha_i t$ for $t \in R_+$, $\alpha_i \in (0,1]$ $(i = 1, \ldots, m)$, and*

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \alpha_i^{n-2} \left(t \int_t^{+\infty} s^{n-2} p_i(s)\,ds + \int_{\alpha_* t}^{t} s^{n-1} p_i(s)\,ds + \frac{1}{\alpha_* t} \int_0^{\alpha_* t} s^n p_i(s)\,ds \right) > \frac{(n-1)!}{\alpha_*} \tag{6.3}$$

with $\alpha_* = \min\{\alpha_i\colon\ i = 1, \ldots, m\}$. Then Eq. (1.4) *has Property* A.

Proof. Suppose on the contrary that the equation does not have Property A. Then by Theorem 2.1 of [5] the equation

$$u^{(n)}(t) + \sum_{i=1}^{m} p_i(t)u(\alpha_i t) = 0 \tag{6.4}$$

does not have Property A. On the other hand, according to (6.3) all the conditions of Theorem 6.1 are fulfilled for Eq. (6.4). Therefore Eq. (6.4) has Property A. The obtained contradiction proves the theorem. □

Corollary 6.1′. *Let $\delta_i(t) \geqslant \alpha_i t$ for $t \in R_+$ with $\alpha_i \in (0,1]$, $p_i, \tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \ldots, m)$, and*

$$\limsup_{t\to+\infty} \sum_{i=2}^{m} \alpha_i^{n-2} \left(t \int_t^{+\infty} s^{n-2}\bigl(p_i(s) - \tilde{p}(s)\bigr)\,ds + \int_{\alpha_* t}^{t} s^{n-1}\bigl(p_i(s) - \tilde{p}(s)\bigr)\,ds \right.$$

$$+\frac{1}{\alpha_* t}\int_0^{\alpha_* t} s^n\big(p_i(s)-\tilde p(s)\big)\,ds\Bigg)\geqslant 0, \tag{6.5}$$

where $\alpha_*=\min\{\alpha_i\colon\ i=1,\dots,m\}$. *Let, moreover,*

$$\limsup_{t\to+\infty}\sum_{i=1}^m\Bigg(t\int_t^{+\infty}s^{n-2}\tilde p(s)\,ds+\int_{\alpha_* t}^t s^n\tilde p(s)\,ds+\frac{1}{\alpha_* t}\int_0^{\alpha_* t}s^n\tilde p(s)\,ds\Bigg)$$
$$>\frac{(n-1)!}{\alpha_*\sum_{i=1}^m\alpha_i^{n-2}}. \tag{6.6}$$

Then Eq. (1.4) *has Property* A.

Since conditions (6.5) and (6.6) imply (6.3), this corollary follows from Corollary 6.1.

Remark 6.1. Inequality (6.5) is obviously fulfilled if $p_i(t)\equiv\tilde p(t)+o(t^{-n})$ $(i=1,\dots,m)$. Even in this case the result obtained in Corollary 6.1$'$ is new.

Theorem 6.2. *Let*

$$\delta_i(t)\geqslant t\quad \text{for } t\in R_+\ (i=1,\dots,m). \tag{6.7}$$

Suppose, moreover, that in the case where n *is even the inequality*

$$\limsup_{t\to+\infty}\sum_{i=1}^m\Bigg(\delta_*(t)\int_{\delta_*(t)}^{+\infty}s^{n-2}p_i(s)\,ds+\int_t^{\delta_*(t)}s^{n-1}p_i(s)\,ds$$
$$+\frac{1}{\delta_*(t)}\int_0^t s^{n-1}\delta_*(s)p_i(s)\,ds\Bigg)>(n-1)!$$

holds, while in the case where n *is odd the following three conditions are fulfilled*:

$$\limsup_{t\to+\infty}\sum_{i=1}^m\Bigg(\delta_*(t)\int_{\delta_*(t)}^{+\infty}s^{n-3}\delta_i(s)p_i(s)\,ds+\int_t^{\delta_*(t)}s^{n-2}\delta_i(s)p_i(s)\,ds$$
$$+\frac{1}{\delta_*(t)}\int_0^t s^{n-2}\delta_*(s)\delta_i(s)p_i(s)\,ds\Bigg)>2(n-2)!,$$

$$\limsup_{t\to+\infty}\sum_{i=1}^m\Bigg(\delta_*(t)\int_{\delta_*(t)}^{+\infty}\delta_i^{n-2}(s)p_i(s)\,ds+\int_t^{\delta_*(t)}s\delta_i^{n-2}(s)p_i(s)\,ds$$
$$+\frac{1}{\delta_*(t)}\int_0^t s\delta_*(s)\delta_i^{n-2}(s)p_i(s)\,ds\Bigg)>(n-1)!,$$

and

$$\int^{+\infty} t^{n-1} \sum_{i=1}^{m} p_i(t)\,dt = +\infty,$$

where $\delta_*(t)$ *is defined by* (6.2). *Then Eq.* (1.4) *has Property* A.

Remark 6.2. As it has been noted above (Remark 5.1), in this case too, none of the three conditions of Theorem 6.2 can be ignored when n is odd.

Similarly to Corollary 6.1 one can prove

Corollary 6.2. *Let* $\delta_i(t) \geqslant \alpha_i t$ *for* $t \in R_+$ *with* $\alpha_i \in [1, +\infty)$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \dots, m)$. *Let for even* n,

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \left(\alpha_* t \int_{\alpha_* t}^{+\infty} s^{n-2} p_i(s)\,ds + \int_{t}^{\alpha_* t} s^{n-1} p_i(s)\,ds + \frac{1}{t}\int_{0}^{t} s^n p_i(s)\,ds \right) > (n-1)!,$$

while for odd n,

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \alpha_i \left(\alpha_* t \int_{\alpha_* t}^{+\infty} s^{n-2} p_i(s)\,ds + \int_{t}^{\alpha_* t} s^{n-1} p_i(s)\,ds + \frac{1}{t}\int_{0}^{t} s^n p_i(s)\,ds \right) > 2(n-2)!$$

and

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \alpha_i^{n-2} \left(\alpha_* t \int_{\alpha_* t}^{+\infty} s^{n-2} p_i(s)\,ds + \int_{t}^{\alpha_* t} s^{n-1} p_i(s)\,ds + \frac{1}{t}\int_{0}^{t} s^n p_i(s)\,ds \right) > (n-1)!,$$

where $\alpha_* = \min\{\alpha_i \colon i = 1, \dots, m\}$. *Then Eq.* (1.4) *has Property* A.

Corollary 6.2′. *Let* $\alpha_i \in [1, +\infty)$, $\tau_i(t) \geqslant \alpha_i t$ *for* $t \in R_+$, $p_i, \tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ $(i = 1, \dots, m)$, *and*

$$\liminf_{t\to+\infty} \int_{t}^{+\infty} s^{n-2}\big(p_i(s) - \tilde{p}(s)\big)\,ds \geqslant 0,$$

$$\liminf_{t\to+\infty} \int_{t}^{\alpha_* t} s^{n-1}\big(p_i(s) - \tilde{p}(s)\big)\,ds \geqslant 0,$$

$$\liminf_{t\to+\infty} \frac{1}{t} \int_{0}^{t} s^{n}\big(p_i(s) - \tilde{p}(s)\big)\,ds \geqslant 0 \quad (i = 1, \dots, m),$$

where $\alpha_* = \min\{\alpha_i\colon\ i = 1, \dots, m\}$. *Let, moreover, for even* n,

$$\limsup_{t\to+\infty}\left(\alpha_* t\int_{\alpha_* t}^{+\infty} s^{n-2}\tilde{p}(s)\,ds + \int_t^{\alpha_* t} s^{n-1}\tilde{p}(s)\,ds + \frac{1}{t}\int_0^t s^n\,\tilde{p}(s)\,ds\right) > \frac{(n-1)!}{m},$$

while for odd n,

$$\limsup_{t\to+\infty}\left(\alpha_* t\int_{\alpha_* t}^{+\infty} s^{n-2}\tilde{p}(s)\,ds + \int_t^{\alpha_* t} s^{n-1}\tilde{p}(s)\,ds + \frac{1}{t}\int_0^t s^n\,\tilde{p}(s)\,ds\right) > \max\left\{\frac{2(n-2)!}{\sum_{i=1}^m \alpha_i}, \frac{(n-1)!}{\sum_{i=1}^m \alpha_i^{n-2}}\right\}.$$

Then Eq. (1.4) *has Property* A.

This corollary can be proved similarly to Corollary 6.1′.

References

[1] V.A. Kondrat'ev, On oscillation of solutions of the equation $y^{(n)} + p(x)y = 0$, Trudy Moskov. Mat. Obshch. 10 (1961) 419–436 (in Russian).

[2] A. Kneser, Untersuchungen über die reelen Nullstellen der Integrale lineare Differentialgleichungen, Math. Ann. 42 (1893) 409–435.

[3] I.T. Kiguradze, T.A. Chanturia, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations, Kluwer Academic, Dordrecht, 1993.

[4] R.G. Koplatadze, T.A. Chanturia, On Oscillatory Properties of Differential Equations with Deviating Arguments, Tbilisi State University Press, 1977, p. 115 (in Russian).

[5] R. Koplatadze, On oscillatory properties of solutions of functional differential equations, Mem. Differential Equations Math. Phys. 3 (1994) 3–179.

[6] R. Koplatadze, G. Kvinikadze, I.P. Stavroulakis, Properties A and B of nth order linear differential equations with deviating argument, Georgian Math. J. 6 (1999) 287–298.